Cyclotrons: Resonances

Rick Baartman

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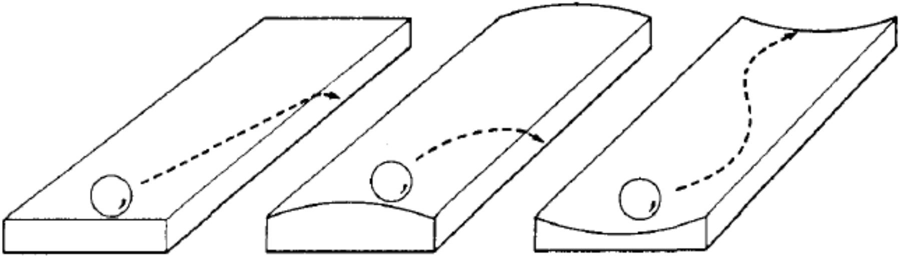
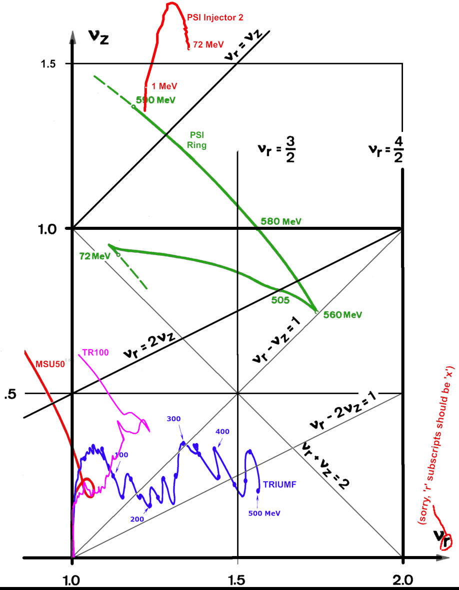
To learn, this lecture

- Compared to other rings, what's special in cyclotron case.
- 1D resonances, their order; integer, half-integer, etc.
- ODEs to describe them
- Hamiltonian approach and perturbation theory
- Fun Mathematica apps, stable/unstable fixed points
- Intrinsic resonances
- Fast passage through resonances; not the same as static case.
- Coupling resonances, their characteristics
- Walkinshaw resonance behaviour (through Mathematica app) and why important for cyclotrons

Bowling Alley Analogue



Tune Diagram



It's a frequency domain space: Horizontal fequency is ν_x (ν_r in the plot, sorry), vertical is ν_z.

- For all integers, (n_x, n_z, m), the condition

$$n_x\nu_x \pm n_z\nu_z = m \qquad (1)$$

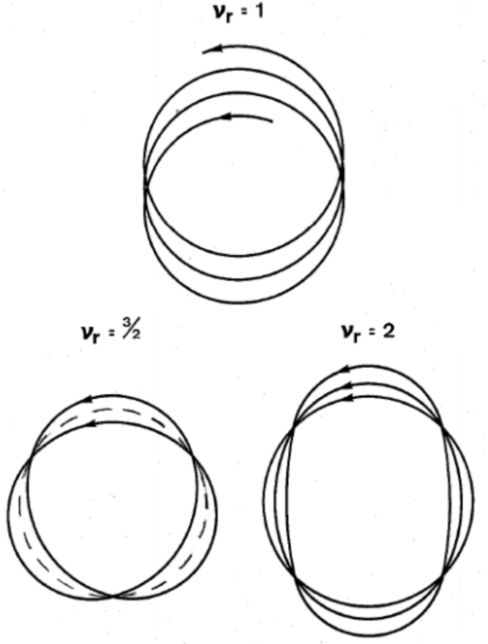
  is a resonance. Why?
- These are lines in this space.
- 5 examples. What's peculiar about cyclotrons?
- The most dangerous lines are the boldest.

Going to explain in 4 different ways:

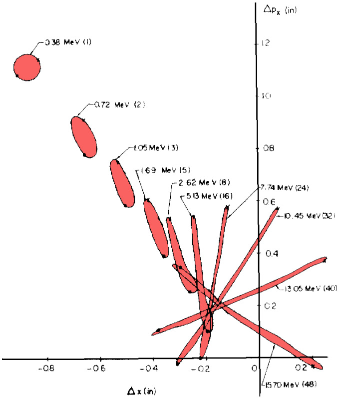
- hand-wavy,
- ODEs roughly,
- Canonical transformations,
- Mathematica.

Hand-wavy

- Circular orbit, integer tune. Closed orbit anywhere.
- If not an integer, excursion crossover points move along orbit. If an integer, they are frozen.
- Half integer seems self cancelling. But wait, what about envelope?


νr = 1
νr = 3/2
νr = 2

e.g. Half-integer



In general, $\nu = m/n$ pulls phase space in n directions...

As cyclotrons start from $\nu_x = 1$, a 2-sector cyclotron is not possible.
Why?
A 3-sector cyclotron can reach $\gamma = 3/2$? No, it can hardly reach 180 MeV for protons.

Some theory

We write the Hamiltonian as (SHM plus perturbation)

$$\mathcal{H} = \frac{\tilde{p}^2}{2} + \frac{\nu^2 x^2}{2} + \left(\frac{m}{n}\, b_{mn}\right)\, x^n\, \cos(m\theta + \theta_m) \tag{2}$$

where

$$\left(\frac{m}{n} b_{mn}\right) := \frac{\mathcal{R}^2}{B\rho}\, \frac{1}{n!}\, \frac{\partial^{n-1} B_m}{\partial x^{n-1}} \tag{3}$$

$x' = \frac{\partial \mathcal{H}}{\partial \tilde{p}}$, $\tilde{p}' = -\frac{\partial \mathcal{H}}{\partial x}$, gives:

$$x'' + \nu^2 x = -\left(m\, b_{mn}\right)\, x^{n-1}\, \cos(m\theta + \theta_m) \tag{4}$$

Action-angle coordinates (J, ψ):

$$x = \sqrt{2J/\nu}\,\cos\psi,\ \tilde{p} = \sqrt{2J\nu}\,\sin\psi \tag{5}$$

The transformed Hamiltonian:

$$\mathcal{H} = \nu J + \frac{mb_{mn}}{n}\ \left(\frac{2J}{\nu}\right)^{n/2}\cos^n\psi\cos(m\theta + \theta_m) \tag{6}$$

Solution

$$J' = -\frac{\partial H}{\partial \psi} = m b_{mn} \left(\frac{2J}{\nu_0}\right)^{n/2} \cos^{n-1}\psi \sin\psi \cos(m\theta + \theta_m), \tag{7}$$

or, expanding the trig powers,

$$J' = m b_{mn} \left(\frac{J}{2\nu_0}\right)^{n/2} \left[\sin(n\psi - m\theta - \theta_m) + \text{ other terms}\right] \tag{8}$$

We retain the designated term because it is the only one with slow variation with θ, and does so when the resonance condition

$$n\nu_0 = m \tag{9}$$

is met; the other terms will vary too rapidly to make a net contribution, because Eqn. 8 is quite easily integrated under the condition of a fixed tune, since

$$\psi' = \frac{\partial H}{\partial J} = \nu + \text{ oscillatory term}, \tag{10}$$

and so to a good approximation, within a constant, $\psi = \nu_0\theta$.

The x^n for order n gives an n times $\nu\theta$ dependence, and the harmonics m give $m\theta$. Product gives terms where if $n\nu = m$, the θ-dependence vanishes for one of the harmonic terms.

So $J' \propto J^{n/2}$ or converting back to $A = \sqrt{J}$, $J = A^2$, $J' = 2AA' \propto J^{n/2} = A^n$, $A' \propto A^{n-1}$. This confirms: $n = 1$ means A grows linearly, $n = 2$ means A grows exponentially.

Also notice that there is a 2^n in the denominator; for every trig power there is a factor 2 reduction. This with $n!$ means resonances fall in importance with n.

But nth derivatives of B grow with n if $B \propto \gamma$. This means very high energy cyclotrons, (in multi-GeV range, none ever built) could get challenging.

Mathematica Demo's...

Intrinsics

Perturbing fields often called "error fields", but that's not always the case. For N sectors, we have intrinsic harmonics for all integer multiples of N. So $n\nu = mN$ are called intrinsics or 'structural' resonances.

Expect a strong resonance at $\nu_x = 3/2$ ($\gamma = 3/2$ or proton energy 470 MeV), but if it's intrinsic like in a 3-sector cyclotron, cannot cross it or even get close.

There is also a zero intrinsic. This is relevant for vertical tune at injection in compact (non-ring) cyclotrons, and will also be shown to be relevant for coupling resonances.

‘Fast’ Resonance passage

See eqn.10. On a resonance, $\psi = \nu\theta$, so easy to calculate growth rate: $A' \propto A^{n-1}$.

For varying tune as the beam passes through a resonance, the naïve approach would be to assume there are some particles at the worst phase and set the sine to 1 and integrate J to find the growth over a finite time, for instance over the time it takes to pass through the stopband. Or half that time since the growth rate varies between zero and maximum through the stopband. This would be correct for fixed tune or for rate of change of tune ($\nu' := \mathrm{d}\nu/\mathrm{d}\theta$) to be very small compared with the amplitude growth rate. It is not true in general. Instead of fixed ψ, we have a quadratic function of θ:

$$\psi \approx \int \nu \mathrm{d}\theta = \nu_0\theta + \nu'\theta^2/2. \tag{11}$$

$$n\psi - m\theta = n[\nu_0\theta + \nu'\theta^2/2] - m\theta = n\nu'\theta^2/2. \tag{12}$$

The action equation is a little simpler if we revert to $A = \sqrt{2J/\nu_0}$, the betatron amplitude:

$$\frac{A'}{A^{n-1}} = \frac{mb_{mn}}{2^n \nu_0}\, \sin(n\nu'\theta^2/2 - \theta_m) \tag{13}$$

We see that we get a Fresnel integral. The largest amplitude gain occurs for phase $\theta_m = \pi/4$:

$$\frac{\Delta(A^{2-n})}{2-n} = \frac{mb_{mn}}{2^n \nu_0} \sqrt{\frac{2\pi}{n\nu'}} \tag{14}$$

Note $\nu_0 = m/n$. Also, we prefer the tune change per turn, $\nu_\tau \equiv 2\pi\nu'$,

$$\frac{\Delta(A^{2-n})}{2-n} = \frac{\pi}{2^{n-1}}\, b_{mn} \sqrt{\frac{n}{\nu_\tau}} \tag{15}$$

Of course this does not hold for $n = 2$; in that case, the LHS is $\Delta(\log A)$.

Exercise: Take a simple case where beam is injected into a compact proton cyclotron. Find the static orbit shift per Gauss of first harmonic field amplitude, assuming an isochronous field. Now also find the fast passage shift given by the above formula, assuming energy gain per turn is 100 keV.

Coupling resonances

This follows same H dynamics, but with extra dimensions. The advantage of the H is that the two dimensions' coupling is known from Maxwell equations. We know e.g. that potentials are like Laplace solutions: $x, z, x^2 - z^2, xz, x^3 - 3xz^2$, etc. It means for example that if field has to rise for isochronism, there is an x^2 term and there will result a z^2. Etc.

The theory above can be generalized to coupling resonances. We define **positive** integers n_x, n_z, m as referring to the resonance

$$n_x\nu_x \pm n_z\nu_z = m. \tag{16}$$

The "order" of the resonance is $n_x + n_z$.

Coupling Hamiltonian term is:

$$\mathcal{H}_1 = a_m \left(\frac{J_x}{2\nu_x}\right)^{n_x/2} \left(\frac{J_z}{2\nu_z}\right)^{n_z/2} \cos(n_x\psi_x \pm n_z\psi_z - m\theta), \tag{17}$$

where:

$$a_m = a_m(n_x, n_z) = \frac{1}{n_x! n_z!} \frac{\mathcal{R}^2}{B\rho} \left.\frac{\partial^{n_x+n_z-1} B_{(x|z)}}{\partial x^{n_x+n_z-1}}\right|_m . \tag{18}$$

The subscript m means the m^{th} Fourier component of the field derivative of B and the notation $B_{(x|z)}$ means we take the radial or x-component of $\vec{B}$ if n_z is odd, and the vertical or z-component if it is even. This can be recognized as respectively the "skew" and "normal" multipole components of the magnetic field.

New invariant follows from $J_x'/n_x \mp J_z'/n_z = 0$ from $J' = \partial H/\partial\psi$. A sum resonance grows without limit, a difference has a limit: one grows, the other shrinks.

First I want to show a real life example of a coupling resonance. 3 masses, 2, and then 1 is where the springing is exactly twice the swinging freq.

Note there is no driving harmonic, no forced oscillation; only coupling.

Back to cyclotrons. Again, no driving harmonic; only a second radial field derivative. Turn down strength to get perfect Lissajous, varying ν_x (don't forget momenta).

Put back onto $\nu_x = 2\nu_z$, show SFP, UFP at $z = 0$, invariance of $J_x + J_z/2$.

Now show the phase space traj's., let "turns" self-run.

Remember: There are particles with all initial conditions, covering their emittance in phase space. Once through a resonance, the phase spaces are correlated.

This is Walkinshaw and why is it important for cyclotrons? See 7.58 and following, 2nd derivative of B_z with R, automatic when $B \propto \gamma$. Secondly, it is intrinsic ($0 \times N$). Lastly, cyclotrons have $\nu_x \sim \gamma$, and avoiding it would keep ν_z other constantly above $\gamma/2$ or below. See tune diagram again.

Walkinshaw_in_cyclotrons.nb

Walkinshaw in EMMA

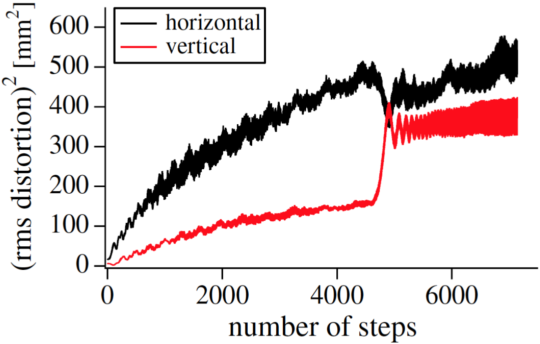


FIG. 14. (Color) Evolution of the square of horizontal and vertical rms orbit distortion when acceleration rate is 5 times slower. Exchange of distortion occurs when a particle crosses $qx - 2qy = 0$ resonance.

Just remember these characteristics:

- $\nu_x = 2\nu_z$ so it's an intrinsic.
- $\Delta J_x = -(1/2)\Delta J_z$ emittance exchange.

Note typical oscillations like diffraction from an edge.

Other Coupling

Of course there are other resonances, for example,

$$\nu_x = \nu_z \tag{19}$$

is also ‘intrinsic’ and has many of the same features as the Walkinshaw. But because of the nature of tunes in cyclotrons, this is rarely crossed, easily avoided.

More common is

$$\nu_x - \nu_z = 1. \tag{20}$$

This requires a first harmonic ‘skew quadrupole’ or tilted median plane field. So it can be corrected with first harmonic correction coils.

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How this relates to your cyclotron design project

The bare minimum is for you to take make a $B(r, \theta)$ for your chosen cyclotron, use `CYCLOPS` to: (1) demonstrate the isochronism and find the minimum energy gain per turn to avoid phase slip to deceleration. (2) Calculate the tunes, plot them on a tune diagram, comment on the most dangerous resonances, avoid the intrinsic (structural) resonances.