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Linearized Hamiltonian and F-Matrix for an RFQ: A Step-by-Step Approach

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Abstract: This report details a step-by-step procedure to manually linearize the Hamiltonian for a Radiofrequency Quadrupole (RFQ) linear accelerator, largely following that outlined in Reference [2]. In the present case it is applied to a 2-term RFQ potential expansion. The F-Matrix, essential to TRANSOPTR implementation, is then presented.

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The generalized longitudinal momentum for a charged particle within an electric field with scalar potential Π may be written:

$$P = \sqrt{\left(\frac{E - q\Pi}{c}\right)^2 - m^2c^2} \tag{1}$$

In the case of an RFQ, we note the two term scalar potential [1]:

$$\Pi(x, y, s, t) = \left[A_{01}\frac{U}{2}(x^2 - y^2) + A_{10}\frac{U}{2}\cos(kz)I_0(k\sqrt{x^2 + y^2})\right]\sin(\omega t + \theta) \tag{2}$$

the parameters A_{01} and A_{10} define RFQ quadrupole focussing and acceleration, respectively, with given 4-rod voltage U, RFQ design wavenumber k, aperture a and modulation factor m:

$$A_{01} = \frac{1}{a^2}\frac{I_0(ka) + I_0(mka)}{m^2I_0(ka) + I_0(mka)} \tag{3}$$

$$A_{10} = \frac{m^2 - 1}{m^2I_0(ka) + I_0(mka)} \tag{4}$$

that is, the two term potential expansion linearly breaks the potential into a transverse focussing term, defined by a hyperbolic term $(x^2 - y^2)$ and a longitudinally oscillating accelerating term dependant upon the modified Bessel function of the first kind. Considering the smallness of the transverse dimensions of an RFQ compared to the length of its longitudinal axis - typically apertures are on the order of 1cm while the length is several meters - it is clear that the potential will be dominated by the longitudinal accelerating term. We further note that the transverse focussing term has no explicit s dependance. We now evaluate the energy-potential term in Equation 1, in the case of a two term RFQ potential:

$$\left(\frac{E - q\Pi(x, y, s, t)}{c}\right)^2 \tag{5}$$

To keep things tidy, we first consider breaking Π into a purely longitudinal component $\phi(s)$, and one which depends on the variables x^2 and y^2, which will be refered to as the transverse component $T(x, y, s)$. This is done by expanding the Bessel function to second order:

$$I_0(kr) = 1 + \frac{k^2}{4}(x^2 + y^2) + \mathcal{O}(r^4) \tag{6}$$

For conciseness, we denote $\mathrm{S} = \sin(\omega \mathrm{t}_0 + \theta)$, $\mathrm{C} = \cos(\omega \mathrm{t}_0 + \theta)$. The time-independent two term potential is expressed as:

$$\Pi(x,y,s,t) \rightarrow \phi(s)S + T(x,y,s)S = \frac{A_{10}U}{2} cos(ks)S + \left[\frac{A_{01}U}{2}(x^2 - y^2) + \frac{A_{10}Uk^2\cos(ks)}{8}(x^2 + y^2)\right]S \tag{7}$$

The energy-potential term becomes:

$$\left(\frac{E - q\Pi}{c}\right)^2 \rightarrow \left(\frac{E - q\phi S - qTS}{c}\right)^2 = \left(\frac{E - q\phi S}{c}\right)^2 - 2qTS\left(\frac{E - q\phi S}{c}\right) + \mathcal{O}(x^4, y^4) \tag{8}$$

The expansion produces a fourth order term in (x,y), which is small and therefore neglected. Notice the emergence of a longitudinal energy-potential like term involving ϕS. We wish to expand time and energy around the reference particle: $t \rightarrow t_0 + \Delta t$, $E \rightarrow E_0 + \Delta E$. The time-expansion of the sine term S produces:

$$\sin(\omega t_0 + \omega \Delta t + \theta) = \sin(\omega t_0 + \theta)\cos(\omega \Delta t) + \cos(\omega t_0 + \theta)\sin(\omega \Delta t) \tag{9}$$

$$\sin(\omega t_0 + \omega \Delta t + \theta) = S\left(1 - \frac{\omega \Delta t^2}{2}\right) + C\omega \Delta t \tag{10}$$

where $\omega \Delta t \ll 1$. The squared numerator of the energy-potential term of Equation 8 expands to:

$$\left(E - q\phi S\right)^2 \rightarrow \left(E_0 + \Delta E - q\phi S + \frac{1}{2}q\phi(\omega\Delta t)^2 S - q\phi\omega\Delta t C\right)^2 \tag{11}$$

As for the Transverse term T of Equation 8, we note that only a factor $\sin(\omega t_0 + \theta)$ survives to second order - all other terms in Equation 10 involving $\omega \Delta t$ multiply an x^2 or y^2, beyond second order, and therefore negligible. Altogether, the second-order expansion of the energy-potential term of Equation 5, using a two-term RFQ potential $\Pi(x,y,s,t)$, is expressed as:

$$\begin{aligned}\left(E - q\Pi\right)^2 \rightarrow \left(E_0 - q\phi S\right)^2 - 2qTS\left(E_0 + \Delta E - q\phi S\right) \\ + \left(E_0 - q\phi S\right)\left(2\Delta \mathrm{E} + (\Delta \mathrm{t})^2 q\omega^2\phi S - 2(\omega\Delta \mathrm{t})q\phi C\right) \\ + \frac{1}{4}\left(2\Delta \mathrm{E} + (\omega\Delta \mathrm{t})^2 q\phi S - 2(\omega\Delta \mathrm{t})q\omega\phi C\right)^2\end{aligned} \tag{12}$$

Note the first term of Equation 12 - it corresponds to the energy-potential term for the reference particle, centered at (t_0,E_0). The above expansion is then put into the generalized longitudinal momentum term from Equation 1, causing it to grow considerably. We may extract the reference momentum from the expanded momentum term, ensuring that every remaining factor within the square root is divided by P^2. We now have an expression for the momentum, expanded about the reference particle in (t,$-E$), which can be taylor expanded to second order:

$$P = P_0\sqrt{1+x} \approx P_0\left(1 + \frac{x}{2} - \frac{x^2}{8}\right) \tag{13}$$

and

$$P_0 = \sqrt{\left(\frac{E_0 - q\phi S}{c}\right)^2 - m^2c^2} \tag{14}$$

where in the above, x represents every term in Equation 12, except the final term in (t_0,E_0). The expansion produces several dozen terms. Only terms up to order $(\Delta t)^2$, $(\Delta E)^2$ and $\Delta t\Delta E$ are kept. The general s-based (longitudinal) Hamiltonian for a charged particle in an electric field described by a scalar potential Π, and in absence of vector potentials A_i may be written:

$$H(x, P_x, y, P_y, t, E) = -\sqrt{\left(\frac{E_0 - q\phi S}{c}\right)^2 - m^2c^2 - 2qTS\left(\frac{E_0 - q\phi S}{c^2}\right) - P_x^2 - P_y^2} \tag{15}$$

Where we find the transverse component T. The Hamiltonian is linearized by expanding to first order:

$$H = -P_0\sqrt{1 - \frac{2qTS}{P_0^2}\left(\frac{E_0 - q\phi S}{c^2}\right) - \frac{P_x^2}{P^2} - \frac{P_y^2}{P^2}} \approx -P_0 + \frac{2qTS}{P_0^2}\left(\frac{E_0 - q\phi S}{c^2}\right) + \frac{P_x^2 + P_y^2}{2P_0} \tag{16}$$

here P_0 corresponds to the time and energy expanded general momentum from Equation 13 following the expansion of Equation 12. From here on, we drop the subscript on P_0 - any division by P is assumed to represent the momentum of Equation 14.

A canonical transformation is performed from (t,$-E$) to (Δt, $-\Delta E$) using the generating function [2]:

$$G = -\left(t - \int \frac{ds}{\beta(s)c}\right)(\Delta E + E_0) \tag{17}$$

which produces an added term to the Hamiltonian:

$$\frac{\partial G}{\partial s} = \frac{\Delta E + E_0(s)}{\beta(s)c} - \Delta t E_0'(s) \tag{18}$$

the term $-\Delta t E_0'(s)$ may be found via one of Hamilton's equations:

$$\frac{dE}{ds} = \frac{\partial H}{\partial t} \tag{19}$$

in this case, we evaluate the partial derivative of the non-expanded Hamiltonian from Equation 15. This produces the term:

$$-\Delta t E_0'(s) = \frac{q\phi\omega\Delta t\Big(E_0 - q\phi S\Big)C}{c^2 P} \tag{20}$$

since the derivative is evaluated at the reference particle, any spatial transverse coordinate is set to zero. Effectively in Equation 20, ϕ corresponds to the longitudinal term from Equation 7. Combining this with the Hamiltonian on the right hand side of Equation 16, we obtain to second order:

$$\begin{aligned} H_{\Delta t} = & \\ & \left(-P_0 + \frac{E_0}{\beta \mathrm{c}}\right) + \frac{P_x^2 + P_y^2}{2P} + \frac{qT(E - q\phi S)}{c^2 P} + \Delta \mathrm{E}\left(\frac{1}{\beta \mathrm{c}} - \frac{E_0}{c^2 P} + \frac{q\phi S}{c^2 P}\right) \\ & + \Delta \mathrm{E}^2\left(\frac{E_0^2}{2c^4 P^3} + \frac{q^2\phi^2 S^2}{2c^4 P^3} - \frac{E_0 q\phi S}{c^4 P^3} - \frac{1}{2c^2 P}\right) \\ & + \Delta \mathrm{E}\Delta \mathrm{t}\left(-\frac{q^3\omega\phi^3 S^2 C}{c^4 P^3} + \frac{2E_0 q^2\omega\phi^2 SC}{c^4 P^3} - \frac{E_0^2 q\omega\phi C}{c^4 P^3} + \frac{q\omega\phi C}{c^2 P}\right) \\ & + \Delta \mathrm{t}^2\left(\frac{q^4\omega^2\phi^4 S^2 C^2}{2c^4 P^3} - \frac{E_0 q^3\omega^2\phi^3 SC^2}{c^4 P^3} + \frac{E_0^2 q^2\omega^2\phi^2 C^2}{2c^4 P^3} + \frac{q^2\omega^2\phi^2 S^2}{2c^2 P} - \frac{q^2\omega^2\phi^2 C^2}{2c^2 P} - \frac{E_0 q\omega^2\phi S}{2c^2 P}\right) \end{aligned} \tag{21}$$

Inspection of the ΔE term reveals it is equal to zero. Further simplification is achieved keeping in mind that:

$$\frac{1}{\beta c} = \frac{(E - q\Phi)}{c^2 P} \tag{22}$$

producing:

$$H_{\Delta t} = \left(\frac{E_0}{\beta \mathrm{c}} - P_0\right) + \frac{P_x^2 + P_y^2}{2P} + \frac{qT}{\beta c} - \Delta\mathrm{E}\Delta\mathrm{t}\frac{q\omega\phi\cos\left(\omega t_0 + \theta\right)}{\beta^3\gamma^3 mc^3} + \frac{(\Delta\mathrm{E})^2}{2\beta^3\gamma^3 mc^3}$$
$$+ \omega^2(\Delta\mathrm{t})^2\left(\frac{q^2\phi^2\cos^2\left(\omega t_0 + \theta\right)}{2\beta^3\gamma^3 mc^3} - \frac{q\phi\sin\left(\omega t_0 + \theta\right)}{2\beta \mathrm{c}}\right) \quad (23)$$

We now transform the Hamiltonian to (z, P_z), variables representing small longitudinal displacements in the longitudinal direction, using the generating function:

$$G = -\beta c \Delta t P_z \quad (24)$$

producing the Hamiltonian-added term:

$$\frac{\partial G}{\partial s} = \frac{\beta'}{\beta} z P_z \quad (25)$$

giving:

$$H_z = \left(\frac{E_0}{\beta \mathrm{c}} - P_0\right) + \frac{P_x^2 + P_y^2}{2P} + \frac{qTS}{\beta c} + zP_z\left(\frac{q\omega\phi C}{\beta^3\gamma^3 mc^3} + \frac{\beta'}{\beta}\right) + \frac{P_z^2}{2\gamma^2 P} + \omega^2\left(\frac{z}{\beta c}\right)^2\left(\frac{q^2\phi^2 C^2}{2\beta^3 c^3\gamma^3 m} - \frac{q\phi S}{2\beta \mathrm{c}}\right) \quad (26)$$

The term $\frac{\beta'}{\beta}$ may further be expanded in terms of ϕ, using the following definitions:

$$P = mc\beta\gamma \quad (27)$$

$$\frac{dP}{ds} = -\frac{\partial H}{\partial s} \quad (28)$$

$$\gamma mc^2 = E_0 - q\phi \quad (29)$$

and since E_0 is constant, taking the derivative of Equation 29 with respect to s:

$$\gamma' = -\frac{q\phi'}{mc^2} \tag{30}$$

We can also take the derivative of the momentum with respect to s:

$$\frac{\partial P}{\partial s} = \frac{\partial}{\partial s}(\beta\gamma mc) = \beta'\gamma mc + \beta\gamma' mc \tag{31}$$

Using Equation 30, the above can be rewritten as:

$$\frac{\partial P}{\partial s} = \beta'\gamma mc - \frac{\beta q\phi'}{c} \tag{32}$$

$$\frac{\beta'}{\beta} = \frac{\partial P}{\partial s}\frac{1}{\beta\gamma mc} + \frac{q\phi'}{\gamma mc^2} \tag{33}$$

Isolating β' we now have:

$$\beta' = \frac{\partial P}{\partial s}\frac{1}{\gamma mc} + \frac{\beta q\phi'}{\gamma mc^2} \tag{34}$$

from which we obtain, after substituting in ϕ':

$$\frac{\beta'}{\beta} = \frac{qkUA_{10}\sin(ks)S}{2\beta^2\gamma^3 mc^2} \tag{35}$$

We may now express the linearized Hamiltonian, expanded to second order in z, P_z:

$$\begin{aligned} H_z = \left(\frac{E_0}{\beta \mathrm{c}} - P_0\right) + \frac{P_x^2 + P_y^2}{2P} + qU\frac{4A_{01}(x^2 - y^2) + A_{10}k^2\cos(ks)(x^2 + y^2)}{8\beta c}S \\ + \frac{qUA_{10}zP_z}{2\beta^2\gamma^3 mc^2}\left(\frac{\omega}{\beta c}\cos(ks)C + k\sin(ks)S\right) + \frac{P_z^2}{2\gamma^2 P} \\ + \omega^2 z^2\left(-\frac{qUA_{10}\cos(ks)S}{4\beta^3 c^3} + \frac{q^2U^2A_{10}^2\cos(ks)^2C^2}{8\beta^5\gamma^3 mc^5}\right) \end{aligned} \tag{36}$$

The corresponding F-Matrix [2] for the RFQ is:

$$\mathcal{F} = \begin{pmatrix} 0 & \frac{1}{P} & 0 & 0 & 0 & 0 \\ \mathcal{T}_1(s) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{P} & 0 & 0 \\ 0 & 0 & \mathcal{T}_2(s) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{B}(s) & \frac{1}{\gamma^2 P} \\ 0 & 0 & 0 & 0 & \mathcal{C}(s) & -\mathcal{B}(s) \end{pmatrix}$$

The functions $\mathcal{T}_1(s)$, $\mathcal{T}_2(s)$ (transverse), $\mathcal{B}(s)$ and $\mathcal{C}(s)$ (longitudinal) are:

$$\mathcal{T}_1(s) = -\frac{qU}{4\beta c}\left(4A_{01} + A_{10}k^2\cos(ks)\right)\sin(\omega t_0 + \theta) \tag{37}$$

$$\mathcal{T}_2(s) = \frac{qU}{4\beta c}\left(4A_{01} - A_{10}k^2\cos(ks)\right)\sin(\omega t_0 + \theta) \tag{38}$$

$$\mathcal{B}(s) = \frac{qUA_{10}}{2\beta^2\gamma^3 mc^2}\left(\frac{\omega}{\beta c}\cos(ks)\cos(\omega t_0 + \theta) + k\sin(ks)\sin(\omega t_0 + \theta)\right) \tag{39}$$

$$\mathcal{C}(s) = \frac{A_{10}\omega^2\cos(ks)}{4\beta^5\gamma^3 mc^5}\left(2qU\beta^2\gamma^3 mc^2\sin(\omega t_0 + \theta) - q^2U^2A_{10}\cos(ks)\cos(\omega t_0 + \theta)^2\right) \tag{40}$$

References

[1] T.P. Wangler. *RF Linear Accelerators, 2nd Edition.* Wiley-VCH, 2008.

[2] R. Baartman. Linac Envelope Optics. Technical Report TRI-BN-15-03, TRIUMF, 2015.