## Cyclotrons

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## What we learn; answers to these questions:

- Why cyclotrons at all? What's their advantage?
- Why must they have complicated magnetic fields? (Instead of the modular dipoles and quadrupoles of synchrotrons)
- Why do no GeV cyclotrons exist?


## RADIOCRAFT, June 1947

(Many first year physics texts and first 20 images on Google are still not much better.)


Quiz: How many errors can you spot?

## ...but modern cyclotrons look like:



## Comparison

| Type | Orbit radius | Magnetic field | Frequency | Phase stable? | Multi-pass? | Pulsed? |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Cyclotron | $\propto \beta$ | fixed | fixed | No | Yes | No |
| FM- or Synchro-Cyclotron | $\propto \beta / \omega$ | fixed | $\propto 1 / \gamma$ | Yes | Yes | Yes |
| Synchrotron | fixed | $\propto \beta \gamma$ | $\propto \beta$ | Yes | Yes | Yes |
| Proton Linac | $\infty$ | fixed | fixed | Yes | No | $?$ |
| Electron Linac | $\infty$ | fixed | fixed | No | No | $?$ |



## The basics

Magnetic field $B$. Particle (mass $m$, charge $q$ ) speed $v$.

$$
\begin{gathered}
F=q v B=\frac{m v^{2}}{R} \\
\omega=\frac{q B}{m}
\end{gathered}
$$

so frequency is constant, independent of speed (energy).
"inch-by-inch..." they discovered... focusing requirements.

## Radially...


well-contained (but on integer resonance). But if field is 'flat', it has no centre; orbital circle can be anywhere in a plane.

What about vertical?

## Vertical...

The "bulging" means there is a radial component $B_{R}$, and the vertical resulting force is $F_{z}=q v B_{R}$. This needs to be linear, with zero force on the mid-plane ( $z=0$ ), so in the Taylor-expansion $F_{z}=q v \frac{\partial B_{R}}{\partial z} z$. But knowing that $\nabla \times B=0$, we also have $\frac{\partial B_{R}}{\partial z}=\frac{\partial B_{z}}{\partial R}$, so finally

$$
\begin{equation*}
F_{z}=q v \frac{\partial B_{z}}{\partial R} z . \tag{1}
\end{equation*}
$$

The force is toward the plane, and simple harmonic motion about it, only if $\frac{\partial B_{z}}{\partial R}<0$; the field must be falling with radius and thus cannot be entirely isochronous.

## Worse... relativity

Field should actually be rising not falling! For momentum $p=\gamma m v$,

$$
B \rho=\frac{p}{q}, \text { or, } B=\gamma \frac{m}{q} \omega
$$

Constant $B$ does not mean constant $\omega$ (unless $v \ll c$ )
So how did $(\gamma=1 \rightarrow 1.5)$ TRIUMF cyclotron get around this?
Ans: Saved by strong (or alternating) focusing.

The magnetic rigidity is the product of the transverse magnetic field and the local radius of curvature of the particle's trajectory.

Take a piece of paper and calculate the $B \rho$ of 500 MeV protons. $m c^{2}=938 \mathrm{MeV}, 500 \mathrm{MeV}=(\gamma-1) m c^{2}$.

## Tunes for constant magnetic field with azimuth

Traditionally, dipoles were characterized by a field index $\kappa$ defined by $\kappa=\frac{R}{B} \frac{\partial B}{\partial R}$. If $\kappa$ is constant, we have $B \propto R^{\kappa}$ : the orbits at different momenta are scaled versions of each other. We can rederive the focal properties from the well-known transfer matrix of a dipole. For both radial and vertical motion, the transfer matrix can be written as

$$
\left(\begin{array}{cc}
\cos k L & \frac{\sin k L}{k}  \tag{2}\\
-k \sin k L & \cos k L
\end{array}\right)
$$

where $L$ is the length of the arc in the dipole, and $k^{2}=k_{x}^{2}:=(1+\kappa) / R^{2}$ for radial motion, and $k^{2}=k_{z}^{2}:=-\kappa / R^{2}$ for vertical. Again, if $\kappa$ is negative, the particles are vertically focused; if positive, $k_{z}$ is imaginary and the matrix becomes, with now $k^{2}=-k_{z}^{2}=\kappa / R^{2}$,

$$
\left(\begin{array}{cc}
\cosh k L & \frac{\sinh k L}{k}  \tag{3}\\
k \sinh k L & \cosh k L
\end{array}\right) .
$$

We can derive tunes now. The tune is the number of oscillations per turn; this is reflected in the argument of the cosine, $k L$. One turn means $L=2 \pi R$, so the number of oscillations per turn is $\nu=k L /(2 \pi)$. So we can read off the tunes:

$$
\begin{equation*}
\nu_{x}=\sqrt{1+\kappa}, \nu_{z}=\sqrt{-\kappa} \tag{4}
\end{equation*}
$$

Notice that $\nu_{x}^{2}+\nu_{z}^{2}=1$ : plotted on a tune diagram, the tunes are confined to the unit circle.

## (to jog the memory)

## From SLAC-91:

$\left[\begin{array}{cccccc}\cos k_{x} L & \frac{1}{k_{z}} \sin k_{x} L & 0 & 0 & 0 & \frac{h}{k_{z}^{z}}\left(1-\cos k_{x} L\right) \\ -k_{z} \sin k_{x} L & \cos k_{x} L & 0 & 0 & 0 & \frac{h}{k_{z}} \sin k_{x} L \\ 0 & 0 & \cos k_{y} L & \frac{1}{k_{y}} \sin k_{y} L & 0 & 0 \\ 0 & 0 & -k_{y} \sin k_{y} L & \cos k_{y} L & 0 & 0 \\ -\frac{h}{k_{z}} \sin k_{x} L & -\frac{h}{k_{z}^{z}}\left(1-\cos k_{x} L\right) & 0 & 0 & 1 & -\frac{h}{k_{z}^{z}}\left(k_{x} L-\sin k_{z} L\right) \\ 0 & 0 & 0 & 0 & 0 & 1\end{array}\right]$

Definitions: $h=1 / \rho_{0}, k_{x}^{2}=(1-n) h^{2}, k_{y}^{2}=n h^{2}$.
$\alpha=h L=$ the angle of bend.
$L=$ path length of the central trajectory.

## Isochronism

As stated, we need $B \propto \gamma$ as well, the orbit radius $R=v / \omega$ has to therefore be $\propto \beta$. The proportionality constants are the two most important constant for cyclotrons:

$$
\begin{gathered}
R=\beta \mathcal{R}_{\infty}, B=\gamma B_{c} \\
\mathcal{R}_{\infty}:=c / \omega, B_{c}:=m \omega / q .
\end{gathered}
$$

So $B$ is function of $R$ as $\gamma$ is a function of $\beta$, and we have this universal curve for cyclotrons:

$$
\begin{equation*}
B(R)=\gamma B_{\mathrm{c}}=\frac{B_{\mathrm{c}}}{\sqrt{1-\beta^{2}}}=\frac{B_{\mathrm{c}}}{\sqrt{1-\left(R / \mathcal{R}_{\infty}\right)^{2}}} \tag{5}
\end{equation*}
$$



The 3 points shown are: (L-R) 12 MeV proton (highest energy according to Bethe), TRIUMF ( 520 MeV ), PSI ( 585 MeV ) cyclotrons.

## What now?

If it's isochronous then vertical tune is imaginary
If vertically stable, it's not isochronous; the best we can do is accelerate quickly enough that particles do not slip in phase to the point of being decelerated.

## Saved: by edge focusing

## Why not quadrupoles?

Split into sectors (weak effect), or spiral sectors (strong focusing)


## What happens to isochronism condition?

(...if the orbit is not circular)

Consider a particle with rigidity $B \rho$ circulating on a non-circular closed orbit: what is the integral of the transverse magnetic field seen by the particle over 1 turn?

The isochronous condition sets a strict relation between the average magnetic field and the length of the closed orbit: can you find it?

## What happens to isochronism condition?

Let us define $\theta$ to be the angle of the reference particle momentum with respect to the lab frame. Orbit length (assumed closed... covered later) $L$ is given by speed and orbit period $T$ :

$$
\begin{equation*}
L=\oint d s=\oint \rho d \theta=\beta c T \tag{6}
\end{equation*}
$$

The local curvature $\rho=\rho(s)$ can vary and for reversed-field bends even changes sign. (Along an orbit, $d s=\rho d \theta>0$ so $d \theta$ is also negative in reversed-field bends.) Of course on one orbit, we always have

$$
\begin{equation*}
\oint d \theta=2 \pi \tag{7}
\end{equation*}
$$

What is the magnetic field averaged over the orbit?

$$
\begin{equation*}
\bar{B}=\frac{\oint B d s}{\oint d s}=\frac{\oint B \rho d \theta}{\beta c T} . \tag{8}
\end{equation*}
$$

But $B$ and $\rho$ change in such a way that $B \rho$ is constant on a closed orbit and in fact is $\beta \gamma \mathrm{mc} / q$. Therefore we see that the original equation (5) is maintained with $B$ replaced by its average, and $\mathcal{R}$ redefined as orbit length divided by $2 \pi ; 2 \pi \mathcal{R} \infty$ being the length of the orbit of the particle had been traveling at the speed of light.

$$
\begin{equation*}
\bar{B}=\frac{2 \pi}{T} \frac{m}{q} \gamma \equiv B_{\mathrm{c}} \gamma=\frac{B_{\mathrm{c}}}{\sqrt{1-\left(\mathcal{R} / \mathcal{R}_{\infty}\right)^{2}}} \tag{9}
\end{equation*}
$$

This shows that isochronism does not require the field to be uniform on an orbit; it can vary by any amount provided the average is correct.

## Let's multiply matrices

Imagine fields are step functions of $\theta$, then there are only drifts and constants, plus thin lenses for magnet edges. (Weirdly, first try, suggested by LH Thomas in 1938, was to make field vary sinusoidally; dipole optics not developed till later.)

$$
\begin{gather*}
M_{x}:=\left(\begin{array}{ll}
1 & d \\
0 & 1
\end{array}\right)\left(\begin{array}{cc}
1 & 0 \\
\frac{1}{f} & 1
\end{array}\right)\left(\begin{array}{cc}
\cos \left[k_{x} s\right] & \sin \left[k_{x} s\right] / k_{x} \\
-k_{x} \sin \left[k_{x} s\right] & \cos \left[k_{x} s\right]
\end{array}\right)\left(\begin{array}{cc}
1 & 0 \\
\frac{1}{f} & 1
\end{array}\right)  \tag{10}\\
M_{z}:=\left(\begin{array}{cc}
1 & d \\
0 & 1
\end{array}\right)\left(\begin{array}{cc}
1 & 0 \\
-\frac{1}{f} & 1
\end{array}\right)\left(\begin{array}{cc}
\cosh \left[k_{z} s\right] & \sinh \left[k_{z} s\right] / k_{z} \\
k_{z} \sinh \left[k_{z} s\right] & \cosh \left[k_{z} s\right]
\end{array}\right)\left(\begin{array}{cc}
1 & 0 \\
-\frac{1}{f} & 1
\end{array}\right) \tag{11}
\end{gather*}
$$

## Add in spiral angle effect

The focal length $f$ of the magnet edges can be read from (e.g.) Brown SLAC-75 as:

$$
\begin{equation*}
\frac{1}{f}=\frac{\tan \alpha}{\rho} \tag{12}
\end{equation*}
$$

The orbit length in the dipole is $s=\rho \vartheta$. Each sector must bend the orbit by $\Theta:=2 \pi / N$, so the orbit enters and exits each sector at an "edge" angle of

$$
\begin{equation*}
\alpha=\frac{1}{2}(\Theta-\vartheta) \tag{13}
\end{equation*}
$$

Let us define the "average radius" $\mathcal{R}$ of the orbit by the total length of the orbit in the sector period $d+s$ :

$$
\begin{equation*}
\frac{2 \pi \mathcal{R}}{N}=d+s, \text { or, } \Theta=\frac{d+s}{\mathcal{R}} \tag{14}
\end{equation*}
$$

Homework: Find tunes for case with no spiral. Now add spiral angles to $\alpha$. What are the tunes?

## How far can we go analytically? (Werner's comment)

PaUI SChERRER IMSTITUT


$$
\begin{aligned}
& Q_{\mathbf{K}}^{2}=\quad v^{2}=\underline{1+K}+i \sum_{n=1}^{5}\left(a_{n}^{2}+b_{n^{2}}^{2}\left\{\left[1+\frac{2 \pi}{\left(N_{n}\right)^{2}} \frac{\left(N_{n}\right)^{2}-2 \sigma^{5}}{\left(N_{n}\right)^{2}-\sigma^{2}}\right]^{\frac{\left(N_{n}\right)^{2}}{\left(\left(N_{n}\right)^{2}-4 \sigma^{2}\right]^{2}}+}\right.\right. \\
& +\frac{2(N m)^{2}}{\left(\left(N^{n} m\right)^{2}-e^{2} j\left(\left(N_{n} n\right)^{2}-4 \sigma^{2}\right]\right.}\left[1+\frac{2^{2}}{\left(N_{m}\right)^{2}} \frac{\left(N_{n}\right)^{2}-2 e^{5}}{\left(N_{n}\right)^{2}-e^{2}}\right]-\frac{1+3 K+K^{\prime}}{2 e^{2}}\left[\frac{3\left(N_{n}\right)^{2}-2+K^{\prime}}{\left(\left(N_{n}\right)^{2}-\theta^{2}\right)^{2}}\right]+ \\
& +\frac{3 K^{*}+K^{*}}{2\left(\left(N_{n}\right)^{2}-e^{2} \tilde{I}^{2}\right.}-4 \sigma^{2} \frac{\left[\left(N_{n}\right)^{2}-1+\left(K^{\prime} / 2\right)\right]}{\left(\left(N_{n}\right)^{2}-e^{2}\right]^{2}\left[\left(N_{n}\right)^{2}-4 \sigma^{2}\right]}-e^{2}\left[\frac{3\left(N_{n}\right)^{2}+2 \lambda^{2}}{N_{n}\left(\left(N_{n}\right)^{2}-4 \sigma^{2}\right]\left[\left(N_{n}\right)^{2}-e^{2}\right]}\right]^{2}+ \\
& \left.+\frac{3 a^{2}}{2}\left[\frac{N_{n}}{\left(N_{n}\right)^{2}-a^{4}}\right]^{2}\right\}+\sum_{*=1}^{n}\left(a_{n} a^{\prime}+b_{*} b_{n}^{\prime}\right)\left\{\frac{(N n)^{2}-2 \sigma^{4}}{(N n)^{2}-4 \sigma^{2}}\left[1+\frac{\lambda^{2}}{\left(N_{n}\right)^{2}} \frac{(N n)^{2}-2 \sigma^{2}}{(N n)^{2}-a^{2}}\right]+\right. \\
& +\frac{\left(\mathrm{N}_{\mathrm{m}}\right)^{2}-2 \sigma^{2}}{\left[\left(\mathrm{~V}_{m}\right)^{2}-\sigma^{2}\right]\left(\left(\mathrm{N}_{n}\right)^{2}-4 \sigma^{2}\right)}-\frac{1+3 K+K^{2}}{2 \sigma^{2}\left(\left(\mathrm{~V}_{n}\right)^{2}-\sigma^{2}\right)}+\frac{1}{\left(\mathrm{~N}_{n}\right)^{2}-\sigma^{2}}\left[1-\frac{\lambda^{2}}{\left(N_{n}\right)^{2}} \frac{\left(N_{n}\right)^{2}-2 \sigma^{2}}{\left(N_{m}\right)^{2}-\sigma^{2}}\right]-
\end{aligned}
$$

$$
\begin{aligned}
& \text { Fig. 3. General formula for radial freqnency, vr useless ! }
\end{aligned}
$$

$$
\begin{aligned}
& Q_{2}^{2}=v_{2}^{2}-K+i \sum_{n=1}^{n}\left(a_{n}^{2}+b_{n}{ }^{2}\right)\left(\frac{\left(N_{n}\right)^{2}}{(N n)^{2}-(1+K)}+i\left(K+\frac{3 K^{\prime}}{1+K}\right)\left[\frac{N_{n}}{(N n)^{2}-(1+K)}\right]^{2}-\right. \\
& \left.-\left[K^{\prime} \frac{2+K}{1+K}+\frac{K^{*}}{2}-\frac{\left(K^{\prime}\right)^{2}}{2(1+K)}\right]\left[\frac{1}{\left(N_{n}\right)^{2}-(1+K)}\right]^{2}+\frac{K^{2}}{\left(N_{n}\right)^{2}\left[\left(N_{n}\right)^{2}-(1+K)\right]^{2}}-\frac{4 K\left(K^{\prime}\right)^{2}}{\left(N_{n}\right)^{2}\left[\left(N_{n}\right)^{2}-\left(1+K^{2}\right)\right]^{2}}\right]+ \\
& \left.+\frac{+}{n} \sum_{=1}^{n}\left(a_{n} a^{\prime} n+b_{n} b_{n}^{\prime}\right) \left\lvert\, \frac{2 K^{\prime}}{\left(N_{n}\right)^{2}\left[\left(N_{n}\right)^{2}-(1+K)\right]}-\frac{1+K-K^{\prime}}{(1+K)\left[\left(N_{n}\right)^{2}-(1+K)\right]}-\frac{8 K K^{\prime}}{\left(N_{n}\right)^{4}\left[\left(N_{n}\right)^{2}-(1+K)\right]}\right.\right)- \\
& -+\sum_{n=1}^{n} \frac{a_{n} a^{n} n+b_{n} b^{n} s}{\left(N_{n}\right)^{2}-(1+K)}+\frac{1}{2} \sum_{n=1}^{n}\left(a^{2} s^{2}+b_{n}^{2}\right)\left(\frac{1}{\left(N_{n}\right)^{2}}-\frac{4 K}{\left(N_{n}\right)^{4}}\right) . \\
& \text { Fig. 4. General formula for axial Irequency, } v_{2} \quad \longrightarrow \text { useless! }
\end{aligned}
$$

## how to scare young students!!

(Al Garren 1962)

> better approach to get focusing frequencies:
> 1. simple approximations
> 2. numerical calculations
> $\mathrm{Q}_{\mathrm{x}}^{2} \approx 1+\mathrm{k}$
> $\mathrm{Q}_{z}^{2} \approx-\mathrm{k}+\mathrm{F}\left(1+2 \tan ^{2} \delta\right)$

End of Lecture 1

## Why should $\partial B / \partial R$ derivative depend on $\bar{B}$ ?

How about Fresnel lens idea?


## Why should $\partial B / \partial R$ derivative depend on $\bar{B}$ ?

How about Fresnel lens idea?



Lighthouse Fresnel


Fig. 28 Schematic view of a separated orbit cyclotron (SOC) with single acceleration gap and individual magnetic channels (dotted) around each orbit.


Fig. 29 Field profile in a separated orbit cyclotron. The magnetic field increases globally with radius to keep the revolution frequency isochronous for relativistic energies. The local gradient around each orbit can be chosen independently and can be used to give transversal and longitudinal focusing as in a synchrotron!

## Why should $\partial B / \partial R$ derivative depend on $\bar{B}$ ?



Fig. 2. Cross section of four pole pairs of an SOC magnet.
R.S. Lord et al., Magnet for an $800-\mathrm{MeV}$ Separated Orbit Cyclotron

## Why should $\mathrm{d} B / \mathrm{d} R$ derivative depend on $\bar{B}$ ?

Quiz: What's good/bad about this?
Look up the TRITRON cyclotron.

## How high E?

Specifically, $\gamma$. EMMA FFA (next slide) reached $\gamma=41.20 \mathrm{MeV}$ electrons but at 2 MeV per turn ( $\Delta \gamma=4$ per turn). TRIUMF and PSI exist since 1974, reach $\gamma=1.6$ with a $\Delta \gamma=0.0003,0.003$ per turn. (Protons are harder to accelerate.) But we have designed up to 12 GeV ( $\gamma=13$ ), but not built. Why not? Maybe linac people had more political clout.


Five sectors from the 40 m radius 42 sector cyclotron showing magnets, RF cavities and proton orbits.

Exercise: From these few parameters plus the rectangular boxes being rf cavities that give 1 MV , calculate the final orbit separation.


End of lecture 1 a .

