Cyclotrons, Space Charge

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Intro to space charge: what to learn

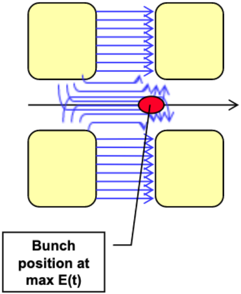
- Basics: Space charge is an internal force, i.e. adaptive: not the same as external.
- Low energy: “Fattens” the beam and therefore where tune is small (vertical at injection), can limit current. Mitigated by ‘electric focusing’.
- ‘Vortex effect’ at high energy: Causes phase stability in cyclotrons, but only beneficial if bunches are very short. Provides a space charge limit.
- Bonus: there’s a paradox.

Compact cyclotron – first turns

In a small or “compact” cyclotron, the magnet gap height is comparable to the radius of the first few turns. The azimuthal field dependence, which is normally the source of vertical focusing, is smoothed out. In that case, and for that region of the cyclotron, it is similar to the “classic” cyclotron originally invented by Ernest Lawrence. The salient feature is that there is essentially no magnetic vertical focusing. By contrast, in a “ring” cyclotron the beam is injected at a radius much larger than the magnet gap, and the azimuthal variation of the magnetic field insures vertical stability right from the first turn.

With no vertical focusing there, how is the beam stabilized?

Electric focusing – Linac case: on rising side means transverse defocusing



→ RF defocusing experienced by particles crossing a gap on a longitudinally stable phase. Increasing field means that the defocusing effect going out of the gap is stronger than the focusing effect going in.

→ In the rest frame of the particle, only electrostatic forces → no stable points (maximum or minimum) → radial defocusing.

→ Lorentz transformation and calculation of radial momentum impulse per period (from electric and magnetic field contribution in the laboratory frame):

$$\Delta p_r = -\frac{\pi e E_0 T L r \sin\varphi}{c \beta^2 \gamma^2 \lambda}$$

→ Transverse defocusing ~ $1/\gamma^2$ disappears at relativistic velocity (transverse magnetic force cancels the transverse RF electric force).

Quiz

In a straight beamline, or a linac, an electric field is used to bunch particles by giving more energy to latecomers and less to earlybirds, effectively bunching the particles.

How is a cyclotron different? What needs to be done to trailing versus leading particles in a bunch?

Electric focusing – Cyclotron case

Answer: Nothing; in isochronous cyclotrons, energy spread does not cause debunching.

In a straight beamline, or a linac an electric field is used to bunch particles by giving more energy to latecomers and less to earlybirds, effectively bunching the particles. Those same particles are defocused transversely because by Maxwell's equations, there cannot be focusing in all 3 directions at once.

Conversely, debunching by accelerating the bunches on the **falling** part of the rf waveform causes transverse focusing.

But in isochronous cyclotrons, energy spread does not cause debunching; it only results in a spread of orbit radii. Thus we can capitalize on the vertical focusing that comes from the falling side of the rf acceleration.

Besides this focusing effect, there is also the effect of the acceleration itself. This is similar to that of an electrostatic or einzel lens, and peaks for on-crest phases.

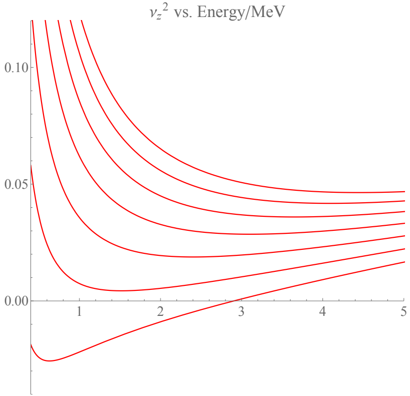
Electric focus vertical tune

The two effects together can be written as terms contributing to the square of the vertical tune ν_z:

$$\nu_z^2 = \frac{(1-\iota)n_\mathrm{g}}{8\pi}\left[h\left(\frac{qV_\mathrm{g}\sin\phi}{E_\mathrm{k}}\right) + \frac{R}{2L_\mathrm{eff}}\left(\frac{qV_\mathrm{g}\cos\phi}{E_\mathrm{k}}\right)^2\right]. \qquad (1)$$

E_k is the kinetic energy, V_g an effective voltage across the gap, n_g the number of gaps per turn, ι is an asymmetry parameter: 0 for circular aperture, -1 for a gap that is extended radially, and $+1$ for a tall and narrow gap. R is the orbit radius and L_eff is an effective length of the gap. The phase ϕ here is zero for peak energy gain, and positive on the falling edge.

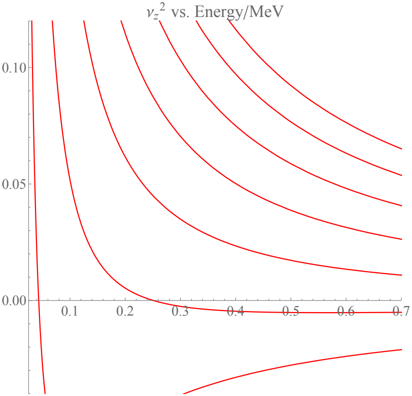
Example – TRIUMF



Shows electric contribution to ν_z^2 for seven phases (from lowest to highest curves) $-18°$ to $+42°$ in steps of $10°$. This is a typical example, where electric $\nu_z \sim 0.2$.

(Also a little magnetic focusing thrown in: $\nu^2_{z\text{mag}} = 0.006(E_\text{k} - 1\,\text{MeV})$.)

Example – TR30



Shows electric contribution to ν_z^2 for seven phases (from lowest to highest curves) $-18°$ to $+42°$ in steps of $10°$. This is a typical example, where electric $\nu_z \sim 0.2$.

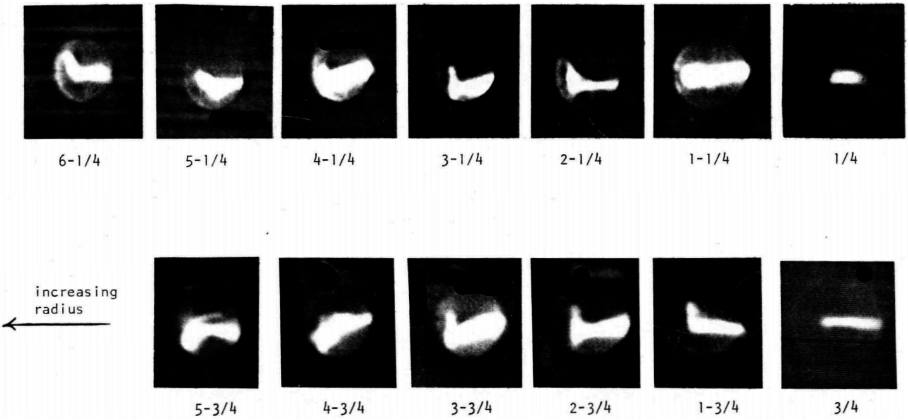
Nice to get this free focusing but there's a problem: What is it?

But

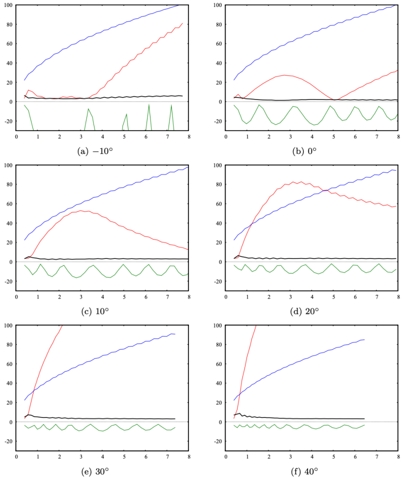
So the vertical focusing arises from the falling part of the rf waveform: The strongest focused particles are the ones with the smallest energy gain. There is thus a phase acceptance arising from a compromise. At the trailing edge of the bunch, particles do not gain sufficient energy to get past the injection gap, and at the leading edge, they are lost by being defocused vertically.

Strictly, this is a highly nonlinear effect: particles have essentially different acceptance ellipses depending on their rf phase. Result is that it is not possible to match the beam transversely. (See slide 4.) There is a large emittance increase between ion source and the circulating beam in the cyclotron; sometimes it's an order of magnitude. It can only be avoided if there is no reliance on electric focusing. That requires a ring cyclotron, as both the 72 MeV and the 585 MeV cyclotrons of PSI are. But $$!

TRIUMF scintillator pix



TRIUMF envelope simulation



(a) $-10°$ (b) $0°$ (c) $10°$ (d) $20°$ (e) $30°$ (f) $40°$

Pay attention to the green curve; it's the vertical envelope, for six different rf phases.

Space Charge: How much ν_z needed?

It is clear now that the first few turns are the “bottleneck”: Too much charge density will depress an already small vertical tune to zero, and particles are lost. We could simply quote the Laslett tune shift and develop the space charge limit comparing it to electric tune calculated above. But I want to start at a more basic level.

$$\ddot{z} + \nu_z^2 \omega^2 z = \frac{1}{m\gamma} \frac{\partial F_z}{\partial z} z \tag{2}$$

so the tune shift is

$$\Delta\nu_z^2 = \frac{-1}{m\gamma\omega^2} \frac{\partial F_z}{\partial z} \tag{3}$$

Space Charge

Now do the typical Gauss' law applied to a cylinder (length L) of charge density (ϱ_q):

$$\iint_{\text{surface of cylinder}} \vec{\mathcal{E}} \cdot \mathrm{d}\vec{S} = \frac{Q_{\text{inside surface}}}{\epsilon_0} \tag{4}$$

$$\mathcal{E}_r \cdot 2\pi r L = \frac{\pi r^2 L \varrho_q}{\epsilon_0} \tag{5}$$

$$\mathcal{E}_r = \frac{\varrho_q}{2\epsilon_0} r = \frac{\lambda_q}{2\pi a^2 \epsilon_0} r \tag{6}$$

where $\lambda_q =$ charge per unit length $= \frac{I}{\beta c}$. Now we remember two things: (1)There is a magnetic component that is $-\beta^2$, and the sum of the two just divides this result by γ^2. (2)For elliptical cylinder, replace a^2 by $a(a+b)/2$ (horizontal) or $b(a+b)/2$ (vertical). We get:

Space Charge

$$\frac{\partial F_z}{\partial z} = \frac{1}{\beta\gamma^2}\,\frac{2qI}{4\pi\epsilon_0 c}\,\frac{2}{b(a+b)} \tag{7}$$

To get the tune shift, we divide by $m\gamma\omega^2$, but use $\omega = \beta c/R$:

$$\Delta\nu_z^2 = \frac{-4}{\beta^3\gamma^3}\,\frac{I\,\frac{1}{4\pi\epsilon_0 c}}{mc^2/q}\,\frac{R^2}{b(a+b)}, \tag{8}$$

may seem a peculiar way to write it until realize:

$$\frac{1}{4\pi\epsilon_0 c} = 30\,\Omega, \text{ and } \frac{mc^2}{q} = V_\mathrm{m}, \tag{9}$$

the rest-energy-to-charge of the particle (in Volts). If desired, roll them together as $I_0 = 4\pi\epsilon_0 c V_\mathrm{m} = 31.3\times 10^6$ Amps for protons.

just to touch base with traditional tuneshift:

$$\Delta Q^{V}_{inc} = -\frac{NRr_0}{\pi Q\gamma\beta^2}\left(\frac{\varepsilon_1^V}{h^2} + \frac{\beta^2\varepsilon_2^V}{g^2} + \frac{1}{\gamma^2 b(a+b)}\right)$$

To make connection, $q\mathrm{N} = \frac{2\pi R}{\beta c} I$, $\mathrm{r}_0 = \frac{qc}{I_0}$.

Space Charge in Cyclotrons

We know $R = \beta R_\infty$, $\gamma \approx 1$ at injection:

$$\Delta\nu_z^2 = \frac{-4}{\beta}\,\frac{I}{I_0}\left(\frac{R_\infty}{b}\right)^2, \tag{10}$$

where we include the consideration that since $\nu_x \gg \nu_z$, $b(a+b) \approx b^2$.

NB: This is local current, therefore use the **peak** current for I. For example, if we want a $1\,$mA output, and the bunch length is 36°, then $I = 10\,$mA.

Space Charge Limit

We need $\nu_z^2 + \Delta\nu_z^2 > 0$. Ignoring the beam's emittance, we allow it to expand vertically to fill the aperture. Then taking for b some effective aperture size we can re-write above for protons as

$$I < \beta_{\mathrm{inj}} \left(\frac{b\nu_z}{R_\infty} \right)^2 7.8\,\mathrm{MA} \tag{11}$$

Exercise: Refine this criterion, using the actual injected emittance and Courant-Snyder function $\beta_z = R/\nu_z$. This would be the appropriate limit if we want to maintain matching without emittance growth, as is the case for the PSI Injector 2 example below.

Space Charge Limit in TR30

$$I < \beta_{\rm inj}\left(\frac{b\nu_z}{R_\infty}\right)^2 7.8\,\text{MA} \tag{12}$$

Example: TR30; $\beta_{\rm inj} = 0.007$, $b \sim 5$ mm, $\nu_z \sim 0.2$, $R_\infty = 2.6$ m. Results in $I < 8$ mA. Coupled with known phase width of about 60° means that the output current average is to be less than 1.3 mA, preferably much less. This agrees with decades of experience. The original design was for 0.5 mA, upgraded to 1 mA, and further upgraded to higher intensity, but nothing greater than 0.75 mA is used for 24/7 running. Let's look at some pictures to see why. These components erode rather than melt and must be replaced annually. To raise intensity, they would be replaced even more often.

Space Charge Limit in TRIUMF

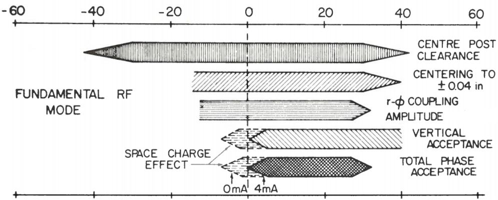
$$I < \beta_{\rm inj}\left(\frac{b\nu_z}{R_\infty}\right)^2 7.8\,\text{MA} \tag{13}$$

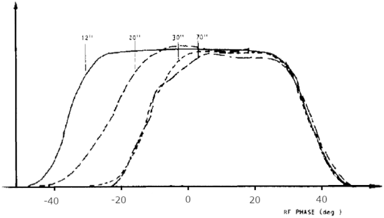
TRIUMF 500 MeV; $\beta_{\rm inj} = 0.025$, $b \sim 10$ mm, $\nu_z \sim 0.15$, $R_\infty = 10.$ m. Results in limit of about 5 mA, or 0.5 mA time average. Maximum demonstrated is 0.42 mA, and centre region cooling was augmented to run continuously at $\bar{I} > 0.3$ mA.

We can edge this towards 0.5 mA by increasing the acceleration voltage since this helps in two ways: It widens phase acceptance, and it raises β at the "bottleneck" of the space cahrge limit, which occurs somewhat beyond the injection energy.

Space Charge Limit in TRIUMF

Left: Design (1972). Right: Commissioning (1975)





If we try to increase peak intensity, we lose phase acceptance, and the average output does not increase. E.g. 30° phase acceptance with $4\,$mA peak $= 333\,\mu$A average.

Injection Space Charge Limit in PSI Injector 2

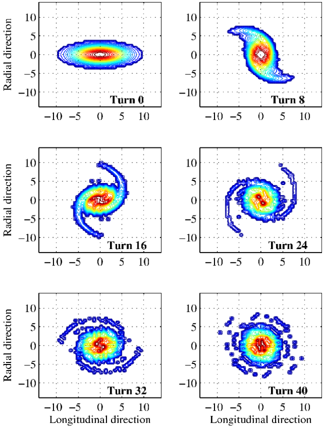
We re-insert the dependence on radial size a since this cyclotron has a large magnetic vertical tune:

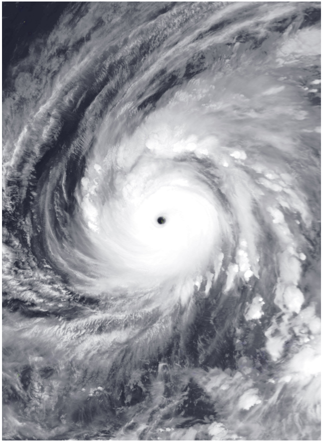
$$I < \beta_{\text{inj}} \left(\frac{b\nu_z}{R_\infty}\right)^2 \left(1 + \frac{a}{b}\right) 7.8\,\text{MA} \tag{14}$$

PSI 72 MeV injector cyclotron; $\beta_{\text{inj}} = 0.043$, $a \sim b \sim 5$ mm, $\nu_z = 1.35$, $R_\infty = 9.4$ m.

The limit is about 350 mA. Effectively, the large vertical tune increases the limit by two orders of magnitude compared with TRIUMF. Even with a very small phase acceptance of a few degrees, we can see that they are not limited by space charge at injection.

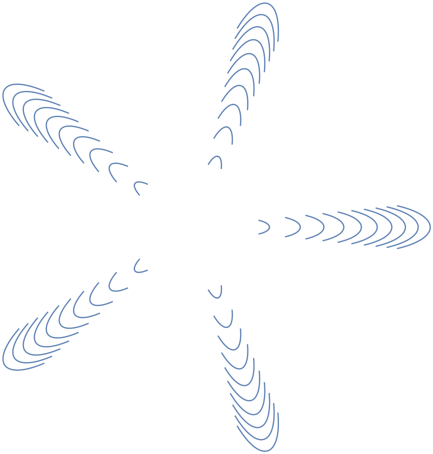
Space Charge Part 2: Vortex effect





Left: figure from Cerfon (2013) showing a simulation of the vortex motion in a cyclotron. Right: satellite image of the Yutu Typhoon, from NASA.

Vortex effect



In a linac, or in a drift space, the particles at the head of the bunch get pushed forward – in the direction of the bunch's motion – gaining an incremental amount of energy. Trailing particles do the reverse: they lose energy. In cyclotrons, this cannot happen: higher energy does **not** mean getting ahead. At first it was thought that this energy gain would lead the head of each bunch to shift to higher radius (and the tail to lower radius) thus tilting the bunches. This would have been easy to counteract: simply accelerate sufficiently off-crest that the tilt is compensated by the phase-dependent energy gain from the rf.

Vortex effect –cont'd

It was progressively understood (1969 – 1988) that the same effect would cause particles at the low-radius side to advance in phase, and the high-radius particles would retard. The overall effect is not to tilt the bunch, but to rotate it around the cyclotron axis. But remember that the bunches are already rotating around this axis: once per turn. The space charge effect is to slow the intrinsic rotation frequency from once per turn to slightly less than 1. This frequency shift when thought of in the radial direction is nothing other than the Laslett tune shift.

Vortex effect

A crucial ingredient in the physics of the evolution of the vortex effect is that these equipotentials are different than the distribution density contours. Specifically, for $a,\ b$ as semi-axes, the form of the equipotentials is

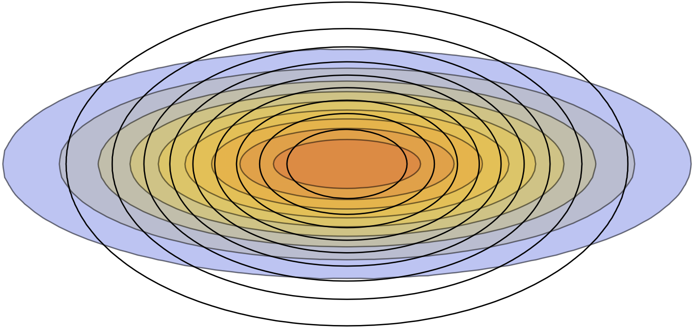
$$\text{constant} \ = \frac{x^2}{a(a+b)} + \frac{y^2}{b(a+b)}, \tag{15}$$

when the distribution has the form

$$\text{constant} \ = \frac{x^2}{a^2} + \frac{y^2}{b^2}. \tag{16}$$

Note that the two only agree when the distribution is circular. The fact that the particle flow follows the equipotentials leads to the intuitive understanding that circular distributions are stationary in cyclotron, a fact that was discovered and first described by Wiel Kleeven (thesis, 1988). With relativistic beams the stationary distribution is actually not circular: its aspect ratio $\frac{\Delta r}{r\Delta\theta} = \gamma$.

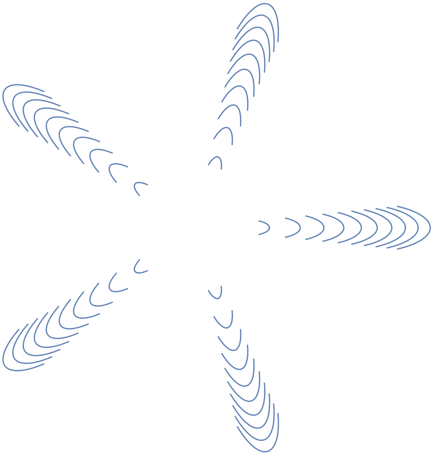
Vortex effect

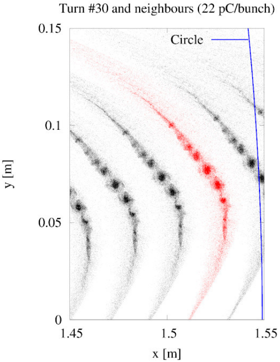


But what if bunch is too long?

short bunch, 1→2 bunch, medium bunch, super long.

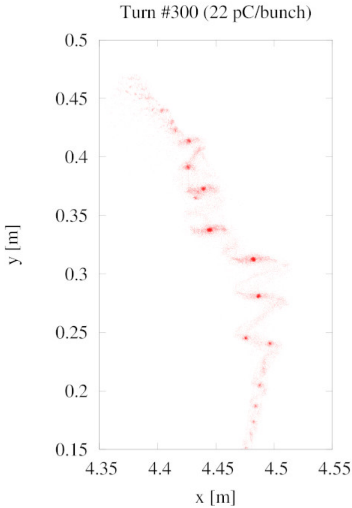
TRIUMF ‘soup’





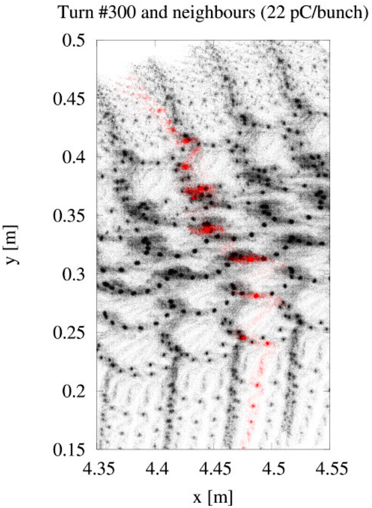
Turn #30 and neighbours (22 pC/bunch)
Circle
0.15
0.1
0.05
0
1.45
1.5
1.55
x [m]
y [m]

TRIUMF ‘soup’



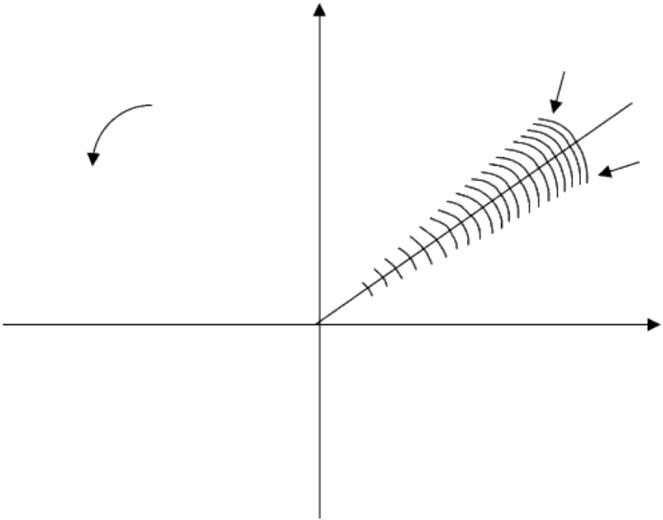
Turn #300 (22 pC/bunch)
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
4.35
4.4
4.45
4.5
4.55
x [m]
y [m]

TRIUMF 'soup'

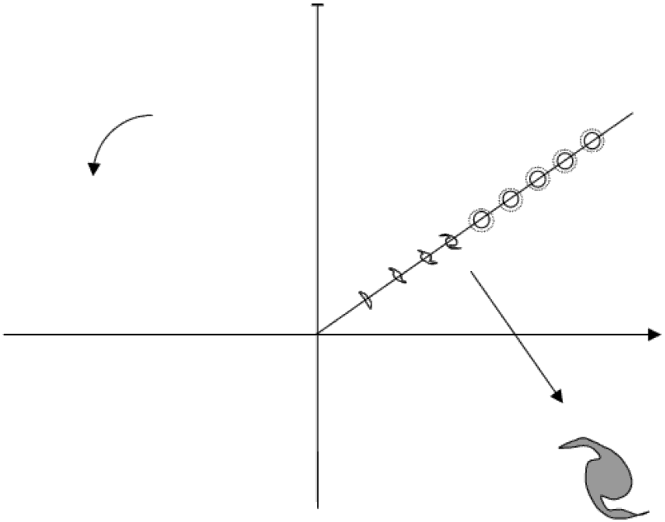


Turn #300 and neighbours (22 pC/bunch)
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
y [m]
4.35
4.4
4.45
4.5
4.55
x [m]

Constant phase width, or...



constant bunch length?



What's going on?

- With space charge, stationary distribution is a circular bunch (relativity changes this slightly though).
- If starting bunch is too long, it will split into multiple bunches.
- One droplet has a space charge tune shift, retarding radial betatron motion; if retarded too much bunch will be unstable. This provides a space charge limit.

Let's repeat the Laslett tune shift equation 8, but now for radial motion:

$$\Delta\nu_x^2 = \frac{-4}{\beta^3\gamma^3}\frac{I}{I_0}\frac{R^2}{a(a+b)}, \tag{17}$$

and assume a spherical droplet ($a = b = \varrho$, avoids elliptic integrals), non-relativistic, and with a contained charge Q use as effective current $I \approx \frac{Q\beta c}{\gamma\varrho}$:

$$\Delta\nu_x^2 = \frac{-2Qc}{I_0}\frac{R_\infty^2}{\varrho^3} = \frac{-4\pi}{h\gamma^2}\frac{\bar{I}}{I_0}\left(\frac{R_\infty}{\varrho}\right)^3. \tag{18}$$

Space charge limit

Note that $Q = \bar{I}/f_{\rm rf}$ expressed here in terms of the average current.

$$\Delta\nu_x^2 = \frac{-4\pi}{h\gamma^2}\frac{\bar{I}}{I_0}\left(\frac{R_\infty}{\varrho}\right)^3. \tag{19}$$

Clearly, if this is absolutely too large, stability will be lost: it would be like the case of vertical tune at injection: total tune vanishes and Twiss β-function diverges. In fact, because of the way in which the radial and longitudinal couple, the tune shift can reach only to one half this limit. Naturally, beam can expand in ϱ, but if extracting by separated turns, there is an upper limit on beam size. The limit on extracted average intensity is

$$\bar{I}_{\rm max} = \frac{hg_x}{8\pi}\left(\frac{\varrho}{R_\infty}\right)^3 \nu_{x0}^2\gamma^2 I_0, \tag{20}$$

here included the proper dependence on relativistic effect, and $g_x \approx 1 - \frac{3}{5}\log\left(\frac{b}{\varrho}\right)$ is a form factor to take into account the aspect ratio height-to-radius. Show the bunch rotating/not rotating.

Space charge paradox

Does the bunch length expand to maintain constant $\Delta\theta = h\Delta\phi$, or not?

Exercise or lab: what parameters determine which tendency dominates? Formulate a criterion.

Conclusion

- Compact cyclotrons have no vertical focusing at injection and must rely on electric focusing requiring to accelerate on falling side of rf (late particles get least energy gain).
- This leaves them vulnerable to space charge effects, and finally sets the limit on intensity (current).
- Vortex effect keeps short bunches short (good for separated-turn acceleration), but long bunches split into ‘droplets’ (does not matter for stripping-extracted beams).