Classical Mechanics → Accelerator Physics

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Summary

- Relativistic dynamics
- Some phase space conservation laws
- Curvilinear coordinate system
- Magnetic fields for planar reference orbit, and resulting Hamiltonian

Note: Green exercises are for your own benefit. Please try them. Blue exercises are to be handed in and graded.

What's the big deal?

- Accelerated particles have known properties: charge, mass, ...
- We impose known electric and magnetic fields to control them
- $\vec{F} = \frac{d\vec{p}}{dt} = q(\vec{\mathcal{E}} + \vec{v} \times \vec{B})$
- Use a good numerical integrator like Runge-Kutta

Seems trivial.

But:

- How to describe motion? What reference frame? “Lab” frame not efficient or practical (in a large storage ring like LHC, $1\,\mu$m matters in a 10 km machine; that’s 10 sig figs!). So we define a reference trajectory and coordinates are w.r.t. that. But this is a non-inertial frame. Classical Mechanics

- How accurately to describe $\vec{\mathcal{E}}$, $\vec{B}$? On a $1\,\mu$m scale? One can use measured fields, calculated fields or idealized fields. When to use which? Measured field have measurement errors that introduce spurious dynamics. E&M

- Further, in this description, fields not intrinsically realistic. Conservation laws not necessarily obeyed. Classical Mechanics

- There are typically 10^{10} particles in a "Bunch". Do we need to track every one? For total storage times (days in cases of storage rings)? This would take more computing time than practical. Instead, the distribution of particles is idealized in some way. e.g. rms parameters are tracked. Other statistical tricks used as well.Stat. Mech.

- The particles collectively create fields that perturb dynamics ("Space Charge"). This is essentially a "plasma in motion"; collective modes of Plasma Physics apply, plus dynamically-induced ("Wake") fields.Plasma Phys.

Relativity

$$E_0 = m_0 c^2,\ E = mc^2 = \gamma m_0 c^2$$

where E is energy[1], $\gamma = \left(1 - v^2/c^2\right)^{-1/2}$, m_0 is rest mass and $m = \gamma m_0$ is taken to be "relativistic mass". These are formulas that (a) simplify to the point of removing the essential Hamiltonian physics, and (b) mislead. Lev Okun, 2006 (CLICK ME: magenta are links) calls it a "pedagogical virus". For good reason. The way it's generally taught has made it on one hand overly simplified, and on the other hand useless for making even simple calculations.

Hereafter, I will not use $m = \gamma m_0$, only the rest mass. The subscript 0 then becomes redundant, so I drop it.

So let's work backwards from there.

$$E = \gamma mc^2$$

[1]Notation: electric field will be denoted $\vec{\mathcal{E}}$ to distinguish it from energy E.

This is better written as

$$E^2 = (mc^2)^2 + (pc)^2$$

(where $p^2 = p_x^2 + p_y^2 + p_z^2$, the square of the momentum magnitude). It's better because we want to track the positions and momenta of our accelerated particles. We need functional, not just numerical formulas. *Excursion: What exactly is E? There is a tendency to subtract rest mass from E and call this the "kinetic" energy. $E - mc^2 = (\gamma - 1)mc^2$. One is tempted to think of mc^2 as potential energy. But this is **wrong**; it is a holdover from Newtonian mechanics. There actually are two parts, but they are not potential and kinetic, rather rest and dynamic, and add together **in quadrature**.*

This is in fact only correct for particles of charge q in absence of electric field. The canonical coordinate E is the total energy, while what appears in the the above equation is only the dynamical (kinetic) part. The electric potential Φ must be subtracted:

$$(E - q\Phi)^2 = (mc^2)^2 + p^2c^2 \tag{1}$$

For cases with magnetic field, the canonical momentum is $\vec{P} = \vec{p} + q\vec{A}$, where

$\vec{A}$ is the vector potential.

$$(E - q\Phi)^2 = (mc^2)^2 + |\vec{P} - q\vec{A}|^2 c^2 \tag{2}$$

We can write more explicitly in a form that exhibits the symmetry:

$$m^2c^2 = (E - q\Phi)^2/c^2 - (P_x - qA_x)^2 - (P_y - qA_y)^2 - (P_z - qA_z)^2 \tag{3}$$

This equation plus the fact that $-E, P_x, P_y, P_z$ are canonical and conjugate to t, x, y, z can be used to derive **all** of the relativistic dynamics of charged particles.

Recall Hamilton's variational principle (or Principle of least action):

$$\delta \int (P_x dx + P_y dy + P_z dz - E dt) = 0 \tag{4}$$

(Maybe you recall this as: $\delta \int (\sum_i P_i \dot{q}_i - H) dt$.)

4-vectors

We can make these even more elegant by defining 4-vectors:

Position	q_μ	x	y	z	ict
Momentum	P_μ	P_x	P_y	P_z	iE/c
Potential	A_μ	A_x	A_y	A_z	$i\Phi/c$

"Energy equation":

$$|P_\mu - qA_\mu|^2 = -m^2c^2 \tag{5}$$

Principle of least action:

$$\delta \int P_\mu \cdot dq_\mu = 0$$

Far from messing up Newtonian mechanics, relativity restores a pleasing symmetry. The unity of electric and magnetic fields, of energy and momentum are not apparent in the Lorentz equation.

Newton and Leibniz

...would argue as to which was the *real* quantity of motion: mv or mv^2.

Look again at 4. It's true even in Newtonian mechanics and was known far before Einstein.

It must have been a puzzle why E and $\vec{P}$ played similar roles when the central equation was

$$E = \frac{1}{2m}\left|\vec{P} - q\vec{A}\right|^2 + q\Phi.$$

E and P in this non-relativistic limit look like completely different concepts. But in relativity they're just different components of the 4-vector.

Equations of motion

If E is taken as the Hamiltonian, we can use the Principle of Least Action to derive the usual equations of motion:

Use the calculus of variations on the integrand

$\mathcal{L} = \sum_i P_i \dot{q}_i - H \Longrightarrow \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q} = 0 \Longrightarrow$

$$\begin{aligned}
\frac{dx}{dt} &= \frac{\partial H}{\partial P_x}, \; \frac{dP_x}{dt} = -\frac{\partial H}{\partial x}, \\
\frac{dy}{dt} &= \frac{\partial H}{\partial P_y}, \; \frac{dP_y}{dt} = -\frac{\partial H}{\partial y}, \\
\frac{dz}{dt} &= \frac{\partial H}{\partial P_z}, \; \frac{dP_z}{dt} = -\frac{\partial H}{\partial z}, \\
\frac{dE}{dt} &= \frac{\partial H}{\partial t}
\end{aligned} \tag{6}$$

Some Exercises

Exercise 1: Use eqns. 6 along with the "Energy equation" (3) to derive the Lorentz equation:

$$\frac{d\vec{p}}{dt} = q(\vec{\mathcal{E}} + \vec{v} \times \vec{B}) \qquad (7)$$

where $\vec{\mathcal{E}} = -\nabla\Phi - \frac{\partial A}{\partial t}$, and $\vec{B} = \nabla \times \vec{A}$. Note that this equation is *unchanged* compared with the non-relativistic case.

Exercise 2: Take the "Energy equation" 3 to the non-relativistic limit $E - mc^2 \ll mc^2$ to confirm the "Newtonian" Hamiltonian

$$H = \frac{1}{2m}\left|\vec{P} - q\vec{A}\right|^2 + q\Phi \qquad (8)$$

Relation between Momentum and Velocity

We can write for H:

$$(H - q\Phi)^2 = m^2c^4 + c^2(P_x - qA_x)^2 + c^2(P_y - qA_y)^2 + c^2(P_z - qA_z)^2 \tag{9}$$

Find

$$\begin{aligned}(H - q\Phi)v_x = (H - q\Phi)\frac{\partial H}{\partial P_x} = (H - q\Phi)\frac{\partial(H - q\Phi)}{\partial P_x} = \\ = \frac{1}{2}\frac{\partial(H - q\Phi)^2}{\partial P_x} = c^2(P_x - qA_x) = c^2 p_x\end{aligned} \tag{10}$$

But H is the energy E. So $p_x = \frac{E-q\Phi}{c^2}\, v_x$ and similar for p_y and p_z, therefore

$$\vec{p} = \frac{E - q\Phi}{c^2}\,\vec{v}. \tag{11}$$

Compare: $\vec{p} = m\vec{v}$. "Simplified" by replacing m with γm. Refrain from this obfuscatory notation.

What are E and $\vec{P}$?

Once again: E is total energy, $q\Phi$ is electrostatic ("potential") energy, so $E - q\Phi$ can be called "kinetic" energy (call it E_k), but it includes the rest mass.

$\vec{P}$ is the total (or canonical) momentum containing both the dynamic part and the spatial (magnetic field) part; $\vec{p} = \vec{P} - q\vec{A}$ is the "kinetic part" of the momentum.

If we use these in eqn. 3, we find $E_k^2 = (mc^2)^2 + p^2c^2 = (mc^2)^2 + E_k^2\beta^2$. Solving,

$$E_k = \frac{mc^2}{\sqrt{1-\beta^2}} \equiv \gamma mc^2,$$

which is where we started.

However, this last is not the fundamental equation. Equation 5 (or 9) is the fundamental equation; it gives ALL the dynamics.

Vernacular Energy

Sloppily, we use the word “energy” without qualification to mean the amount of energy added to the rest energy. IOW,

$$\text{“Energy”} = E - mc^2 = (\gamma - 1)mc^2 \tag{12}$$

I will never call this energy E, but always in quotes: “Energy”.

Why bother with H?

Since eqn. 7 appears to be a simple equation, why bother with Hamiltonians and canonical momenta etc?

The Canonical approach gives deeper insights, makes symmetries more obvious, allows easier coordinate transformations, is more amenable to numerical calculation, provides relations among seemingly-independent dynamic parameters, etc. State vector, motion, flow in phase space.

Conservation laws are built into the Hamiltonian approach, making computation optimally efficient. For example, one can numerically (inefficiently) solve the Lorentz equation without knowing or using conservation of energy. Or angular momentum, etc. One conservation law not at all obvious from the force equation is Liouville's theorem.

Example 1: No fields

$\Phi = 0,\ \vec{A} = \vec{0}$. Then $H(x, P_x, y, P_y, z, P_z; t) = \sqrt{m^2c^4 + c^2P_x^2 + c^2P_y^2 + c^2P_z^2}$.

No x-dependence, so $\frac{dP_x}{dt} = -\frac{\partial H}{\partial x} = 0$, P_x is conserved. Same for P_y and P_z.
No t-dependence, so $\frac{dE}{dt} = \frac{\partial H}{\partial t} = 0$, E is conserved.

$\frac{dx}{dt} = \frac{\partial H}{\partial P_x} = \frac{cP_x}{\sqrt{m^2c^4+c^2P_x^2+c^2P_y^2+c^2P_z^2}} = \frac{c^2P_x}{E} = \frac{P_x}{\gamma m}$, same for y and z, so $\gamma m\vec{v} = \vec{P}$.

And since $\vec{P}$ is conserved, $\vec{v}$ is a constant.

This is Newton's first law.

Example 2: Electric field only

Then $H(x, P_x, y, P_y, z, P_z; t) = q\Phi + \sqrt{m^2c^4 + c^2P_x^2 + c^2P_y^2 + c^2P_z^2}$.

E is still conserved, but $\vec{P}$ is not. $\frac{d\vec{P}}{dt} = -\nabla\Phi$.

$\vec{v} = \frac{c^2\vec{P}}{E-q\Phi}$, as shown previously (11).

E is conserved, but $E - q\Phi$ is not.

There is still tendency to think of $(E-q\Phi)/c^2$ as a redefined, energy-dependent "mass", but this is not useful computationally and not helpful pedagogically.

Example 3: Constant Magnetic Field, in y direction

An $\vec{A}$ that works is $(0, 0, -xB)$. ($(zB, 0, 0)$ also works ...) Then $H(x, P_x, y, P_y, z, P_z; t) = \sqrt{m^2c^4 + c^2P_x^2 + c^2P_y^2 + c^2(P_z + qxB)^2}$.

E, P_y, and P_z are conserved. P_x is not: $\frac{dP_x}{dt} = -\frac{\partial H}{\partial x} = -\frac{c^2(P_z+qxB)}{E}qB$, and we still have $\vec{v} = \frac{c^2(\vec{P}-q\vec{A})}{E} = \frac{c^2\vec{p}}{E}$ (equation 11). So in terms of $\vec{p}$,

$$\begin{aligned}
\frac{dp_x}{dt} &= -qB\tfrac{dz}{dt} \\
\frac{dp_y}{dt} &= 0 \\
\frac{dp_z}{dt} &= qB\tfrac{dx}{dt}
\end{aligned}$$

(or $\frac{d\vec{p}}{dt} = q\vec{v} \times \vec{B}$). Reminder: $\vec{p}$ is NOT a canonical variable.

Exercise 3: Repeat the same for the alternative $\vec{A} = (zB, 0, 0)$.

or, using Lorentz Equation:

Combine eqn. 11 with $\Phi = 0$, with the Lorentz equation:

$$\dot{\vec{v}} = \frac{qc^2}{E}\vec{v} \times \vec{B} \tag{13}$$

For $\vec{v} \perp \vec{B}$, this results in a circular trajectory, of radius ρ, meaning $\dot{v} = \frac{v^2}{\rho}$.

Finally,

$$B\rho = \frac{p}{q} \tag{14}$$

remarkably, the same as for non-relativistic.

Exercise 4: For the LHC top energy (7 TeV) and magnetic field in the dipoles (8.33 Tesla), find ρ.

Example 4: Paraxial Case

The most common case in accelerators and transport is where the momentum is dominated by one component. $p_z \gg p_x, p_y$. In

$$H(x, P_x, y, P_y, z, P_z; t) = q\Phi + c\sqrt{m^2c^2 + (P_x - qA_x)^2 + (P_y - qA_y)^2 + (P_z - qA_z)^2} \tag{15}$$

let's define γ_0 to abbreviate: $\gamma_0 mc = \sqrt{m^2c^2 + p_z^2} \approx (E - q\Phi)/c$. Then

$$H = q\Phi + \gamma_0 mc^2\sqrt{1 + \left(\frac{P_x - qA_x}{\gamma_0 mc}\right)^2 + \left(\frac{P_y - qA_y}{\gamma_0 mc}\right)^2} \tag{16}$$

$$H \approx q\Phi + \gamma_0 mc^2 + \frac{(P_x - qA_x)^2}{2\gamma_0 m} + \frac{(P_y - qA_y)^2}{2\gamma_0 m} \tag{17}$$

and for example, $\frac{dx}{dt} = \frac{\partial H}{\partial P_x} = \frac{P_x - qA_x}{\gamma_0 m}$. Harmonic equations are common

Liouville's Theorem

Look again at eqns. 6. Take partial derivatives $\frac{\partial}{\partial x}\frac{dx}{dt}$, $\frac{\partial}{\partial P_x}\frac{dP_x}{dt}$, etc. Note that

$$\frac{\partial}{\partial x}\frac{dx}{dt} + \frac{\partial}{\partial P_x}\frac{dP_x}{dt} = \frac{\partial}{\partial y}\frac{dy}{dt} + \frac{\partial}{\partial P_y}\frac{dP_y}{dt} = \frac{\partial}{\partial z}\frac{dz}{dt} + \frac{\partial}{\partial P_z}\frac{dP_z}{dt} = 0.$$

A weaker condition is that the sum of these 3 is zero:

$$\frac{\partial}{\partial x}\frac{dx}{dt} + \frac{\partial}{\partial P_x}\frac{dP_x}{dt} + \frac{\partial}{\partial y}\frac{dy}{dt} + \frac{\partial}{\partial P_y}\frac{dP_y}{dt} + \frac{\partial}{\partial z}\frac{dz}{dt} + \frac{\partial}{\partial P_z}\frac{dP_z}{dt} = 0.$$

But this can be written as a 6-dimensional divergence of the 6-dimensional state vector $\vec{r_6} = (x, P_x, y, P_y, z, P_z)$:

$$\nabla_6 \cdot \frac{d\vec{r_6}}{dt} = 0 \qquad (18)$$

So the flow of points in phase space is divergenceless and therefore the 6-D volume of a set of points is conserved. This is Liouville's theorem.

Stated in mathematical form, let $f(x, P_x, y, P_y, z, P_z)$ be the distribution function in phase space, then

$$\frac{df}{dt} = 0. \tag{19}$$

The total derivative can be expanded: $df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial P_x}dP_x + \frac{\partial f}{\partial P_y}dP_y + \frac{\partial f}{\partial P_z}dP_z$, which we can write more compactly as ($\vec{r} = (x, y, z)$, $\vec{P} = (P_x, P_y, P_z)$)

$$df = \frac{\partial f}{\partial t}dt + \nabla f \cdot d\vec{r} + \nabla_P f \cdot d\vec{P}$$

into which we can substitute the equations of motion 6, so finally the differential equation 19 is:

$$\frac{\partial f}{\partial t} + \nabla f \cdot \frac{\partial H}{\partial \vec{P}} - \nabla_P f \cdot \frac{\partial H}{\partial \vec{r}} = 0. \tag{20}$$

This is in fact the collisionless Boltzmann equation.

Poincaré-Cartan integral invariant

Clearly, if H can be separated into $H_x(x, P_x) + H_y(y, P_y) + H_z(z, P_z)$, then each subspace area is conserved. But what about if the motion is coupled? The stronger form of phase space conservation is that even with coupling, the sum of the subspace areas is conserved. In particular, (in 4-vector form)

$$\oint P_\mu \cdot dq_\mu = \text{constant}$$

where the closed path is arbitrary but moves with the motion of the phase space points. This is called the Poincaré-Cartan integral invariant. Click here for a proof. For the case of time-independent H, simply

$$\oint \vec{P} \cdot d\vec{q} = \text{constant.}$$

That is to say, the sum of the signed projections onto the 3 (q_i, P_i)-planes is preserved.

Emittance

If there are no coupling terms, then each of these is conserved. E.g. $\oint P_y dy$, the area of the closed loop in the 2D "phase space" y-P_y is a constant as a particle's coordinates change with time. Thus if this is the boundary of a group of particles, then this area that the particles occupy in phase space is constant.

This is proportional to what we call the normalized emittance $\epsilon_{\mathrm{n}y}$. Specifically, if the loop is elliptical,

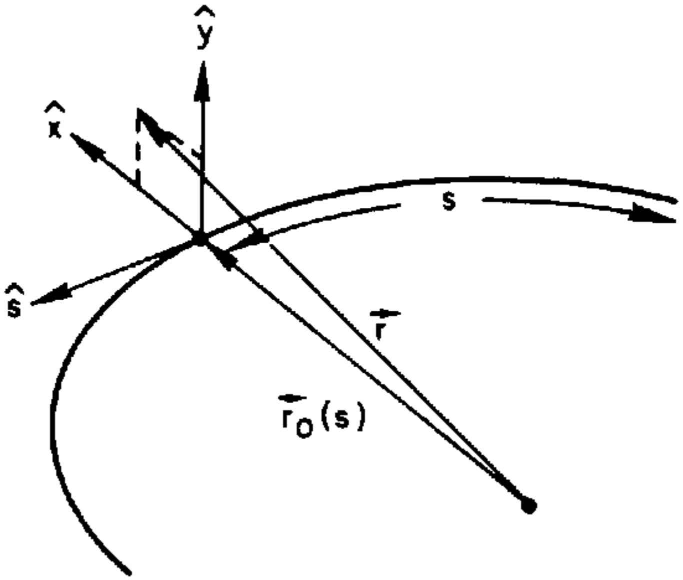
$$\text{constant} = \oint P_y dy = P_0 \oint y' dy = \beta\gamma m c \pi \epsilon_y \equiv m c \pi \epsilon_{\mathrm{n}y}.$$

So the normalized emittance $\epsilon_{\mathrm{n}y} = \beta\gamma\epsilon_y$ is a constant (if motion is uncoupled).

Curvilinear coordinate system

Often in accelerator physics, we do not use time as independent variable. Instead of asking *What are the particles' position, momentum after time* t*?*, we want to know *What are the particles' generalized coordinates when they reach a certain point in space?*. Generally, we want to use s, the distance along the beamline or around the ring as independent variable.

This means 2 things: (1) the reference orbit is not a straight line, so we have to transform to a non-inertial coordinate system, (2) we want to change independent variable.



For a curve in a plane, we take s along the orbit, x the orthogonal in-plane, y orthogonal to the plane, let ρ be the radius of curvature (it may depend upon s). In other words, the coordinate system is defined by the trajectory of a reference particle. Such a coordinate system is called Frenet-Serret. Then the canonical coordinate conjugate to s is

$$P_s = \vec{P} \cdot \hat{s} \left(1 + \frac{x}{\rho} \right) \tag{21}$$

This is proved below. (a Goldstein refresher... click here for a review)

Canonical Transformation to Frenet-Serret System

Let us denote $\vec{R} = (X, Y, Z)$ as the absolute Cartesian system. Momentum is $\vec{P} = (P_X, P_Y, P_Z)$. With time as independent variable, the Hamiltonian H is implicit in the following:

$$m^2c^2 = (H - q\Phi)^2/c^2 - |\vec{P} - q\vec{A}|^2 \tag{22}$$

The Frenet-Serret reference orbit is assumed to lie in a plane and given by $\vec{R} = \vec{R_0}(s)$; the new coordinates are (x, y, s) and we wish to transform to this system. s is along (tangent to) the orbit, x is orthogonal in the plane, and y is orthogonal to the plane. In terms of unit vectors denoted by $\hat{}$, $\hat{x} \times \hat{y} = \hat{s}$.

The general space vector in terms of the new coordinates is therefore:

$$\vec{R}(x, y, s) = x\hat{x} + y\hat{y} + \vec{R_0}(s) \tag{23}$$

The local curvature radius of the orbit is ρ, so simply by inspection we find the following relations to hold:

$$\hat{s} = \frac{d\vec{R}_0}{ds}, \ \frac{d\hat{x}}{ds} = \frac{\hat{s}}{\rho}, \ \frac{d\hat{y}}{ds} = 0, \ \frac{d\hat{s}}{ds} = -\frac{\hat{x}}{\rho}. \tag{24}$$

I state without derivation that the generating function for the transformation is

$$F_3(x, y, s; P_X, P_Y, P_Z) = \vec{P} \cdot \vec{R}(x, y, s) = x\vec{P} \cdot \hat{x} + y\vec{P} \cdot \hat{y} + \vec{P} \cdot \vec{R}_0 \tag{25}$$

This is of the type “new positions, old momenta”, so require derivatives w.r.t. old momenta are the old positions. Trivially, we see that $\frac{\partial F_3}{\partial P_X} = X$ and similarly for Y and Z.

It remains only to discover the new momenta conjugate to the chosen coordinates (x, y, s). We have:

$$P_x = \frac{\partial F_3}{\partial x} \quad = \quad \vec{P} \cdot \hat{x} \tag{26}$$

$$P_y = \frac{\partial F_3}{\partial y} \quad = \quad \vec{P} \cdot \hat{y} \tag{27}$$

$$P_s = \frac{\partial F_3}{\partial s} \quad = \quad \vec{P} \cdot \left(x\,\frac{d\hat{x}}{ds} + \frac{d\vec{R}_0}{ds} \right) = \left(1 + \frac{x}{\rho}\right) \vec{P} \cdot \hat{s} \tag{28}$$

We also define the canonical vector potential to have

$$A_s = \left(1 + \frac{x}{\rho}\right) \vec{A} \cdot \hat{s} \tag{29}$$

Notice that the generating function has no explicit time dependence. It therefore follows that the new Hamiltonian is simply the old one with the new canonical coordinates. The vectors in 22 appear as a vector norm, which is unchanged in going from one orthogonal system to another. Let $\vec{P} - q\vec{A} \equiv \vec{\mathcal{P}}$. Then

$$\begin{aligned}
\mathcal{P}^2 &= (\vec{\mathcal{P}} \cdot \hat{X})^2 + (\vec{\mathcal{P}} \cdot \hat{Y})^2 + (\vec{\mathcal{P}} \cdot \hat{Z})^2 \\
&= (\vec{\mathcal{P}} \cdot \hat{x})^2 + (\vec{\mathcal{P}} \cdot \hat{y})^2 + (\vec{\mathcal{P}} \cdot \hat{s})^2 \\
&= \mathcal{P}_x^2 + \mathcal{P}_y^2 + \left(\frac{\mathcal{P}_s}{1 + x/\rho}\right)^2
\end{aligned} \tag{30}$$

Also see Courant & Snyder (1958), p. 406[2]. A general very good reference is the textbook by S.Y. Lee (2004) (beware, the link does not allow one to see every page).

It is also handy to define

$$A_s = \vec{A} \cdot \hat{s} \left(1 + \frac{x}{\rho}\right). \tag{31}$$

Then (with $h \equiv 1 + x/\rho$)

$$\vec{B} = \nabla \times \vec{A} = \left(\frac{1}{h}\frac{\partial A_s}{\partial y} - \frac{\partial A_y}{\partial s}, \frac{\partial A_x}{\partial s} - \frac{1}{h}\frac{\partial A_s}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \tag{32}$$

[2]Note that there is an error in B10; a closing square bracket is missing. Further, note that the case considered is more general. In our case, the "torsion" ω is zero.

Making a canonical transformation to this system, the equation 22 becomes:

$$m^2c^2 = \left(\frac{H - q\Phi}{c}\right)^2 - (P_x - qA_x)^2 - (P_y - qA_y)^2 - \left(\frac{P_s - qA_s}{1 + x/\rho}\right)^2 \qquad (33)$$

Remarkably, the addition of the one simple term $x/\rho(s)$ gives all the new non-inertial dynamics, commonly ascribed to “centrifugal” and “Coriolis” forces. (More on this later.)

Change of independent variable

All canonical variables have the same status (eqn. 4). The pairs are (x, P_x), (y, P_y), (s, P_s), $(t, -E)$. We can circularly permute from one pair to another.

Therefore, to obtain the t-based Hamiltonian, simply Let $H_t = E$ and solve for H_t:

$$H_t = q\Phi + c\sqrt{m^2c^2 + (P_x - qA_x)^2 + (P_y - qA_y)^2 + \left(\frac{P_s - qA_s}{1 + x/\rho}\right)^2} \tag{34}$$

To obtain the s-based Hamiltonian, simply Let $H_s = -P_s$ and solve for H_s:

$$H_s = -qA_s - \left(1 + \frac{x}{\rho}\right)\sqrt{-m^2c^2 - (P_x - qA_x)^2 - (P_y - qA_y)^2 + \left(\frac{E - q\Phi}{c}\right)^2} \tag{35}$$