Magnet Relations

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Basic formulas

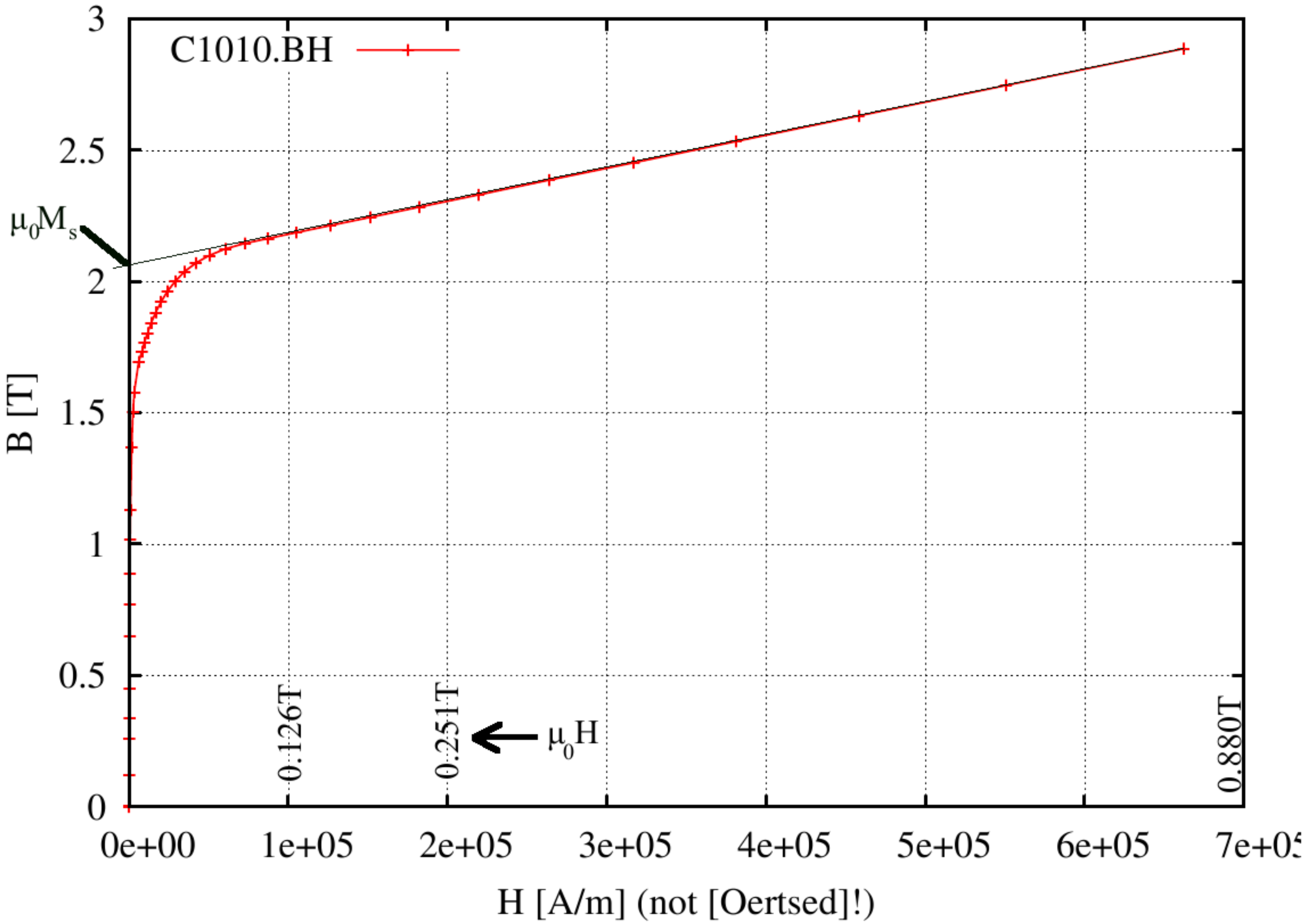
MKS units:

$$B = \mu_0(H + M)$$
$$B = \mu H$$

- Below saturation, $\mu/\mu_0 \gg 1$, so this means that inside a magnet, $H \approx 0$.
- Above saturation, B simply is incremented above saturation magnetization by $\mu_0 H$:

$$B = \mu_0(H + M_s)$$

B-H curve



C1010.BH
B [T]
H [A/m] (not [Oertsed]!)
$\mu_0 M_s$
$\mu_0 H$
0.126T
0.251T
0.880T
3
2.5
2
1.5
1
0.5
0
0e+00
1e+05
2e+05
3e+05
4e+05
5e+05
6e+05
7e+05

Magnetic "charges"

It is useful to think in terms of magnetic charges:

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \cdot (\vec{H} + \vec{M}) = 0$$

$$\nabla \cdot \vec{H} = -\nabla \cdot \vec{M}$$

$$\nabla \cdot \vec{H} = \rho_m$$

where $\rho_m = -\nabla \cdot \vec{M}$, this is just as in electric charges:

$$\nabla \cdot \vec{D} = \rho$$

Ferromagnet in external field B_0

B and H inside depend upon the shape of the material.

$$H_{\text{inside}} = B_0/\mu_0 - N_z M = H_0 - N_z M$$

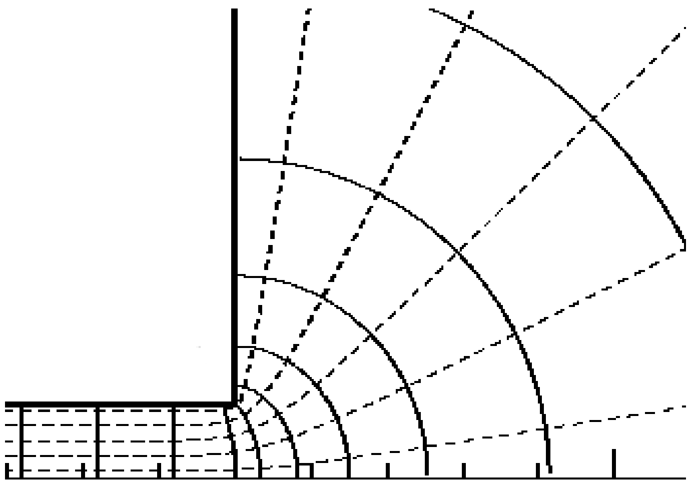
N_z is the "demagnetizing factor". Actually, N is a tensor, but for uniform ellipsoids, the field inside is also uniform and the tensor is diagonal if x, y, z are aligned along the ellipsoid's axes.

$N_x + N_y + N_z = 1$, so by symmetries,

- for a sphere, $N_z = 1/3$.
- For a disc (coin), $N_z \longrightarrow 1$; obvious, considering B_n is continuous,
- for the same disc, $N_x = N_y = 0$.
- For a needle, $N_z \longrightarrow 0$; obvious, considering H_t is continuous, and
- for the same needle, $N_x = N_y = 1/2$.

Magnetic Potential

Key relation: $\oint \vec{H} \cdot d\vec{l} = I$, from $\nabla \times \vec{H} = \vec{J}$.



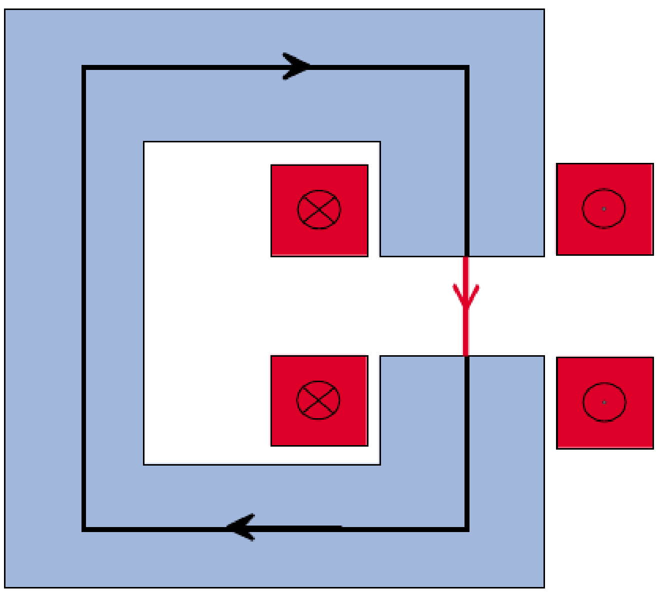
A diagram of 1/4 of a dipole is on the left. In analogy to electrostatics, one can take $\int_1^2 \vec{H} \cdot d\vec{l}$ as the magnetostatic potential difference between locations 1 and 2, and define *magnetostatic potential* $\phi_m(\vec{r}) = \int \vec{H} \cdot d\vec{l}$ where the integral is from a reference location to $\vec{r}$. This is path independent since $\nabla \times \vec{H} = \vec{0}$.

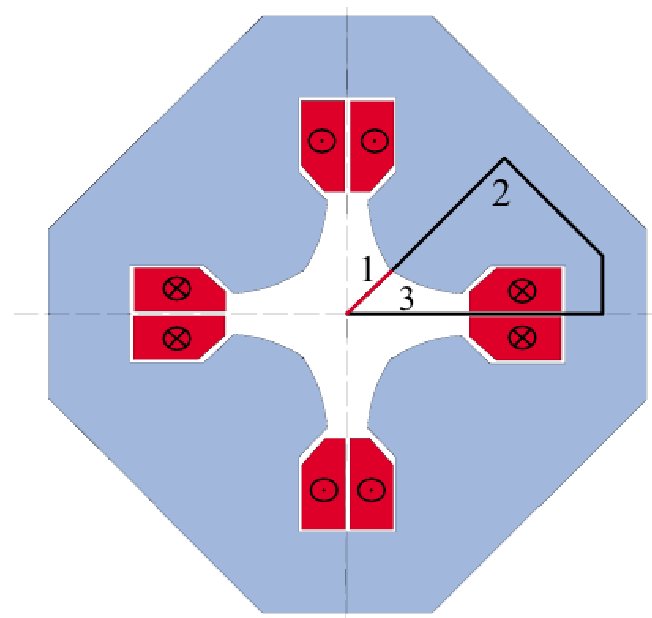
As $H = 0$ inside a ferromagnet far from saturation, the surface of the ferromagnet is an equipotential surface. But this is true iff **the magnetization is below saturation** and $\mu_r \gg 1$. As one approaches saturation, the equipotential surface shape changes and multipole errors arise.

Magnet design

$$\oint \vec{H} \cdot d\vec{l} = I$$

Do the path integral around a closed loop.





Assume magnets are far below saturation.

Dipole:

$$H_{\text{iron}}s_{\text{iron}} + H_{\text{gap}}s_{\text{gap}} = I$$

Far below saturation, $H_{\text{iron}} = 0$, so we have

$$B_{\text{gap}} = \frac{\mu_0 I}{s_{\text{gap}}}$$

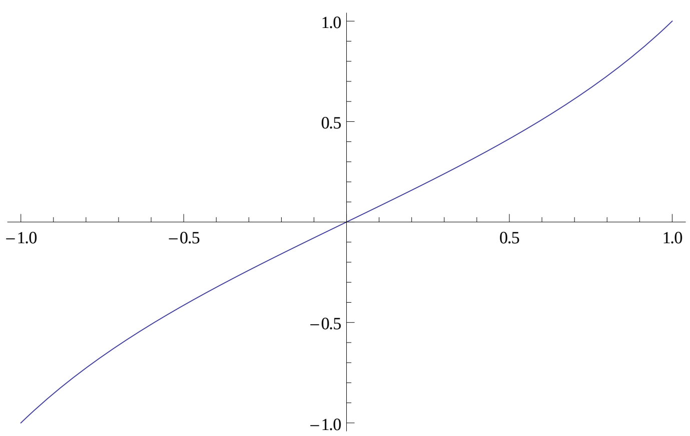
Quadrupole: Path 2 has $H_{\text{iron}} = 0$, on path 3 $\vec{H}$ and $d\vec{l}$ are orthogonal, so only path 1 contributes, and for this, $B = B'r$ where B' is a constant and r runs from 0 to a, the aperture radius. Therefore,

$$\mu_0 I = \int_0^a B dr = B'a^2/2$$

and

$$B' = \frac{2\mu_0 I}{a^2}, \text{ or, current required: } I = \frac{B'a^2}{2\mu_0} = \frac{B_p a}{2\mu_0}.$$

Short Magnets: Note that the above apply to cases where the poles are sufficiently long. If they are not, the field will not have the assumed functional form (constant for dipole and linear for quadrupole), because the flux expands outwards on leaving the poles. This makes the field smaller or conversely the required currents higher.



An example is the short quad with spherical poles. For this case, the magnetic field is not linear, but $B = B_p \tan\left(\frac{\pi}{4}\frac{r}{a}\right)$ (plot is on the left).

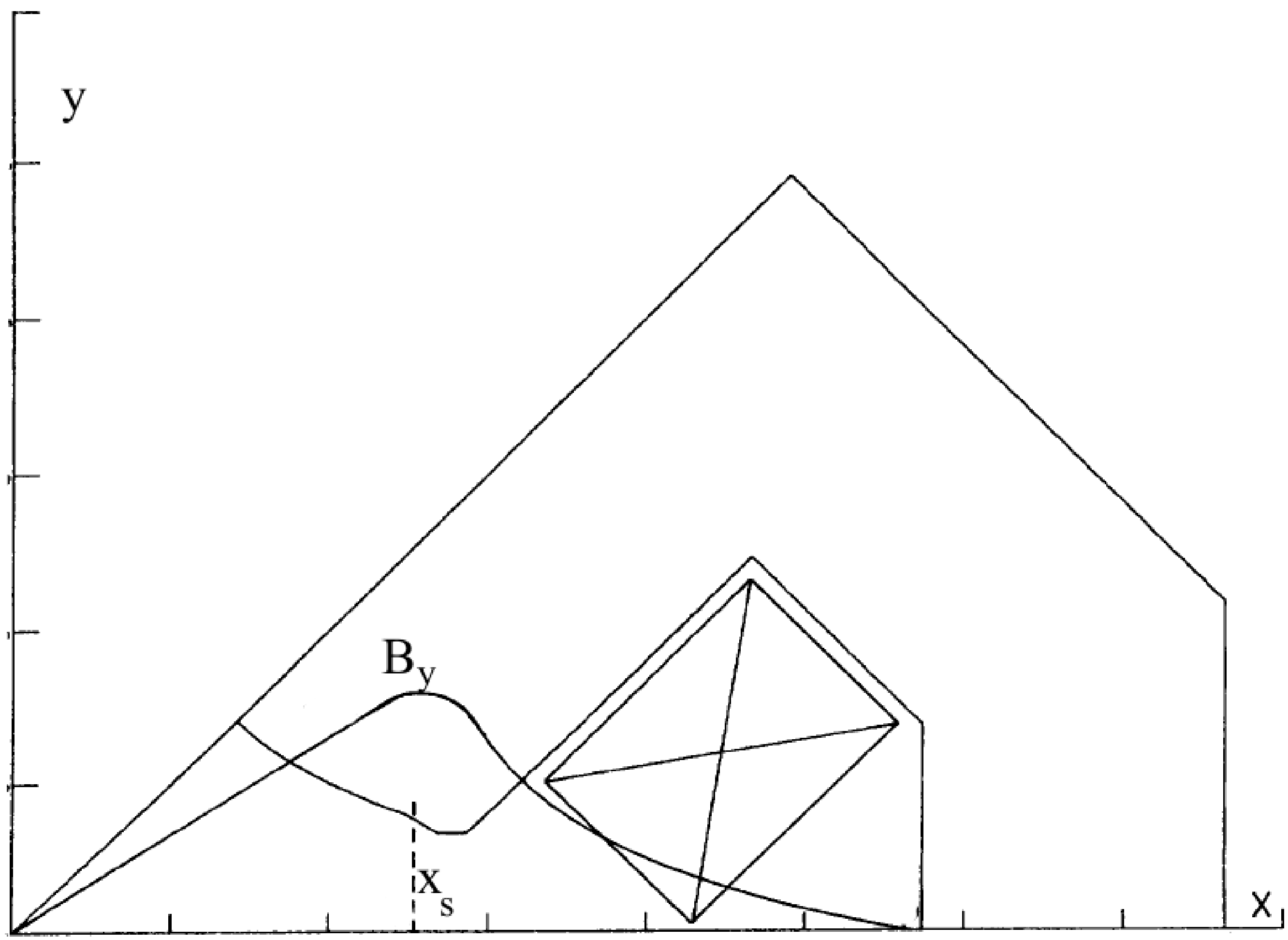
Thus the pole tip field B_p is not $B'a$, and

$$I = \frac{4\log 2}{\pi}\,\frac{B_p a}{2\mu_0} = 0.883\,\frac{B_p a}{2\mu_0}.$$

If we define B' as the gradient for vanishing r, then

$$I = \frac{16\log 2}{\pi^2}\,\frac{B'a^2}{2\mu_0} = 1.124\,\frac{B'a^2}{2\mu_0}.$$

As an exercise, let us try to calculate the flux density $\hat{B}$ inside the pole for this short quad.



Referring to the figure, by symmetry we only need $\int \vec{B} \cdot d\vec{S}$ on the right half of the median plane xz. The total flux Φ through the pole is

$$\Phi = 2 \int_{x=0}^{\infty} \int_{z=-\infty}^{\infty} B_y dz dx$$

For the region of the beam, we already know the first integral[1]:

$$\int_{z=-\infty}^{\infty} B_y dz = Kx,$$

where K is the quad's integrated strength $\int B' dz$.

However, this only works for x up to some value x_s, which is like $1.3 - 2$ times the aperture radius a, depending in general on the pole shape. For the short quads with spherical poles, it is $1.75a$. In the region $x > x_s$, B_y falls to

[1]Though B is nonlinear, $\int B dz$ is linear in x, as pointed out in my paper on short quads.

zero again, and I simply assume this gives an equal contribution to the integral. Thus

$$\Phi = 2Kx_s^2$$

The pole has radius a also, and cross-section πa^2, thus

$$\hat{B} = \frac{2}{\pi}\left(\frac{x_s}{a}\right)^2 K.$$

Taking $x_s \approx 1.75a$, we have

$$\hat{B} \approx 2K = 4B_p.$$

For the strong ARIEL quads, $K = 1.2$ Tesla and $B_p = 0.6$ Tesla, so $\hat{B} \approx 2.4$ Tesla; far above saturation.