Homework Exercise Set #4 PHYS 560

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Exercise 11: Expand the "Energy equation" to the non-relativistic limit $E - m_0c^2 \ll m_0c^2$ to confirm the "Newtonian" Hamiltonian

$$H = \frac{1}{2m}\left|\vec{P} - q\vec{A}\right|^2 + q\Phi,$$

and also find the next-higher-order relativistic correction term.

Solution:

Let $E = E_V + m_0c^2$ where $E_V \ll m_0c^2$. In other words, the vernacular "energy", the energy with rest energy subtracted, is much less than the rest energy. Then

$$m_0^2c^2 = \left(m_0c + \frac{E_V - q\Phi}{c}\right)^2 - |\vec{P} - q\vec{A}|^2.$$

To simplify notation, define $\varepsilon \equiv \frac{E_V - q\Phi}{m_0c^2}$, $\eta = \frac{|\vec{P} - q\vec{A}|}{m_0c}$:

$$(1 + \varepsilon)^2 = 1 + \eta^2$$

Clearly, $\varepsilon \ll 1$, $\eta^2 \ll 1$. So expand $\sqrt{1 + \eta^2} = 1 + \frac{\eta^2}{2} - \frac{\eta^4}{8} + ...$, to get to second order:

$$\varepsilon \approx \frac{\eta^2}{2} - \frac{\eta^4}{8}$$

Now substitute from the definitions:

$$E_V \approx q\Phi + \frac{|\vec{P} - q\vec{A}|^2}{2m_0} - \frac{|\vec{P} - q\vec{A}|^4}{8m_0^3c^2}$$

The Hamiltonian H can be either E or E_V, it does not matter since m_0c^2 is a constant and the motion only depends upon derivatives of H.

Exercise 12: Use equations of motion along with the "Energy equation" to derive the Lorentz equation:

$$\frac{d\vec{p}}{dt} = q(\vec{\mathcal{E}} + \vec{v} \times \vec{B})$$

where $\vec{\mathcal{E}} = -\nabla\Phi - \frac{\partial A}{\partial t}$, and $\vec{B} = \nabla \times \vec{A}$. Notice that this equation is *unchanged* compared with the non-relativistic case.

Solution:

$$H = q\Phi + \sqrt{(m_0c^2)^2 + |\vec{P} - q\vec{A}|^2c^2}$$

$$v_x = \frac{\partial H}{\partial P_x} = \frac{c^2}{\sqrt{\cdots}}(P_x - qA_x),$$

same for y and z, so

$$\vec{v} = \frac{c^2}{\sqrt{\cdots}}\,\vec{p},$$

$$\frac{dP_x}{dt} = -\frac{\partial H}{\partial x} = -q\frac{\partial \Phi}{\partial x} + \frac{qc^2}{\sqrt{\cdots}}\left[(P_x - qA_x)\frac{\partial A_x}{\partial x} + (P_y - qA_y)\frac{\partial A_y}{\partial x} + (P_z - qA_z)\frac{\partial A_z}{\partial x}\right]$$

$$= -q\frac{\partial \Phi}{\partial x} + q\left(v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x}\right)$$

Total derivative of A_x:

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial x}v_x + \frac{\partial A_x}{\partial y}v_y + \frac{\partial A_x}{\partial z}v_z + \frac{\partial A_x}{\partial t}$$

Subtract the last two:

$$\frac{1}{q}\frac{dp_x}{dt} = -\frac{\partial \Phi}{\partial x} - \frac{\partial A_x}{\partial t} + v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + v_z\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)$$

$$= \mathcal{E}_x + v_yB_z - v_zB_y = \mathcal{E}_x + (\vec{v} \times \vec{B})_x$$

and similarly for the y and z components. QED.

There is a quicker method if one knows vector identities. Since $\vec{v}$ is not a field depending on x, y, z, we can write:

$$\nabla(\vec{v} \cdot \vec{A}) = (\vec{v} \cdot \nabla)\vec{A} + \vec{v} \times (\nabla \times \vec{A}),$$

or

$$\vec{v} \times \vec{B} = \nabla(\vec{v} \cdot \vec{A}) - (\vec{v} \cdot \nabla)\vec{A}$$

From the $\frac{dP_x}{dt}$ above,

$$\frac{1}{q}\frac{d\vec{P}}{dt} = -\nabla\Phi + \nabla(\vec{v} \cdot \vec{A}),\ \ \frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial t} + (\vec{v} \cdot \nabla)\vec{A}$$

Subtracting these two using $\vec{\mathcal{E}} = -\nabla\Phi - \frac{\partial A}{\partial t}$ and the above vector identity gives the desired result.

Exercise 13: Demonstrate that the vector potential (54) correctly gives the magnetic field (53).

Solution:

The given vector potential:

$$A_s = -B_0\rho\left[\frac{x}{\rho} + \frac{1+k}{\rho^2}\frac{x^2}{2} - \frac{k}{\rho^2}\frac{y^2}{2}\right],$$

$$B_x = \frac{1}{h}\frac{\partial A_s}{\partial y} = \frac{1}{h}B_0\rho\frac{ky}{\rho^2} = \frac{B_0 ky}{h\rho} \approx B_0 ky/\rho$$

The last step is justified because we are only using linear forces and the neglected term is higher order.

$$B_y = -\frac{1}{h}\frac{\partial A_s}{\partial x} = \frac{B_0\rho}{h}\left[\frac{1}{\rho} + \frac{1+k}{\rho^2}x\right] = \frac{B_0}{h}\left[1 + \frac{x}{\rho} + \frac{kx}{\rho}\right] \approx B_0 + B_0 kx/\rho,$$

the last step because h cancels the $1 + x/\rho$, and again the approximate sign because we neglect forces higher than linear order.