Cyclotrons 2: Magnetic Field and Transverse Dynamics

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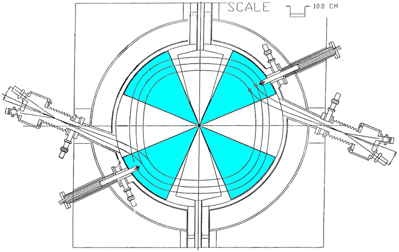
what we learn

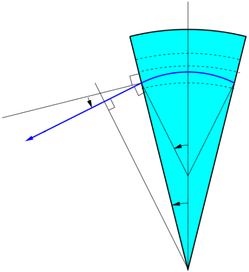
- How to describe magnetic fields.
- How to calculate orbits in a cyclotron.
- How to find **closed** orbits.
- How to find Courant Snyder parameters at all azimuths, energies.

Why we can't have nice simple dipoles

Combined function to get the focusing, field index to get the gradient necessary for isochronism, ... should be simple, right?

Look at the dashed line and the blue line.





...

Key differences is if we know the orbit, we design around it. But for cyclotrons we don't: we have to find it. What about just dispersion? Limited range; orbits cannot change in character, unless scaled. What is meant by scaling?

Scaling rules... If $B(R,\theta) = B_0(\theta)(R/R_0)^\kappa$, then all orbits have exact same character and exact same tune.

But this violates the isochronism rule that $\overline{B(R)} \propto \left[1-(R/R_\infty)^2\right]^{-1/2}$.

So cannot have scaling isochronous machines.

That means isochronous machines have non-simple (non-integrable) magnetic fields.

What do we need?

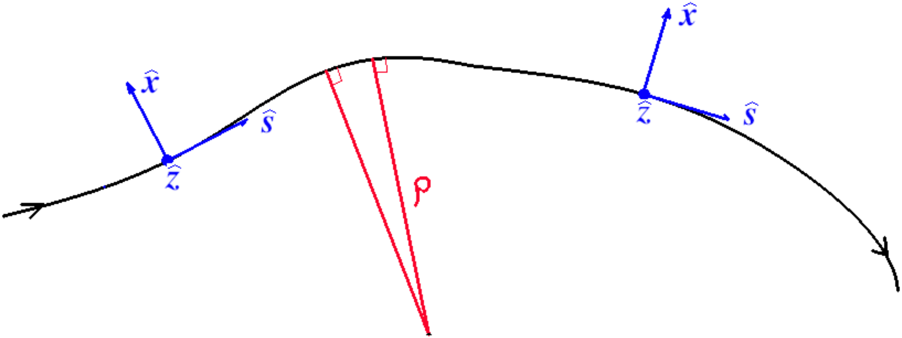
Hamiltonian mechanics to optimally and efficiently numerically integrate through the periodic sector fields.

A primer on Hamiltonian dynamics in accelerators can be found in two of my lectures for a different school.

Frenet-Serret

(looking down on) coordinate system used for synchrotrons.

N.B.: We continue to use z for vertical, x for horizontal.



What do we need?

In a synchrotron, one uses the Frenet-Serret coordinate system.
Q:What is it? Let z be vertical, i.e. the direction of the magnetic field on the median plane. Let s be direction of the reference orbit $\vec{v}$. Then x is the direction of the force ($\vec{v} \times \vec{B}$). But why can we do this? Because we **start** with this orbit. In the cyclotron case, this orbit is **not known**.

What we do know is the field on some grid (and that's all we know).

That grid is often, especially for ‘compact’ cyclotrons, on a polar grid. Let us assume this is so.

In case of “median plane symmetry”, if we have: $B(r, \theta)$ on the median plane, we have enough. Maxwell does the rest. If we do not, the field components in the plane drive the particles away from the plane.

Magnetic Field

Since we are interested in orbits on or near the median plane $z = 0$, we expand the potential and, hence, the field in powers of z. Since $\nabla \times \vec{B} = \vec{0}$, we can set $\vec{B} = -\nabla\Psi$, and hence, since $\nabla \cdot \vec{B} = 0$ also, we have:

$$\nabla^2\Psi = \frac{\partial^2\Psi}{\partial z^2} + \nabla_2^2\Psi = 0, \tag{1}$$

where ∇_2^2 is the 2-dimensional Laplace operator

$$\nabla_2^2\Psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}. \tag{2}$$

Solution for median plane symmetry

So we just solve Laplace equation with $B(r,\theta)$ being the $z=0$ boundary condition. The solution is

$$\Psi_{\rm o} := zB - \frac{z^3}{3!}\nabla_2^2 B + \frac{z^5}{5!}\nabla_2^4 B - ..., \tag{3}$$

(subscript meaning “odd”)

Scary but: For most purposes, need only the first term.

...and the field's three components are given by:

$$B_z = -B + \frac{z^2}{2!}\nabla_2^2 B - \frac{z^4}{4!}\nabla_2^4 B + ... \quad (4)$$

$$B_r = -z\frac{\partial B}{\partial r} + \frac{z^3}{3!}\frac{\partial \nabla_2^2 B}{\partial r} - ... \quad (5)$$

$$rB_\theta = -z\frac{\partial B}{\partial \theta} + \frac{z^3}{3!}\frac{\partial \nabla_2^2 B}{\partial \theta} - ... \quad (6)$$

(Again, likely need only the first term of each.)

Thus, the entire field off the median plane can be expressed in terms of B and its derivatives.

What if no median plane symmetry?

Then on the median plane, there is a companion field $C(r,\theta)$, and an asymmetric part to the scalar potential:

$$\Psi_{\mathrm{e}} := C - \frac{z^2}{2!}\nabla_2^2 C + \frac{z^4}{4!}\nabla_2^4 C - \ldots \tag{7}$$

Taking only first two terms,

$$\begin{aligned} B_z &= -B + \nabla_2^2 C z && (8)\\ B_r &= -\frac{\partial C}{\partial r} - \frac{\partial B}{\partial r} z && (9)\\ rB_\theta &= -\frac{\partial C}{\partial \theta} - \frac{\partial B}{\partial \theta} z && (10) \end{aligned}$$

For magnetic measurement purposes, it is to be noted that $\frac{\partial B_z}{\partial z} = \nabla_2^2 C$. Thus from a measurement of B_z at the geometric median plane and a distance Δz above and below it, the derivatives of $C(r,\theta)$ can be estimated.

Quiz

Which of the following statements are correct?

1. Any magnetic field distribution in a 2-D plane can be extrapolated to a 3-D distribution that satisfies Maxwell's equations (with no charge, no current source).
2. A 2-D distribution can be extrapolates to 3-D only when assuming median plane symmetry.
3. The way to extrapolate the magnetic field from 2-D to 3-D is **not unique**, and depends on the choice of gauge.
4. Extrapolation far off a 2-D plane is extremely sensitive to the noise in the 2-D field data.

Quiz

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4. Extrapolation far off a 2-D plane is extremely sensitive to the noise in the 2-D field data.

Motivating $\mathcal{H}$

Why can't we just use $\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = q(\vec{\mathcal{E}} + \vec{v} \times \vec{B})$? Even J.D. Jackson in his E&M textbook uses it in the following form

$$\frac{\mathrm{d}\vec{\beta}}{\mathrm{d}t} = \frac{q}{\gamma mc}\left[\vec{\mathcal{E}} + \vec{\beta} \times \vec{B} - \vec{\beta}(\vec{\beta} \cdot \vec{\mathcal{E}})\right] \tag{11}$$

(Perhaps not obvious until one realizes that $\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = m\gamma\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} + m\vec{v}\frac{\mathrm{d}\gamma}{\mathrm{d}t}$.)

This formulation neglects that the fields are derivable from potentials. These potentials are included in the definitions of the canonical momenta/energy, and so in this formulation, the conservation laws are not intrinsic to the dynamics. Not only can conservation laws be violated, but also, the description is not efficient, not optimally parsimonious: over-described fields can be self-contradictory.

Equations of Motion

From the basic

$$(E - q\Phi)^2 - p^2c^2 = m^2c^4\,, \tag{12}$$

knowing the canonical momenta

$$\begin{aligned} p_x &= P_x - qA_x & (13)\\ p_y &= P_y - qA_y & (14)\\ p_z &= P_z - qA_z, & (15) \end{aligned}$$

we find the usual Hamiltonian as $\mathcal{H}_t = E$:

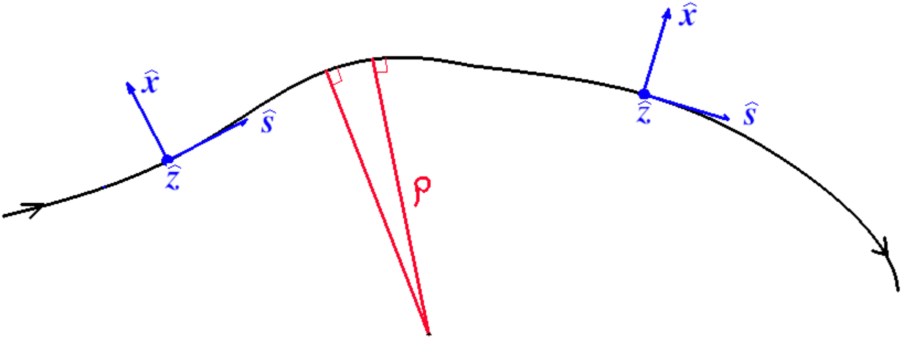
$$\begin{aligned} &\mathcal{H}_t(x, P_x, y, P_y, z, P_z) = \\ &q\Phi + c\sqrt{m^2c^2 + (P_x - qA_x)^2 + (P_y - qA_y)^2 + (P_z - qA_z)^2} \end{aligned} \tag{16}$$

but this isn't the one we want. Why?

Frenet-Serret

(looking down on) coordinate system used for synchrotrons.

N.B.: We continue to use z for vertical, x for horizontal.



In the case of a synchrotron, we already know the particles are near some axis of a vacuum tube, we want **differential** values, w.r.t. the intended orbit, and time of arrival, and energy deviation. There we use the Frenet-Serret coordinate system, where the fundamental relation among the canonical variables is

$$\left(\frac{E - q\Phi}{c}\right)^2 = m^2c^2 + \frac{(P_s - qA_s)^2}{\left(1 + \frac{x}{\rho}\right)^2} + (P_x - qA_x)^2 + (P_z - qA_z)^2 \quad (17)$$

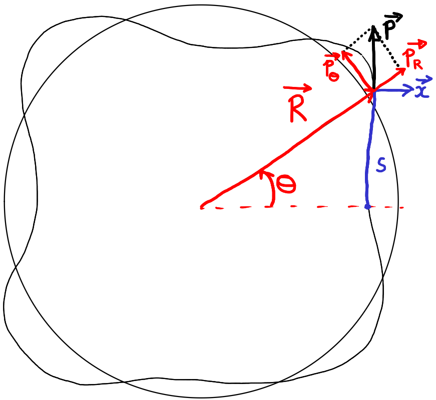
where $\rho = \rho(s)$ is the radius of curvature of the reference trajectory at location s.

We want these at distance s along the reference orbit. We're not asking what are the particles' coordinates at given time, but rather What are they as well as time t, at some s. IOW, we want the independent variable to be s. In that case, the Hamiltonian is

$$\mathcal{H}_s(x, P_x, z, P_z, t, E) = -P_s =$$
$$-qA_s - \left(1 + \frac{x}{\rho}\right)\sqrt{\left(\frac{E - q\Phi}{c}\right)^2 - m^2c^2 - (P_x - qA_x)^2 - (P_z - qA_z)^2} \tag{18}$$

But. We don't want this one either, since we do not know the reference orbit.

θ, not s



We want some reference close to the eventual closed orbit, so we choose a circle, IOW, go to polar coordinates. In any case, the magnetic field is on a polar grid already! In polar coordinates (r, θ, z), the ordinary momentum consists of the components along these three directions (p_r, p_θ, p_z) but the corresponding canonical momenta (P_r, P_θ, P_z) are related to the ordinary ones as follows:

$$
\begin{aligned}
p_r &= P_r - qA_r && (19)\\
p_\theta &= P_\theta/r - qA_\theta && (20)\\
p_z &= P_z - qA_z. && (21)
\end{aligned}
$$

The Hamiltonian with θ as the independent variable is $\mathcal{H}_\theta = -P_\theta = -r(p_\theta + qA_\theta)$ and hence solving eqn. 12 for P_θ:

$$\begin{aligned} &\mathcal{H}_\theta(r, P_r, z, P_z, t, E) = \\ &-r\left[p(E)^2 - (P_r - qA_r)^2 - (P_z - qA_z)^2\right]^{1/2} - q\,r\,A_\theta. \end{aligned} \tag{22}$$

In this case the conjugate pairs are (r, P_r), (z, P_z), and $(t, -E)$. Where we have defined p:

$$p(E)^2 = \frac{1}{c^2}(E - q\,\Phi)^2 - m^2c^2 = (\gamma^2 - 1)m^2c^2 = 2mE_\mathsf{k}\left(1 + \frac{E_\mathsf{k}}{2mc^2}\right) \tag{23}$$

(since one may want to parameterize using $E_\mathsf{k} = (\gamma - 1)mc^2$ rather than E).

Finally, the eom's

$$r' = \frac{r\,p_r}{\sqrt{p^2 - p_r^2 - p_z^2}} \tag{24}$$

$$p_r' = \sqrt{p^2 - p_r^2 - p_z^2} + q(r\,B_z - z'\,B_\theta) + q\,t'\,\mathcal{E}_r \tag{25}$$

$$z' = \frac{r\,p_z}{\sqrt{p^2 - p_r^2 - p_z^2}} \tag{26}$$

$$p_z' = q(r'\,B_\theta - r\,B_r) + q\,t'\,\mathcal{E}_z \tag{27}$$

$$t' = \frac{\gamma\,m\,r}{\sqrt{p^2 - p_r^2 - p_z^2}} \tag{28}$$

$$E' = q(r'\,\mathcal{E}_r + r\,\mathcal{E}_\theta + z'\,\mathcal{E}_z) \tag{29}$$

where $\vec{B}$ is the magnetic field, and $\vec{\mathcal{E}}$ is the electric field. (For finding closed orbits, we set $\vec{\mathcal{E}} = 0$.)

eom Notes

- These are **exact** and can be used in general orbit codes (e.g. `GOBLIN`). The only limit is knowledge of $\vec{B}$.
- We used canonical momenta to derive the equations, but then reverted to using p_r, p_z. It's simpler.
- For median plane symmetry, $B_r = B_\theta = 0$ when $z = 0$, and then $z = p_z = 0$ is a solution. Particles stay on median plane,
- so then z and p_z are differential quantities with respect to the closed orbit. But r and p_r are **not.**
- With no electric field, E is constant, orbit time (isochronism) is given by simple integral $T = \gamma m \oint \frac{r\mathrm{d}\theta}{p_\theta}$.

Equations of motion

Try to derive $r' = \frac{\mathrm{d}r}{\mathrm{d}\theta}$

Equations of motion

Try to derive $z' = \frac{\mathrm{d}z}{\mathrm{d}\theta}$

Equations of motion

A little more complicated: try to derive $p_r' = \frac{\mathsf{d}P_r}{\mathsf{d}\theta} - q\frac{\mathsf{d}A_r}{\mathsf{d}\theta}$

Closed Orbit

On the (assumed flat) median plane, we have simply

$$r' = \frac{r\,p_r}{p_\theta} \tag{30}$$

$$p_r' = p_\theta + qr\,B_z, \tag{31}$$

where $p_\theta := \sqrt{p^2 - p_r^2}$.

Exercise: Use these equations to show the expected behaviour for (1) Flat field B=constant independent of (r, θ) and $p_r \ll p$, and (2) Exact solution for $B = 0$ (hint: try a substitution $p_r = p\sin\psi$.)

We can integrate through 2π, but the orbit will not close, so we simultaneously integrate the differentials of the orbit. We let

$$r \to r + x, \quad p_r \to p_r + p_x \tag{32}$$

Closed Orbit...

where (x, p_x) are first differentials of (r, p_r); they will essentially be coordinates of betatron motion about the main orbit. By differentiating, we find:

$$x' = \frac{p_r}{p_\theta}x + \frac{rp^2}{p_\theta^3}p_x \qquad (33)$$

$$p_x' = q\left[B_z + r\frac{\partial B_z}{\partial r}\right]x - \frac{p_r}{p_\theta}p_x, \qquad (34)$$

The coefficients of (x, p_x) are here **not** evaluated locally, but are taken from the (r, p_r) orbit and so these equations are linear as desired, and therefore generate the transfer matrix for the betatron motion. We simply start with initial conditions $(x, p_x) = (1, 0)$ and $(0, 1)$ to generate the matrix elements.

Numerical Procedure:

Integrate r and p_r for some chosen initial values, but at same time integrate x and p_x for initial values $x = 1$, $p_x = 0$, AND $x = 0$ and $p_x = 1$.

r and p_r after one turn tell you how far from closed you are, while the other 4 values tell you the matrix to be used to correct the initial r and p_r to try again. Then do it again. Converges very quickly, just like Newton's method.

Closed Orbit...

As we have both the (non-closed) orbit and its 2D differential the transfer matrix, we apply Newton's method and find the closed orbit. Since there may be small nonlinearities in the (r, p_r) equations of motion, only a few iterations required.

Explicitly, if the closed orbit at $\theta = 0$ and $\theta = \theta_\mathrm{f}$ is denoted $(r_\mathrm{c}, p_{r\mathrm{c}})$, the expected behaviour is

$$\begin{pmatrix} r_\mathrm{c} \\ p_{r\mathrm{c}} \end{pmatrix} = \begin{pmatrix} r(\theta_\mathrm{f}) \\ p_r(\theta_\mathrm{f}) \end{pmatrix} + M_x \left[\begin{pmatrix} r_\mathrm{c} \\ p_{r\mathrm{c}} \end{pmatrix} - \begin{pmatrix} r(0) \\ p_r(0) \end{pmatrix} \right]. \tag{35}$$

Solving for the closed orbit, we get:

$$r_\mathrm{c} = r(0) + \frac{(M_{22} - 1)\epsilon_r - M_{12}\epsilon_p}{M_{11} + M_{22} - 2} \tag{36}$$

$$p_{r\mathrm{c}} = p_r(0) + \frac{(M_{11} - 1)\epsilon_p - M_{21}\epsilon_r}{M_{11} + M_{22} - 2} \tag{37}$$

and the vertical matrix...

Apply the same treatment to vertical motion: We linearize equations 26 and 27. This omits the p_z^2 in the denominator since it's higher order, and the fields come from only the first terms in eqns. 5, 6, which are linear in z. The result is

$$z' = \frac{r}{p_\theta} p_z \tag{38}$$

$$p_z' = q\left[-r\frac{\partial B_z}{\partial r} + \frac{p_r}{p_\theta}\frac{\partial B_z}{\partial \theta}\right] z \tag{39}$$

again, r, p_r and the field derivatives are evaluated at the (r, p_r), $z = p_z = 0$ orbit, not the local one $(r + x, p_r + p_x)$.

That's a total of how many ODEs? Count them.

Exercise: Using eqns. 30, 31, 33, 34, 38, 39 show that when B is a function of r only, we recover the tunes of the classical cyclotron.

Courant-Snyder parameters

We integrate the equations (usually using Runge-Kutta technique) and then have the actual horizontal and vertical transfer matrices for one period (either sector or whole turn); in fact outputting at every Runge-Kutta step, we have matrices for all θ. From them we first of all get the betatron phase advance (hence, the tune) from the trace of the matrix for the period, then can distill the Courant-Snyder parameters $\alpha_x, \beta_x, \alpha_z, \beta_z$ in the usual way as done for example for synchrotrons or periodic transfer lines.

Courant-Snyder parameters

We want the periodic matrix $M_{\rm p}(\theta)$ which takes the coordinates from θ to $\theta+\theta_{\rm f}$. The missing piece is the matrix that takes one from θ to $\theta_{\rm f}$. This is clearly $M(\theta_{\rm f})M^{-1}(\theta)$. (Remember: right to left, first backtrack to zero, then forward-track through the complete sector.) Then we forward-track a further distance of θ, finally giving:

$$M_{\rm p}(\theta) = M(\theta)M(\theta_{\rm f})M^{-1}(\theta) \tag{40}$$

CS parameters

Courant-Snyder parameters $\alpha(\theta), \beta(\theta), \gamma(\theta)$ are then found by equating:

$$M_{\rm p}(\theta) = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix} \tag{41}$$

where μ is the phase advance per period, $\nu\theta_{\rm f}$.

One tiny detail about Frenet-Serret

Note on symplecticity: The equations of motion for x, p_x arise from the Hamiltonian:

$$\mathcal{H}_\theta^x = -q\left[B_z + r\frac{\partial B_z}{\partial r}\right]\frac{x^2}{2} + \left[\frac{p_r}{p_\theta}\right]xp_x + \left[\frac{rp^2}{p_\theta^3}\right]\frac{p_x^2}{2}. \qquad (42)$$

The CS parameters that this generates are not quite the same as the usual ones. Why? Think of the x direction. In θ-picture, x is always radial, but in s-picture, x is always perpendicular to the momentum on the reference orbit. This will mean the β-function while agreeing at $r_{\min}$ and $r_{\max}$, it oscillates slightly about it at other values.

Further, we could switch to s as independent variable using $(\mathrm{d}s)^2 = (\mathrm{d}r)^2 + (r\mathrm{d}\theta)^2$. We could re-derive the Courant-Snyder Hamiltonian that results in the simpler $x'' + K(s)x = 0$, but there's little point.

Integration Details

- If measured B_z is on a polar grid, the Runge-Kutta steps can be the azimuth increments and no interpolation is needed. This is a real advantage.
- If noisy, the field data can be improved by filtering out unphysically high harmonics. Since there can be no field fluctuations on a length scale short compared with the magnet gap, Fourier analyze the field with azimuth, and throw away Fourier components above a certain limit.
- In the radial direction, we do need interpolation that gives accurate values for the field and its derivative. Commonly, the cubic spline is used, applied to either the fields at constant r, or to the Fourier components. The same considerations can be applied to the interpolation regarding noise as for the azimuthal.
- The result of field noise will be that the tunes calculated in the orbit code fluctuate as a function of radius more rapidly than about half the magnet gap. This can be used to fine tune the field and the interpolation routine.