Solenoids at 300 keV

Rick Baartman, TRIUMF

Sept. 9, 2010

Expansions

It is useful to know that the magnetic field from a solenoid is completely given by the following expansions:

$$B_z(r,z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!^2} \left(\frac{r}{2}\right)^{2n} \left(\frac{d}{dz}\right)^{2n} B_z(0,z) \tag{1}$$

$$B_r(r,z) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)!n!} \left(\frac{r}{2}\right)^{2n+1} \left(\frac{d}{dz}\right)^{2n+1} B_z(0,z) \tag{2}$$

These follow directly from Maxwell's equations plus the symmetry:

$$\nabla \cdot \vec{B} = 0 \quad : \quad \frac{1}{r}\frac{\partial(rB_r)}{\partial r} + \frac{\partial B_z}{\partial z} = 0 \tag{3}$$

$$\nabla \times \vec{B} = \vec{0} \quad : \quad \frac{\partial B_z}{\partial r} - \frac{\partial B_r}{\partial z} = 0 \tag{4}$$

This establishes that the fields can be completely derived knowing only the on-axis field function.

For small r, we have:

$$B_z = B_0 - \frac{r^2}{4} B'' \quad (5)$$

$$B_r = -\frac{r}{2} B' + \frac{r^3}{16} B''' \quad (6)$$

($B_0(z)$ is the on axis field $B_z(0, z)$, and B' etc. are derivatives of $B_0(z)$.)

To the same order,

$$B_z^2 = B_0^2 - \frac{r^2}{2} B_0 B'' \quad (7)$$

Focal Power

The equation of motion through the solenoid has radial part:

$$r'' + KB_z^2 r = 0 \tag{8}$$

(K is a constant containing the magnetic rigidity: $K = \frac{1}{(2B\rho)^2}$.)

Expanding to cubic force order gives

$$r'' + K\left(B^2 r - \frac{BB''}{2} r^3\right) = 0 \tag{9}$$

So the contribution that the linear part gives to r' ($\Delta_1 r'$) is:

$$\Delta_1 r' = K \int r B^2 dz \tag{10}$$

$$= Kr \int B^2 dz \text{ (thin lens approx.)} \tag{11}$$

The nonlinear term gives an r' error, denoted by $\Delta_3 r'$,

$$\Delta_3 r' = \frac{K}{2} \int r^3 B B'' dz \tag{12}$$

$$= \frac{K}{2} r^3 \int B B'' dz \text{ (thin lens approx.)} \tag{13}$$

$$= -\frac{K}{2} r^3 \int B'^2 dz \tag{14}$$

Thus we see that **optimizing a solenoid by minimizing the off-axis focal deviation is exactly the same as minimizing the mean-squared value of** B'.

The figure of merit for a solenoid is therefore the following quantity:

$$M \equiv \frac{\int B'^2 \, dz}{\int B^2 \, dz} \tag{15}$$

$Mr^2/2$ is the fractional error in focal strength ($\Delta_3 r'/\Delta_1 r'$) for a particle at radius r.

Scaling

In a long solenoid, B' is nonzero in only a region at both ends, whose length scales as the aperture. Thus extending the solenoid would scale $\int B^2\,dz$ but not $\int B'^2\,dz$. So we expect $M \propto 1/l$.

By dimensional analysis, scaling all lengths by a factor s results in M scaled by factor s^2. Thus we expect

$$M \propto \frac{1}{al} \tag{16}$$

The application below indicates the proportionality constant is in the range 0.8 to 0.9.

Application to 300 keV electron transport

The current design calls for 3 solenoids of roughly the same strength. The strongest of these requires a focal power $\frac{1}{f} = -\frac{\Delta_1 r'}{r} = 5.4\,\mathrm{m}^{-1}$. Allowing for some contingency, we specify a focal power of 6.3/m.

COSY-∞ was used for the detailed optics calculations. There are a number of solenoid types in this code, but the only types derived from the physics are air core. These are inappropriate for our use as the magnetic fields fall off too slowly. Steel-encapsulated solenoids of the type usually called “Glaser lenses” have more rapidly falling fields. For these, the hyperbolic tangent field form called CMS in COSY-∞ is, while not exact, a somewhat better representation. This has the on-axis field:

$$B(z) = \frac{B_c}{2\tanh\frac{l}{2a}}\left(\tanh\frac{2z+l}{2a} - \tanh\frac{2z-l}{2a}\right) \tag{17}$$

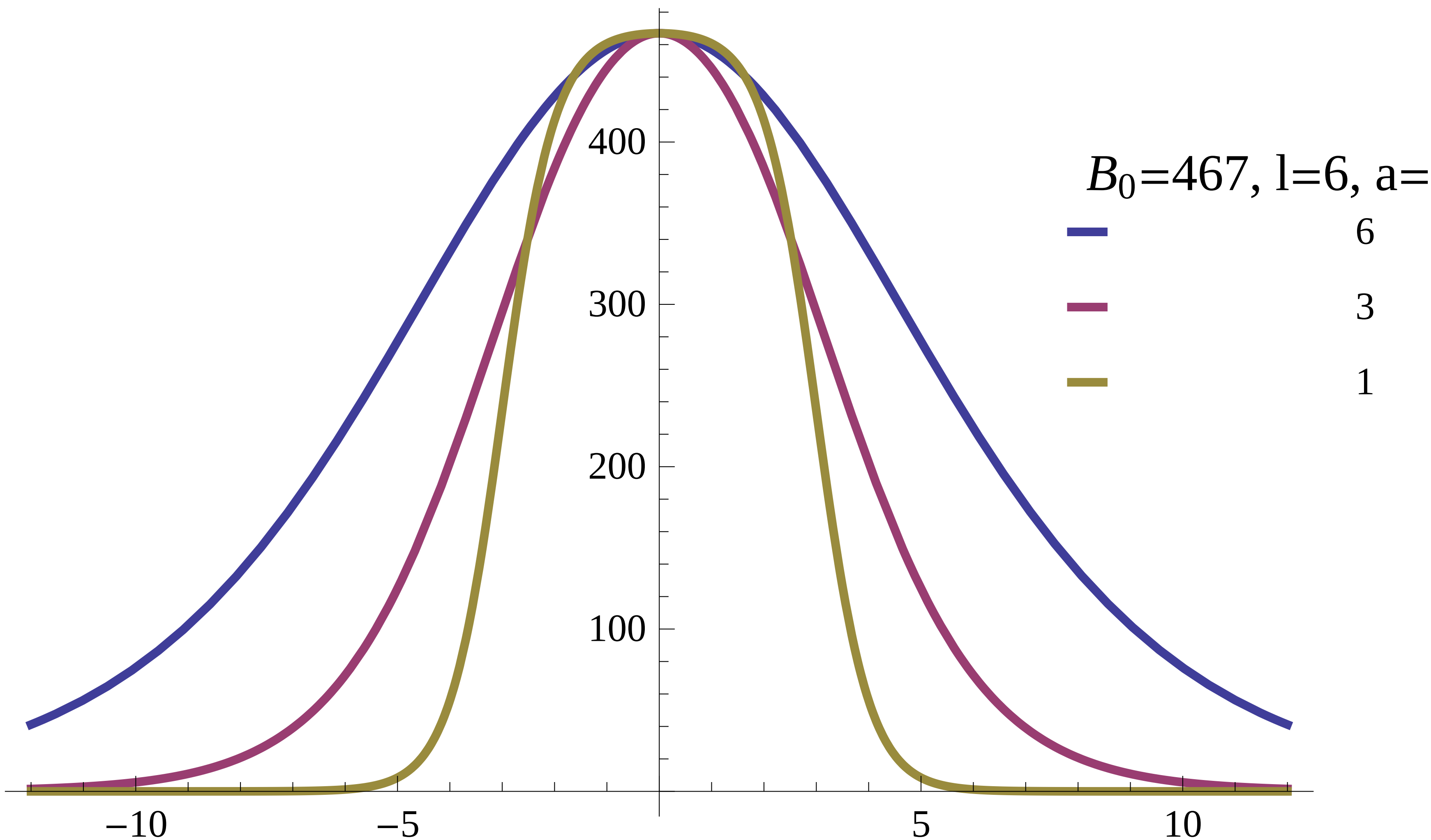

Figure 1: $B(z)$ for $l = 6$, $a = 6, 3, 1$

8 example solenoids are shown in the table below. All have $\frac{1}{f} = 6.3$/m. M is the thin-lens formula (15). The COSY-∞ value for M is found from the third order transfer map, being $2 * (x'|300000)/(x'|100000)$. As one can see, agree fairly well. Discrepancies for longer lenses can be attributed to the breakdown of the thin-lens approximation.

l/cm	a/cm	B_c/G	$\frac{\int B^2 dz}{10^6 \text{ G}^2\text{-cm}}$	$M_{\text{COSY}} *$ cm^2	$M *$ cm^2	Mal
20.	3.	292.	1.457	0.0075	0.0131	0.79
10.	3.	394.	1.258	0.0295	0.0305	0.91
10.	1.	372.	1.244	0.0543	0.0741	0.74
6.	4.5	421.	1.250	0.0321	0.0301	0.81
6.	3.	467.	1.212	0.0543	0.0520	0.94
6.	1.	484.	1.183	0.121	0.133	0.80
4.	3.	505.	1.199	0.072	0.068	0.81
4.	1.	599.	1.160	0.216	0.220	0.88

Emittance Growth

We use the results of TRI-DN-99-8. In the notation of that note, aberration coefficients (sorry for duplication of symbol B), $A = B = M/(2f) \approx 0.4/(alf)$. From equation 10 of that note,

$$\Delta(\epsilon_{\rm rms}^2) = 8A^2\sigma_x^8 \tag{18}$$

where we assume a round beam $\sigma_x = \sigma_y$. This is intended to be small compared with the square of the rms emittance, and thus leads to the following criterion for beam size:

$$\sigma_x^4 \ll 1.2\, f\, a\, l\, \epsilon_{\rm rms} \tag{19}$$

We use $f = 160$ mm, $\epsilon_{\rm rms} = 0.005$ mm (commonly referred to as "5πmm-mrad"). The current design foresees beam pipe radius of 32 mm, so a must be greater than this. But we can choose l. For $l = 40$ mm, we find

$\sigma_x^4 \ll (6.1\,\text{mm})^4$. Matched and aligned, our beam size is $\sigma_x = 2.5\,$mm, so the emittance growth is roughly 1 part in 36 for the largest amplitude particles ($1/36 = (2.5/6.1)^4$). This is OK if misalignment is no larger than a couple of mm.