Isochronous and Scaling FFAGs

R. Baartman

TRIUMF

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Abstract: If an FFAG can be made isochronous, acceleration can occur without ramping the rf frequency. But this is of course simply a cyclotron, of which many examples exist worldwide: 2 of the largest are PSI (580 MeV p, up to 1 MW beam power), TRIUMF (500 MeV H^-, up to 0.1 MW beam power). How do the beam dynamics change if an FFAG is isochronous? E.g. can we still achieve large acceptance?

Isochronism

Let us define θ to be the angle of the reference particle momentum w.r.t. the lab frame. Orbit length L is given by speed and orbit period T:

$$L = \oint ds = \oint \rho d\theta = \beta c T.$$

The local curvature $\rho = \rho(s)$ can vary and for reversed-field bends even changes sign. (Along an orbit, $ds = \rho d\theta > 0$ so $d\theta$ is also negative in reversed-field bends.) Of course on one orbit, we always have

$$\oint d\theta = 2\pi.$$

What is the magnetic field averaged over the orbit?

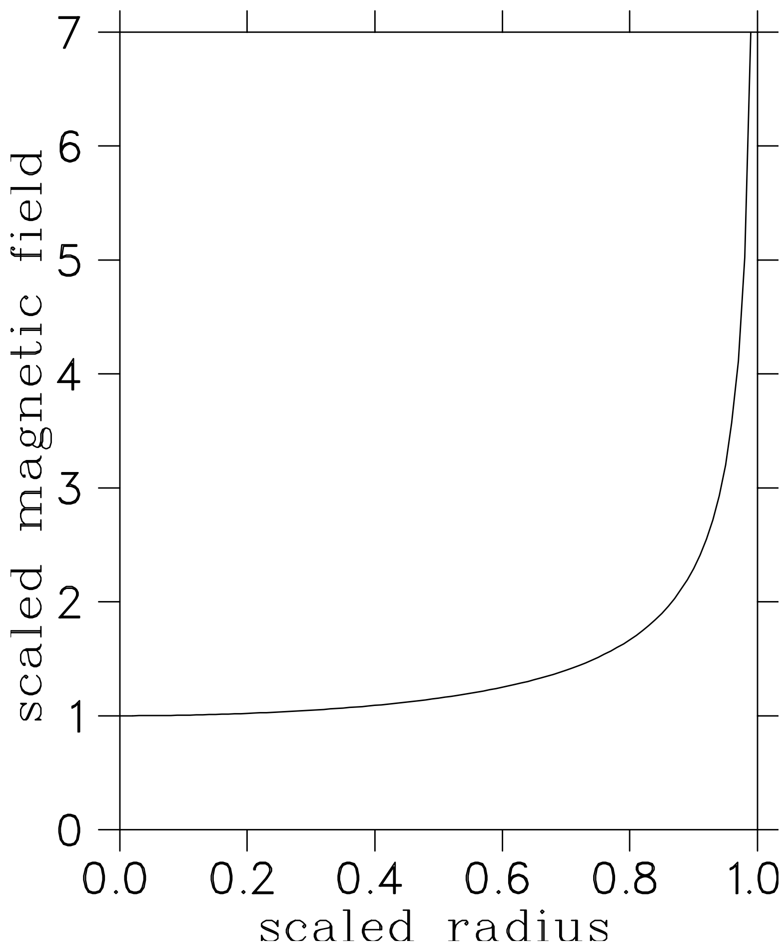
$$\overline{B} = \frac{\oint B ds}{\oint ds} = \frac{\oint B\rho\, d\theta}{\beta c T}.$$

But $B\rho$ is constant and in fact is $\beta\gamma m_0 c/q$. Therefore

$$\overline{B} = \frac{2\pi}{T}\frac{m_0}{q}\gamma \quad \equiv B_{\mathrm{c}}\,\gamma = \frac{B_{\mathrm{c}}}{\sqrt{1-\beta^2}}.$$

Remember, β is related to the orbit length:
$\beta = L/(cT) = 2\pi R/(cT) \equiv R/R_\infty$. So

$$\overline{B} = \frac{B_{\mathrm{c}}}{\sqrt{1-(R/R_\infty)^2}}.$$



Of course, this means the field index is
$k = \frac{R}{B}\frac{dB}{dR} = \frac{\beta}{\gamma}\frac{d\gamma}{d\beta} = \beta^2\gamma^2 \neq$ constant.

So isochronism $\Rightarrow$ non-scaling.

Focusing (flat field)

For the moment, let us imagine that there are no sectors, no azimuthal field variation, just radial variation with field index $k = \beta^2\gamma^2$.

We know the transfer matrix in such a dipole, and can write directly:

$$\begin{aligned} \nu_r^2 &= 1 + k \\ &= 1 + \beta^2\gamma^2 \\ &= \gamma^2 \end{aligned} \qquad \begin{aligned} \nu_z^2 &= -k \\ &= -\beta^2\gamma^2 \\ &< 0 \end{aligned}$$

$$\nu_r = \gamma \qquad \nu_z = \text{imaginary}$$

So why do such cyclotrons work at all? Lawrence's first cyclotrons, and those built before the early 50s, had no sector focusing. To obtain vertical focusing, k was made slightly negative. This resulted in phase slippage, so the rf voltage was made as large as possible to achieve the final energy before an accumulated phase slip of $\pi/2$. In addition, some vertical focusing was achieved by accelerating on the falling side of the rf voltage. In this way, such cyclotrons achieved maximum energies of 20 to 30 MeV (protons).

To reach higher energy, it was necessary to release isochronism, ramp the rf frequency, and pulse the machine. Unfortunately, this meant that the maximum intensity was 2 or 3 orders of magnitude lower than for the (cw) cyclotron.

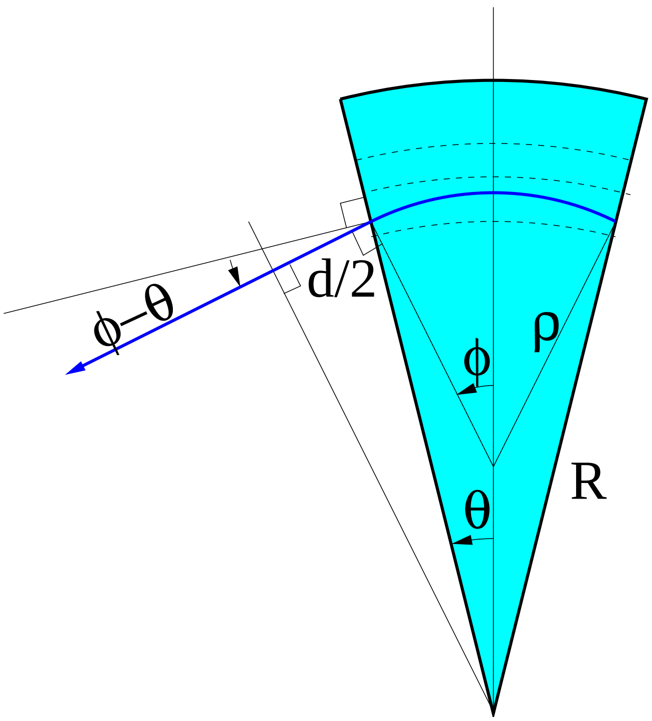
Focusing (AVF or FFAG, regular lattice)

Strong focusing was invented/developed in the 50s. This had implications for

- **cyclotrons**: they could now be isochronous AND vertically focusing
- **synchro-cyclotrons**: tunes need be no longer near 1, so beams were smaller, space charge limits higher
- and **synchrotrons**: same advantages as for synchro-cyclotrons

The application was by far simplest for synchrotrons. For FF machines, the extended nature of the field was a calculational headache. That is why synchrotrons developed rapidly in the 50s (when large computing power was unavailable), while synchro-cyclotrons of the FFAG type did not. Nowadays, large computing power is freely available.

For Azimuthally-Varying Field (AVF) or the special case of Alternating Gradient (FFAG), let us use **Mathematica** to calculate the tunes. To make it transparent, let us consider all identical dipoles and drifts; no reverse bends. We have drifts d, dipoles with index k, radius ρ, bend angle ϕ, and edge angles $\phi - \theta$:



$$\frac{\sin(\theta)}{\rho} = \frac{\sin(\phi)}{R}$$

$$d/2 = R\sin(\phi - \theta)$$

In addition, imagine that the edges are inclined by an extra angle ξ. This is called the “spiral angle” (hard to draw).

In this hard-edged case, the “flutter” $F^2 \equiv \langle (B - \overline{B})^2 \rangle / \overline{B}^2 = R/\rho - 1$.

Aside: Notice that the particle trajectory (blue curve) does not coincide with a contour of constant B (dashed curves). This has large implications for using existing transport codes to describe FFAGs.

```
s = ρ * ϕ;
θ = ArcSin[(ρ / R) Sin[ϕ]];
f1 = ρ / Tan[ϕ - θ + ξ];
f2 = ρ / Tan[ϕ - θ - ξ];
d = 2 R * Sin[ϕ - θ];
R = ρ (1 + F^2);
k = κ * ρ / R;
kx = Sqrt[1 + k] / ρ;
ky = Sqrt[k] / ρ;
Mz := ({{Cosh[ky s], Sinh[ky s] / ky}, {ky Sinh[ky s], Cosh[ky s]}} .
   {{1, 0}, {-1/f2, 1}} . {{1, d}, {0, 1}} . {{1, 0}, {-1/f1, 1}} .
   {{Cosh[ky s], Sinh[ky s] / ky}, {ky Sinh[ky s], Cosh[ky s]}})

νz^2 = FullSimplify[
   (ArcCos[Series[
       (Mz[[1, 1]] + Mz[[2, 2]]) / 2,
       {ϕ, 0, 3}]]) / (2 ϕ)]^2
```

$$\left(-\kappa + F^2\,(1 + 2\,\mathrm{Tan}[\xi]^2)\right) + O[\phi]^2$$

```
Mr := ({{Cos[kx s], Sin[kx s] / kx}, {-kx Sin[kx s], Cos[kx s]}} .
   {{1, 0}, {1/f2, 1}} . {{1, d}, {0, 1}} . {{1, 0}, {1/f1, 1}} .
   {{Cos[kx s], Sin[kx s] / kx}, {-kx Sin[kx s], Cos[kx s]}})

νr^2 = FullSimplify[
   ArcCos[Series[
       (Mr[[1, 1]] + Mr[[2, 2]]) / 2,
       {ϕ, 0, 3}]] / (2 ϕ)]^2
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$$(1 + \kappa) + O[\phi]^2$$

These expressions are identical to those originally derived by Symon et al. in the original 1956 Phys. Rev. paper about FFAGs.

But beware! Since there is now a distinction between local curvature (ρ) and global (R), the definition of field index is ambiguous. The local index, used in the dipole transfer matrix, is

$$k = \frac{\rho}{B}\frac{dB}{d\rho},$$

while the Symon formula uses

$$\kappa = \frac{R}{B}\frac{dB}{dR} \approx k\,\frac{R}{\rho}$$

As we proved, it is in fact this latter quantity which must be equal to $\beta^2\gamma^2$ for isochronism. We therefore still have

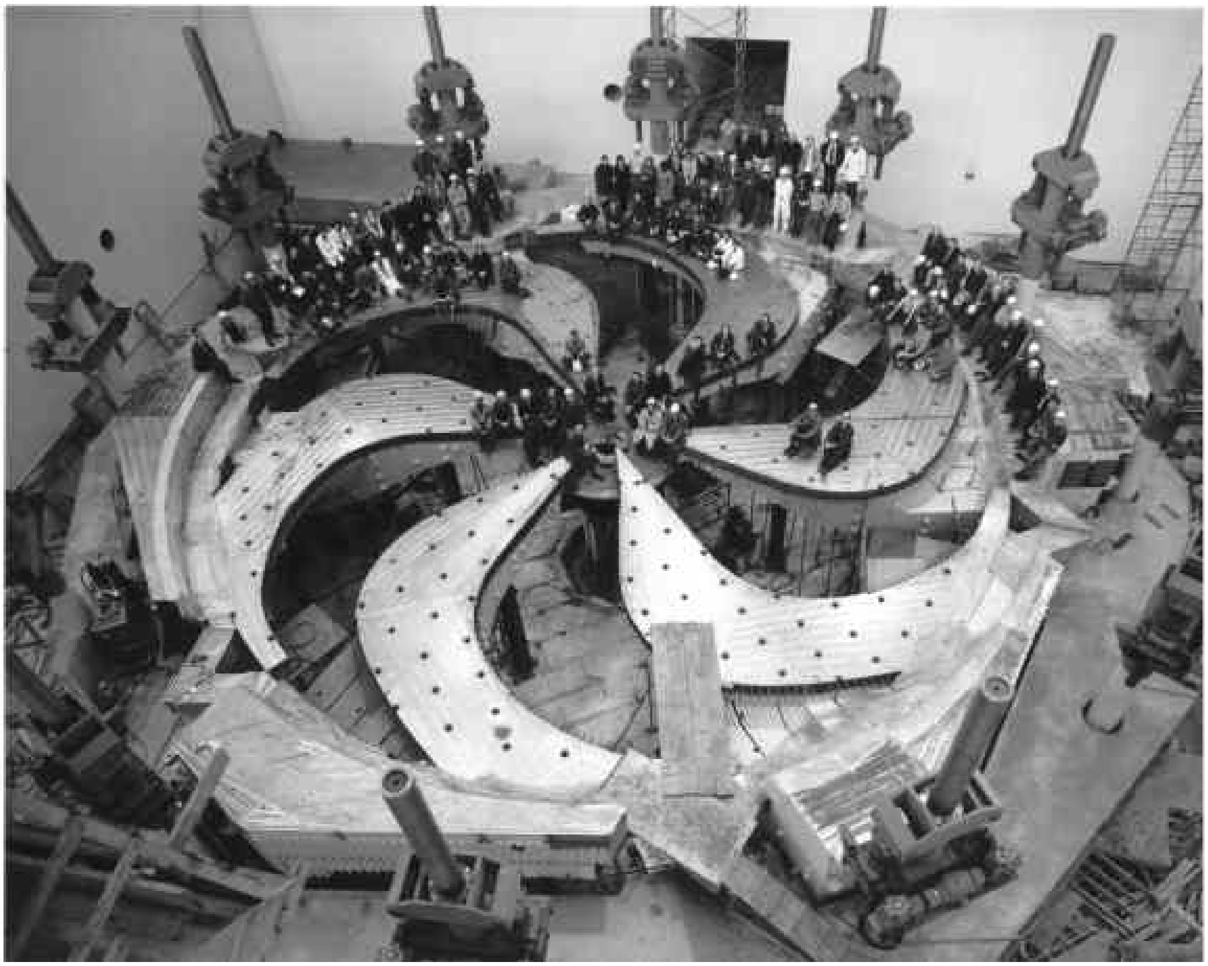
$$\nu_r = \gamma \qquad \text{(isochronous)}$$

But

$$\nu_z^2 = -\beta^2\gamma^2 + F^2\left(1 + 2\tan^2\xi\right) \qquad \text{(isochronous)}$$

Aside: We could have **Mathematica** print out the next higher order, but it would be in error because the trajectories are not circular arcs.

Example: TRIUMF cyclotron



Energy	R	$\beta\gamma$	ξ	$1 + 2\tan^2\xi$	F^2	ν_z
100 MeV	175 in.	0.47	0°	0.0	0.30	0.28
250 MeV	251 in.	0.78	47°	3.3	0.20	0.24
505 MeV	311 in.	1.17	72°	20.0	0.07	0.24

Irregular FFAG cyclotrons

By adjusting the flutter F and spiral angle ξ as functions of R, we can arrange to make ν_z constant.

But what about ν_r? Can we change it by departing from the regular N-cell lattice? Perhaps, but no one has ever tried it.

In synchrotron language, isochronism means $\gamma = \gamma_{\mathrm{t}}$. But we know that for a “regular” lattice, $\gamma_{\mathrm{t}} = \nu_r$. Hence, $\nu_r = \gamma$. For an irregular lattice,

$$\frac{1}{\gamma_{\mathrm{t}}^2} = \frac{\nu_r^3}{R} \sum_n \frac{|a_n|^2}{\nu_r^2 - n^2}$$

where a_n is the Fourier transform of $\beta_x^{3/2}/\rho$. For N identical cells where $N \gg \nu_r$, this gives $\gamma_{\mathrm{t}} = \nu_r$ because the $n = 0$ term gives by far the largest contribution to the sum.

If the lattice is arranged to have a superperiodicity $n = \pm S$ where $S \sim \nu_r$, γ_{t} can be moved away from ν_r:

$$\frac{1}{\gamma_{\mathrm{t}}^2} = \frac{1}{\nu_r^2} + \frac{2|a_S|^2}{R} \frac{\nu_r^3}{\nu_r^2 - S^2}$$

This is the approach used to make γ_{t} imaginary in e.g. the JHF main ring. Whether this can be used to fix ν_r in a cyclotron where necessarily $\gamma_{\mathrm{t}} = \gamma$, is not clear, since it prescribes a peculiar variation of a_S with R.

Conclusions:

The chief virtue that FFAG scaling machines have over non-scaling isochronous machines is the constancy of the tunes. This results in huge acceptance ($> 1000\pi$ mm-mrad). But because they cannot be isochronous, they must be pulsed.

The chief virtue of isochronous FFAGs is that they can run cw and reach very high intensity. However, because $\nu_r = \gamma$, many resonances must be traversed to reach high energy. This reduces acceptance to only a few π mm-mrad.