Iterative Learning Control for Two-Pole System

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This work builds on, and is a continuation of, the results reported in TRI-BN-16-17.
http://lin12.triumf.ca/text/design_notes/TRI-BN-16-17.pdf

One-pole closed loop transfer function $\frac{A}{\mathrm{z}-B}$

Two-pole closed loop transfer function $\frac{A(\mathrm{z}-1)}{(\mathrm{z}-\mathrm{B1})(z-\mathrm{B2})}$

Z-domain conditions for stability under iteration of 1, 2 & 3 term learning functions for the single-pole plant, corresponding to integral gain Ki=0, were derived/reported in TRI-BN-16-17 and are repeated here.
In this report, we obtain the analogous "monotonic convergence" stability criteria for the two-pole plant, corresponding to integral gain Ki=a.Kp.
Stability criteria are given first for the generic variables A, B, B1 & B2;
And then for the particular relations between A, B, B1 &B2 that pertain to the system described in TRI-DN-13-23 with proportional and integral control loop gains Kp & Ki.

Summary of Results

# poles	1 term ILC	2 term look back	2 term look ahead	3 term look back	3 term look head
1	$v < 2(1 + \mathrm{Kp})$	$v < (1 + \mathrm{Kp})$	$v < (1 + \mathrm{Kp})$	$\mathrm{B} \rightarrow -1/2$	$\mathrm{B} \rightarrow 1$
2	$v < 2(1 + \mathrm{Kp})$	$v < (1 + \mathrm{Kp})$	$v < (1 + \mathrm{Kp}) - \mathrm{KpU}$	$\mathrm{B2} \rightarrow -\mathrm{B1}/(1+\mathrm{B1})$	$\mathrm{B2} \rightarrow 1/\mathrm{B1}$

v = learning gain. Kp = phase loop gain.
For the 2-pole case, neither of the 3-term ILC is stable when the gain (Kp) and delay (T) values are those given in TRI-DN-13-23.

Stability Analysis in Z-domain

- System is stable if the poles of the closed loop transfer function P(z) are inside or on the unit circle.
- Corollary: system stable if the zeros of the denominator of P(z) are inside or on the unit circle.

$$P(z)=\frac{G(z)}{1+C(z)G(z)}$$

Analysis is vastly simplified if we transform from variables U=aT & Kp to A & B

$$\mathrm{A} = 1 - \mathrm{e}^{-U} \geq 0$$

$$\mathrm{B} = \mathrm{e}^{-U}(1 + \mathrm{Kp}) - \mathrm{Kp}$$

$$\mathrm{B} = \mathrm{e}^{-U} - A\mathrm{Kp}$$

$$\mathrm{B} = 1 - A(1 + \mathrm{Kp})$$

sampling rate $\frac{1}{T}$, the cavity time constant $\frac{1}{a}$

The following slides 27-30 deal with the single pole problem when Ki=0.
See slides 146-149 for the 2-pole case when Ki=a.Kp

See Roy Smith for simple pole sandwiched inside a ZOH:
https://www.coursehero.com/file/pe2squ/Roy-Smith-ECE-147b-5-3-Zero-order-hold-equivalence-Discrete-time-transfer/

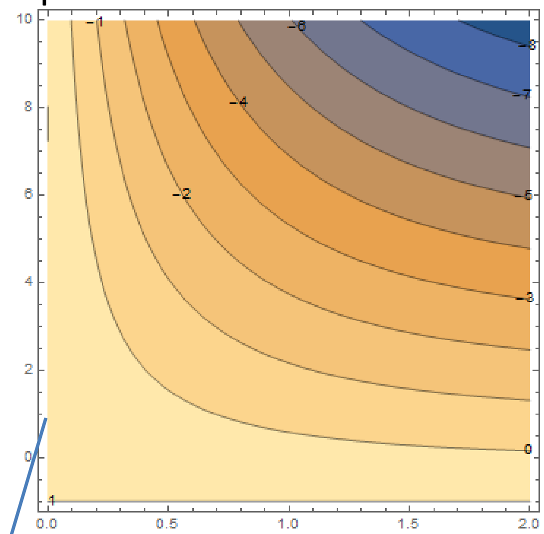
Proportional Control
Closed Loop Gain Function

$$P(z)=\frac{-1+e^{U}}{-1-Kp+e^{U}Kp+e^{U}z}$$

Can be written: $\frac{A}{z-B}$

$U > 0$

Kp

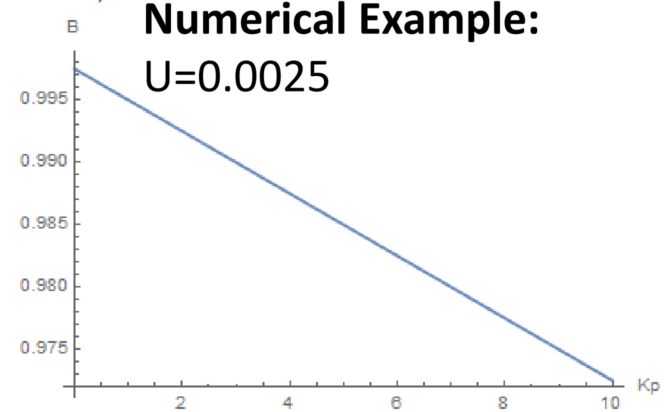


U

B(U,Kp)

Stability conditions:

- B >-1 implies → $Kp < \frac{1+e^{U}}{-1+e^{U}}$
- B >0 implies → $Kp < \frac{1}{-1+e^{U}}$
- B <1 implies → $Kp \geq -1$ & U>0
- U=0 implies → B=1 for all Kp



Numerical Example:
U=0.0025

1-TERM LEARNING; Ki=0

L(z)=v: use only information from the present time step within a trial

$$T = 1 - \frac{Ave^{i\theta}}{-B + e^{i\theta}}$$

T and R=|T|is largest at phase θ= 0,π
θ=0 →DC; θ=π → Nyquist frequency = (sample freq)/2
And dR/dϴ =0 at those phases.

$$(-1 < B \leq 0 \,\&\, 0 < Av < 2 + 2B) \text{ or } (0 < B < 1 \,\&\, 0 < Av < 2 - 2B)$$

$$0 < Av < 2 \,\&\, \frac{1}{2}(-2 + Av) < B < \frac{2 - Av}{2}$$

$$0 < v < 2 + 2\mathrm{Kp} \quad \text{and} \quad -1 < \mathrm{Kp} < \frac{1}{-1 + e^{U}}$$

Which is the gain limit with no ILC

U=aT; v = learning gain

2-TERM, LOOK BACK 1 STEP; Ki=0

L = v(1+1/z); causal = use only information from present and previous time step

$$\mathrm{T} = 1 - \frac{\mathrm{A}v\mathrm{e}^{\mathrm{i}\theta}\left(1 + \mathrm{e}^{-\mathrm{i}\theta}\right)}{-B + \mathrm{e}^{\mathrm{i}\theta}}$$

T and R=|T| is largest at phases θ=0,π
And dR/dΘ =0 at those phases

$$-1 < B < 1 \,\&\, 0 < \mathrm{A}v < 1 - B$$

$$0 < \mathrm{A}v < 2 \,\&\, -1 < B < 1 - \mathrm{A}v$$

We add the constraint B>0, leading to

$v < (1 + \mathrm{Kp})$ and $-1 < \mathrm{Kp} < \dfrac{1}{-1 + \mathrm{e}^{U}}$

2-TERM, LOOK AHEAD 1 STEP; Ki=0

L = v(1+z); non-causal = use information from present and next time step

$$\mathrm{T} = 1 - \frac{\mathrm{A}v\mathrm{e}^{\mathrm{i}\theta}\left(1 + \mathrm{e}^{\mathrm{i}\theta}\right)}{-B + \mathrm{e}^{\mathrm{i}\theta}}$$

T is large in four directions, including the directions θ=0,π.
And dR/dΘ =0 at those phases.
We consider θ=0, π and minimize Av

After some work, we find the sufficient condition:

$$0 < \mathrm{A}v < 1 \;\&\; (-1 + \mathrm{A}v)/3 \leq B \leq 1 - \mathrm{A}v$$

We add the constraint B>0, leading to

$$v \leq (1 + \mathrm{Kp}) \qquad -1 < \mathrm{Kp} < \frac{1}{-1 + \mathrm{e}^{U}}$$

So, the iterations are "monotonic convergent/stable" under those conditions

3-Term Time Symmetric Learning function; Ki=0

L = v(1+z+1/z); use information from present, previous and next steps within a trial

$$\mathrm{T} = 1 - \frac{\mathrm{Ave}^{\mathrm{i}\theta}\left(1 + \mathrm{e}^{-\mathrm{i}\theta} + \mathrm{e}^{\mathrm{i}\theta}\right)}{-B + \mathrm{e}^{\mathrm{i}\theta}}$$

T is large in four directions, including the directions θ=0,π.

We consider θ=0, π, π/2 leading to the contradiction: A*v*>0 & A*v*<0

The iterations are definitely NOT “monotonic stable”

3-Term, "Look Back 2 steps"; Ki=0

L = v(1+1/z+1/z²); causal = use information from present and 2 previous time steps

$$\mathrm{T} = 1 - \frac{\mathrm{A}v\mathrm{e}^{\mathrm{i}\theta}\left(1 + \mathrm{e}^{-\mathrm{i}\theta} + \mathrm{e}^{-2\mathrm{i}\theta}\right)}{-B + \mathrm{e}^{\mathrm{i}\theta}}$$

T and R=|T| is large at four phases, including θ=0,π. R=1 when θ= 2π/3.
We consider θ=0, π, π/2 leading to a necessary, but insufficient condition:

$$\left(-1 < B \leq -\frac{1}{2} \,\&\, 0 < \mathrm{A}v < 2 + 2B\right) \mathrm{or} \left(-\frac{1}{2} < B < 0 \,\&\, 0 < \mathrm{A}v < -2B\right)$$

$$0 < \mathrm{A}v < 1 \,\&\, \frac{1}{2}(\mathrm{A}v - 2) < B < -\frac{\mathrm{A}v}{2}$$

Note, this excludes B>0; whereas we are interested in 0<B<1

Condition is insufficient because dR/dΘ NOT =0 when phase θ= 2π/3

unless $B = -\frac{1}{2}$

3-Term, "Look Back 2 steps"; Ki=0

If we work harder, we obtain the almost sufficient condition:

$$\left(\mathrm{A}v < 1\ \&\ \frac{1}{42}(-29 + 8\mathrm{A}v) \le B \le \frac{1}{10}(-3 - 2\mathrm{A}v)\right) \text{ or } \left(\mathrm{A}v = 1 \& B - \frac{1}{2}\right)$$

$$\left(B \le -\frac{1}{2}\ \&\ \mathrm{A}v \le \frac{1}{8}(29 + 42B)\right) \text{ or } \left(B > -\frac{1}{2}\ \&\ \mathrm{A}v \le \frac{1}{2}(-3 - 10B)\right)$$

$$v \le 5 - \frac{13}{2A} + 5\mathrm{Kp}$$ and

$(2\mathrm{Kp} \le 1\ \&\ 0 < A < 1)$ or $2\mathrm{Kp} > 1\ \&\ 0<\mathrm{A}<3/(2+2\ \mathrm{Kp})$

Note, the stable area is less than given above. It is possible that the monotonic convergent shrinks to the line B=-1/2 & 0<Av<1

3-Term "Look Ahead 2 steps"; Ki=0

L = v(1+z+z^2); non-causal = use information from present and next two time steps

$$\mathrm{T} = 1 - \frac{\mathrm{A}v\mathrm{e}^{\mathrm{i}\theta}\left(1 + \mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{2\mathrm{i}\theta}\right)}{-B + \mathrm{e}^{\mathrm{i}\theta}}$$

T is large at six phases. R=1 when θ= 2π/3

We consider θ=0, π, π/2 leading to a necessary but insufficient condition:

$$\left(0 < B \le \frac{1}{4}\ \&\ 0 < \mathrm{A}v < 2B\right) \text{or} \left(\frac{1}{4} < B < 1\ \&\ 0 < \mathrm{A}v \le \frac{1}{3}(2 - 2B)\right)$$

$$0 < \mathrm{A}v < \frac{1}{2}\ \&\ \frac{\mathrm{A}v}{2} < B \le \frac{1}{2}(2 - 3\mathrm{A}v)$$

Condition is insufficient because dR/dϴ NOT =0 when phase θ= 2π/3

unless $B = 1$

3-Term "Look Ahead 2 steps"; Ki=0

If we work harder, we obtain the almost sufficient condition:

$$\left(\frac{1}{3} < B \leq \frac{5}{11} \;\&\; 0 < \mathrm{A}v < -1 + 3B\right) \text{or} \left(\frac{5}{11} < B < 1 \;\&\; 0 < \mathrm{A}v \leq \frac{1}{3}(2 - 2B)\right)$$

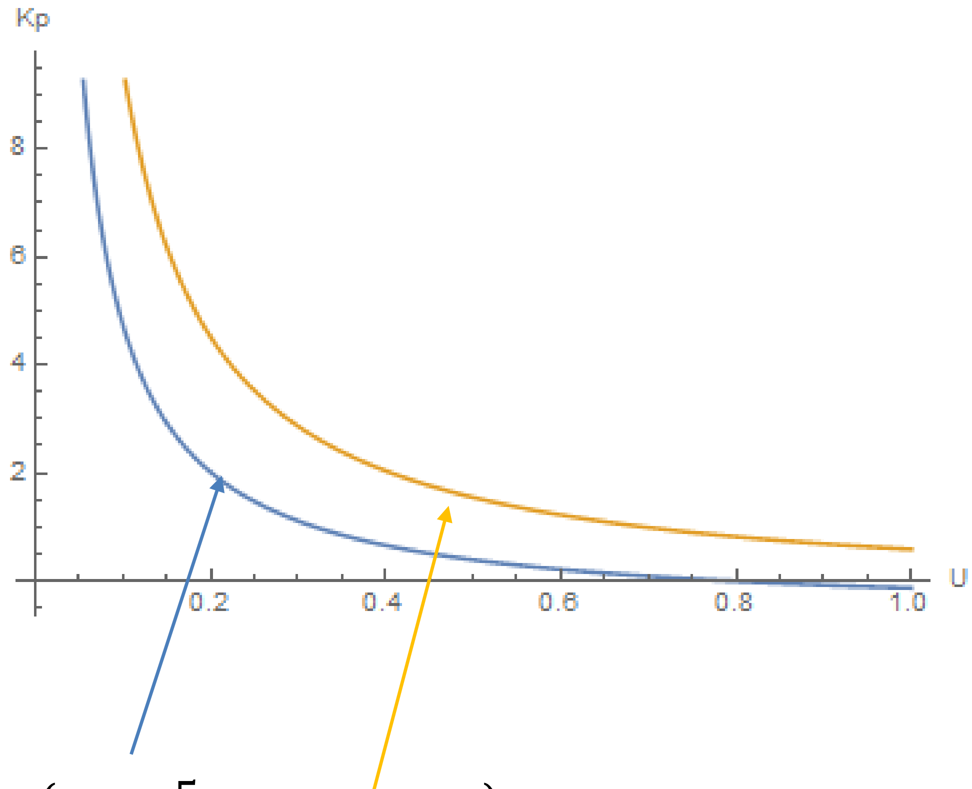
$$0 < \mathrm{A}v < \frac{4}{11} \;\&\; \frac{1 + \mathrm{A}v}{3} < B \leq \frac{1}{2}(2 - 3\mathrm{A}v)$$

$$0 < v < \frac{1}{3}(2 + 2\mathrm{Kp}) \qquad 0 < A < \frac{6}{11 + 11\mathrm{Kp}}$$

$$\frac{5}{11} < \mathrm{e}^{-U} < 1 \;\&\; \mathrm{Kp} > 0 \;\&\; 5 + 11\mathrm{Kp} < \frac{6}{-1 + \mathrm{e}^{U}}$$

Note, the stable area is less than given above. It is possible that the monotonic convergent shrinks to the point B→1 and A*v*→0.

3-Term “Look Ahead 2 steps”; Ki=0



$$\left\{\frac{1-(\frac{5}{11})e^{U}}{(-1+e^{U})}, \frac{1}{-1+e^{U}}\right\}$$

STABILITY OF THE 2-POLE SYSTEM <u>WITHOUT</u> ITERATIVE LEARNING CONTROL

PI Control & Ki=a.Kp
Closed Loop Gain Function

Sampling rate = 1/T; cavity time constant = 1/a

U=aT; ν = learning gain

$$P(z)=\frac{(-1+e^{U})(-1+z)}{1+Kp-e^{U}Kp-z-e^{U}z-Kpz+e^{U}Kpz-KpUz+e^{U}KpUz+e^{U}z^{2}}$$

Can be written: $$\frac{A(z-1)}{(z-B1)(z-B2)}$$

$$\left\{\left\{B1=\frac{1}{2}\left(b-\sqrt{b^{2}-4B}\right)\right\},\left\{B2=\frac{1}{2}\left(b+\sqrt{b^{2}-4B}\right)\right\}\right\}$$

Always √>0, so roots always pure real

$$b=1+e^{-U}-AKp(1+U)$$

Stability conditions:

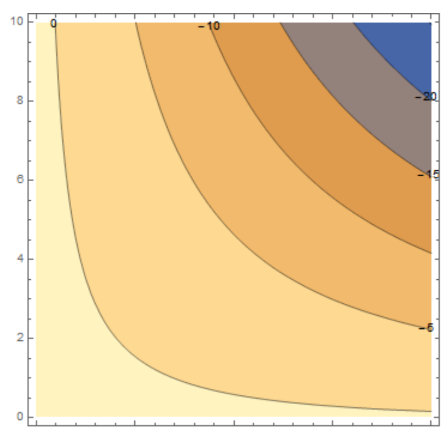
B1 >0 implies

$$U>0\ \&\ Kp\le\frac{1}{-1+e^{U}}$$

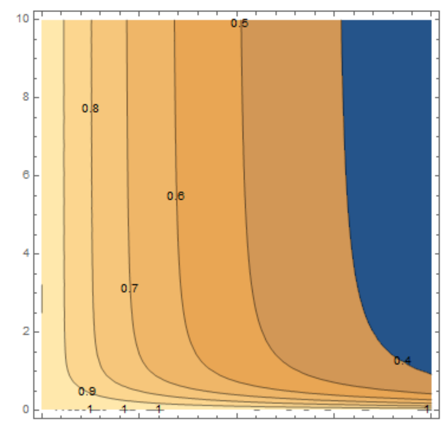
B2 <1 implies

$$U>0\ \&\ Kp\ge 0$$

B=B1=0 & B2= $1-e^{-U}U$ when $$Kp=\frac{1}{-1+e^{U}}$$



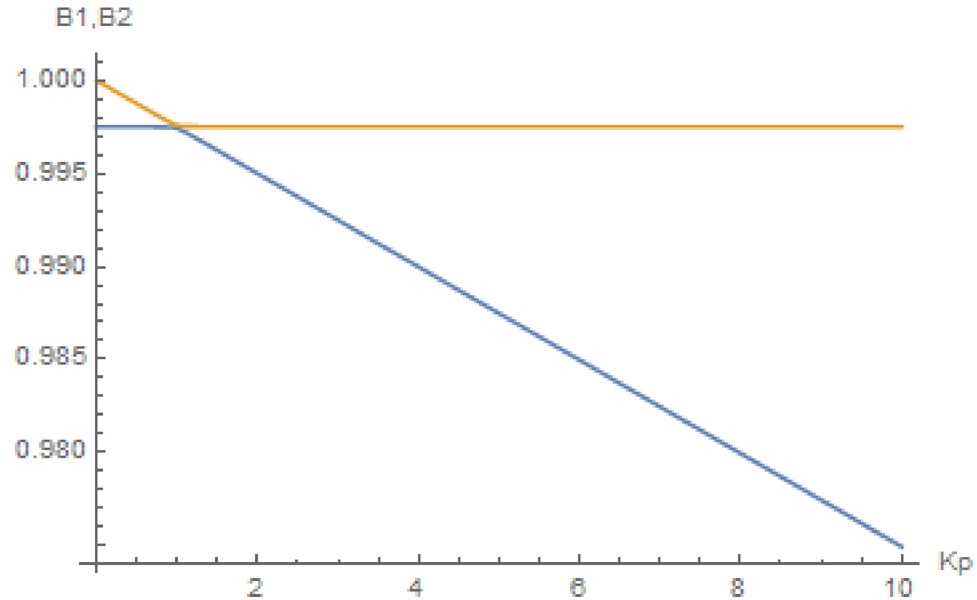
B1(U,Kp)



B2(U,Kp)

PI Control & Ki=a.Kp
Closed Loop Gain Function

Numerical Example:
U=0.0025



Some useful properties:

$\mathrm{B1} + \mathrm{B2} = b$

B2 - B1 = $\sqrt{b^2 - 4B}$

$\mathrm{B1} \times \mathrm{B2}$ = B

1-TERM LEARNING; Ki=a.Kp

L(z)=v: use only information from the present time step within a trial

$$\mathrm{T} = 1 - \frac{\mathrm{A}v\mathrm{e}^{\mathrm{i}\theta}\left(-1 + \mathrm{e}^{\mathrm{i}\theta}\right)}{(-\mathrm{B1} + \mathrm{e}^{\mathrm{i}\theta})(-\mathrm{B2} + \mathrm{e}^{\mathrm{i}\theta})}$$

T and R=|T| is large at phases θ=0,π
dR/dΘ=0 at those phases

There are 4 cases depending on the signs (±) of B1 and B2.
We impose the restriction: $0 < \mathrm{B2} < 1 \;\&\; 0 < \mathrm{B1} < 1$

Leading to:

$$0 < \mathrm{A}v < 1 + \mathrm{B1} + \mathrm{B2} - 3\mathrm{B1B2}$$

$$\mathrm{A}v < 1 + b - 3B$$

$$0 < A < 1 \;\&\; 0 < U < 1 \;\&\; \mathrm{Kp} > 0 \;\&\; 0 < v < 2(1 + \mathrm{Kp}) - \mathrm{Kp}\,\mathrm{U}$$

$$0 < \mathrm{Kp} < \frac{1}{-1 + \mathrm{e}^{U}}$$

1-TERM LEARNING; Ki=a.Kp

We impose the restriction:

$$(\mathrm{B1} < 0\ \&\ \mathrm{B2} > 0)\ \text{or}\ (\mathrm{B1} > 0\ \&\ \mathrm{B2} < 0)$$

Leading to:

$$\mathrm{A}v < (1 + \mathrm{B1})(1 + \mathrm{B2})$$

$$\mathrm{A}v < 1 + b + B$$

$$v < 4/\mathrm{A} - \ (2\text{+Kp }(2\text{+U}))$$

2-TERM, LOOK BACK 1 STEP; Ki= a.Kp

L = v(1+1/z); causal = use only information from present and previous time step

$$1-\frac{\mathrm{A}v\mathrm{e}^{\mathrm{i}\theta}\left(1+\mathrm{e}^{-\mathrm{i}\theta}\right)\left(-1+\mathrm{e}^{\mathrm{i}\theta}\right)}{\left(-\mathrm{B1}+\mathrm{e}^{\mathrm{i}\theta}\right)\left(-\mathrm{B2}+\mathrm{e}^{\mathrm{i}\theta}\right)}$$

R=|T|=1 at phases θ=0,π

dR/dΘ=0 at those values

$$-1<\mathrm{B2}<1\ \&\ -1<\mathrm{B1}<1\ \&\ 0<\mathrm{A}v<1-\mathrm{B1B2}$$

$$0<\mathrm{A}v<1-B$$

$$v<(1+\mathrm{Kp})\qquad -1<\mathrm{Kp}<\frac{1}{-1+\mathrm{e}^{U}}$$

2-TERM, LOOK AHEAD 1 STEP; Ki=a.Kp

L = v(1+z); non-causal = use information from present and next time step

$$T=1-\frac{Av e^{i\theta}(-1+e^{i\theta})(1+e^{i\theta})}{(-B1+e^{i\theta})(-B2+e^{i\theta})}$$

R=|T|=1 at phases θ=0,π
dR/dΘ=0 at those values

There are 4 cases depending on the signs (±) of B1 and B2.
We impose the restriction: $0 < \mathrm{B1} < 1 \;\&\; 0 < \mathrm{B2} < 1$

$$0 < \mathrm{A}v < \mathrm{B1} + \mathrm{B2} - 2\mathrm{B1B2}$$

$$0 < \mathrm{A}v < b - 2B$$

$$0 < U \leq 1 \;\&\; 0 < A < 1 \;\&\; 0 < v < 1 + \mathrm{Kp} - \mathrm{Kp}U$$

$$0 < \mathrm{Kp} < \frac{1}{-1 + e^{U}}$$

So, the iterations are "monotonic stable" under those conditions

2-TERM, LOOK AHEAD 1 STEP; Ki=a.Kp

We impose the restriction: $(-1 < \mathrm{B1} < 0 \;\&\; -1 < \mathrm{B2} < 0)$

$$0 < \mathrm{A}v < \mathrm{B1} + \mathrm{B2} + 2\mathrm{B1B2}$$

$$0 < \mathrm{A}v < b + 2B$$

$$v < -3 + \frac{4}{A} - \mathrm{Kp}(3 + U)$$

3-Term Time Symmetric; Ki=a.Kp

L = v(1+z+1/z); use information from present, previous and next steps within a trial

$$T = 1 - \frac{Av e^{i\theta}\left(-1 + e^{i\theta}\right)\left(1 + e^{-i\theta} + e^{i\theta}\right)}{\left(-B1 + e^{i\theta}\right)\left(-B2 + e^{i\theta}\right)}$$

T is large in four directions, including the directions θ=0,π.

We consider θ=0, π, π/2 leading to the contradiction: Av>0 & Av<0

The iterations are definitely NOT “monotonic stable”

3-Term, “Look Back 2 steps”; Ki=a.Kp

L = v(1+1/z+1/z²); causal = use information from present and 2 previous time steps

$$\mathrm{T} = 1 - \frac{\mathrm{A}v\mathrm{e}^{\mathrm{i}\theta}\left(-1 + \mathrm{e}^{\mathrm{i}\theta}\right)\left(1 + \mathrm{e}^{-\mathrm{i}\theta} + \mathrm{e}^{-2\mathrm{i}\theta}\right)}{\left(-\mathrm{B1} + \mathrm{e}^{\mathrm{i}\theta}\right)\left(-\mathrm{B2} + \mathrm{e}^{\mathrm{i}\theta}\right)}$$

R=|T|=1 when Θ =0 or 2π/3.
dR/dΘ =0 when Θ =0 or π.

The condition for dR/dΘ =0 when Θ = 2π/3 is the split $\mathrm{B2} = -\frac{\mathrm{B1}}{1 + \mathrm{B1}}$

Introducing this leads to the sufficient condition $\mathrm{A}v \leq 1$

For the particular relation between B1 and B2, the split implies $\mathrm{B} = -\mathrm{b} < -1$

$$\mathrm{Kp} \rightarrow \frac{2 + \mathrm{e}^{U}}{\left(-1 + \mathrm{e}^{U}\right)(2 + U)}$$

This is not suitable for operations!
Integral control renders this form of ILC unstable

3-Term "Look Ahead 2 steps"; Ki=a.Kp

L = v(1+z+z²); non-causal = use information from present and next two time steps

$$T = 1 - \frac{Av e^{i\theta}\left(-1 + e^{3i\theta}\right)}{(-B1 + e^{i\theta})(-B2 + e^{i\theta})}$$

R=|T|=1 when Θ= 0 or 2π/3.
dR/dΘ=0 when Θ= 0 or π.

dR/dΘ NOT =0 at Θ= 2π/3 unless B1×B2=1 Introducing, this condition

There are 2 cases depending on the signs (±) of B1 and B2.
We impose the restriction: $0 < B1 < 1 \; \& \; 0 < B2 < 1$

$$\left\{Av < -2 + \frac{1}{B1} + B1\right\} \& \; 0 < B1 < 1$$

$$Av < -2 + b$$

$$v < -1 - Kp(1 + U)$$

For our particular choice of B1,B2, the condition B1×B2=1 implies Kp= − 1
Hence *v*<U

This is not suitable for operations! Integral control renders this form of ILC unstable

3-Term "Look Ahead 2 steps"; Ki=a.Kp

We impose the restriction: $-1 < \mathrm{B1} < 0 \,\&\, -1 < \mathrm{B2} < 0$

$$\left\{\mathrm{A}v < 2 + \mathrm{B1} + \frac{1}{\mathrm{B1}}\right\} \,\&\, -1 < \mathrm{B1} < 0$$

$$\mathrm{A}v < 2 + b$$

$$v < 4/A - 1 - \mathrm{Kp}(1 + U)$$

B1×B2=1 implies Kp= $-$ 1. Hence $v < \frac{4}{A} + U$

This is not suitable for operations!
Integral control renders this form of ILC unstable

Caveats

The Z-domain stability criterion is for "monotonic convergence of the norm of the error vector $y^T y$" (MCNV).

There are other types of convergence, and these are not necessarily excluded by the conditions above.

The two types of eigen-value test are not necessarily identical with MCNV

1. Geometric convergence of the norm of the eigenvectors under iteration
2. Geometric convergence of the eigenvectors under iteration.

Fixed Points

Note, we have not given the fixed points (solution vectors) for ILC iterations of the 2-pole system. It is expected that they may indicate suitable values for B1 & B2 that do not lead to excessive gain values.

This will be a future work.