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# Magnetic Field Representation in Cylindrical Coordinate System

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**Abstract:** The magnetic field representation is derived in the cylindrical coordinate system up to the order of $z^4$.

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# 1 B-field Representation

The Ampere's Law (i.e. the last of Maxwell's Equations) is given in equation:

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} \tag{1}$$

where $\vec{H}$ denotes the magnetic field (it's related to the magnetic flux density $\vec{B}$ by $\vec{B} = \mu\,\vec{H}$, where $\mu$ is the permeability of the medium (material)), $\vec{D}$ denotes the electric flux density (it's related to the electric field $\vec{E}$ by $\vec{D} = \epsilon\,\vec{E}$, where $\epsilon$ is the permittivity of the medium (material)), and $\vec{J}$ denotes the electric current density.

Here we only consider the case of magnetostatic field and electrostatic field. In the vacuum, for instance inside the vacuum pipe, where there is no electric current lead, we have

$$\nabla \times \vec{B} = 0 \tag{2}$$

This means that the $\vec{B}$ can be expressed as a gradient of a scalar potential $\Phi$:

$$\vec{B} = -\nabla\Phi \tag{3}$$

where the negative sign is just a convention. In cylindrical coordinate system, $\Phi = \Phi(r, \theta, z)$ can be Taylor-expanded with respect to the plane where $z = 0$:

$$\Phi = \Phi_0 + z\,\Phi_1 + z^2\,\Phi_2 + z^3\,\Phi_3 + z^4\,\Phi_4 + +z^5\,\Phi_5 + ... \tag{4}$$

where

$$\Phi_n = \left[\frac{1}{n!}\,\frac{\partial^n\Phi}{\partial z^n}\right]_{z=0} \qquad (n = 0, 1, 2, 3, ...)$$

is independent of $z$.

In cylindrical coordinate frame, the $\vec{B}$ is expressed as

$$\vec{B} \equiv\; B_r(r,\theta,z)\,\hat{r}^o + B_\theta(r,\theta,z)\,\hat{\theta}^o + B_z(r,\theta,z)\,\hat{z}^o \tag{5}$$

Meanwhile,

$$\vec{B} = -\nabla\Phi = -\frac{\partial\Phi}{\partial r}\,\hat{r}^o - \frac{\partial\Phi}{r\partial\theta}\,\hat{\theta}^o - \frac{\partial\Phi}{\partial z}\,\hat{z}^o \tag{6}$$

So we have

$$B_z(r,\theta,z) = -\frac{\partial\Phi}{\partial z} \tag{7}$$

$$B_r(r,\theta,z) = -\frac{\partial\Phi}{\partial r} \tag{8}$$

$$B_\theta(r,\theta,z) = -\frac{\partial\Phi}{r\partial\theta} \tag{9}$$

## 1.1 $B_z$ component

Inserting Eq.(4) into Eq.(7) gives

$$B_z(r,\theta,z) = -(\Phi_1 + 2z\Phi_2 + 3z^2\Phi_3 + 4z^3\Phi_4 + 5z^4\Phi_5 + ...)$$

$$= -\left[\left(\frac{\partial\Phi}{\partial z}\right)_{z=0} + z\left(\frac{\partial^2\Phi}{\partial z^2}\right)_{z=0} + \frac{z^2}{2}\left(\frac{\partial^3\Phi}{\partial z^3}\right)_{z=0} + \frac{z^3}{6}\left(\frac{\partial^4\Phi}{\partial z^4}\right)_{z=0} + \frac{z^4}{24}\left(\frac{\partial^5\Phi}{\partial z^5}\right)_{z=0} + ...\right]$$

That is

$$B_z(r,\theta,z) = -\left(\frac{\partial\Phi}{\partial z}\right)_{z=0} + z\left[\frac{\partial}{\partial z}\left(-\frac{\partial\Phi}{\partial z}\right)\right]_{z=0} + \frac{z^2}{2}\left[\frac{\partial}{\partial z}\left(-\frac{\partial^2\Phi}{\partial z^2}\right)\right]_{z=0}$$

$$+\frac{z^3}{6}\left[\frac{\partial^2}{\partial z^2}\left(-\frac{\partial^2\Phi}{\partial z^2}\right)\right]_{z=0} + \frac{z^4}{24}\left[\frac{\partial^3}{\partial z^3}\left(-\frac{\partial^2\Phi}{\partial z^2}\right)\right]_{z=0} + ... \tag{10}$$

Remember Gauss's law for magnetism: $\nabla\cdot\vec{B} = 0$. Inserting Eq.(3) gives

$$\nabla\cdot(-\nabla\Phi) = -\nabla^2\Phi = 0$$

Under the cylindrical coordinate system, it is

$$\left(\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}\right) + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{11}$$

Denote

$$\Gamma^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \tag{12}$$

Then the Eq.(11) reads

$$\Gamma^2\Phi = -\frac{\partial^2\Phi}{\partial z^2} \tag{13}$$

Utilizing Eq.(7) and Eq.(13), we get

$$\frac{\partial}{\partial z}\left(-\frac{\partial\Phi}{\partial z}\right) = \frac{\partial B_z(r,\theta,z)}{\partial z} \tag{14}$$

$$\frac{\partial}{\partial z}\left(-\frac{\partial^2\Phi}{\partial z^2}\right) = \frac{\partial}{\partial z}\left(\Gamma^2\Phi\right) = \Gamma^2\left(\frac{\partial\Phi}{\partial z}\right) = -\Gamma^2 B_z(r,\theta,z) \tag{15}$$

$$\frac{\partial^2}{\partial z^2}\left(-\frac{\partial^2\Phi}{\partial z^2}\right) = \frac{\partial^2}{\partial z^2}\left(\Gamma^2\Phi\right) = \Gamma^2\left[\frac{\partial}{\partial z}\left(\frac{\partial\Phi}{\partial z}\right)\right] = -\Gamma^2\left[\frac{\partial B_z(r,\theta,z)}{\partial z}\right] \tag{16}$$

and

$$\frac{\partial^3}{\partial z^3}\left(-\frac{\partial^2\Phi}{\partial z^2}\right) = \frac{\partial^3}{\partial z^3}\left(\Gamma^2\Phi\right) = \Gamma^2\left[\frac{\partial}{\partial z}\left(\frac{\partial^2\Phi}{\partial z^2}\right)\right] = \Gamma^2\left[\frac{\partial}{\partial z}\left(-\Gamma^2\Phi\right)\right] = \Gamma^2\left[\Gamma^2 B_z(r,\theta,z)\right] \tag{17}$$

where taking partial derivatives in $r$ and $\theta$ are independent of taking partial derivative in $z$, thus they are commutative.

Further, we write

$$\begin{aligned}
\Gamma^4 \equiv \Gamma^2\Gamma^2 &= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) \\
&= \frac{\partial^4}{\partial r^4} + \frac{\partial^2}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial r^2}\left(\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) \\
&+ \frac{1}{r}\frac{\partial^3}{\partial r^3} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) \\
&+ \frac{1}{r^2}\frac{\partial^4}{\partial r^2 \partial \theta^2} + \frac{1}{r^3}\frac{\partial^3}{\partial r \partial \theta^2} + \frac{1}{r^4}\frac{\partial^4}{\partial \theta^4}
\end{aligned} \tag{18}$$

where

$$\frac{\partial^2}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}\right) = \frac{2}{r^3}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial^3}{\partial r^3} \ , \tag{19$a$}$$

$$\frac{\partial^2}{\partial r^2}\left(\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) = \frac{6}{r^4}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2}\frac{\partial^4}{\partial^2 r \partial \theta^2} \ , \tag{19$b$}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right) = -\frac{1}{r^3}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial r^2} \ , \tag{20$a$}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) = -\frac{2}{r^4}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^3}\frac{\partial^3}{\partial r \partial \theta^2} \tag{20$b$}$$

Inserting Eqs.(19)-(20) into Eq.(18) and merging the similar terms, we get

$$\Gamma^4 = \frac{\partial^4}{\partial r^4} + \frac{1}{r^4}\frac{\partial^4}{\partial \theta^4} + \frac{1}{r^2}\frac{\partial^2}{\partial r^2} + \frac{1}{r^3}\frac{\partial}{\partial r} + \frac{4}{r^4}\frac{\partial^2}{\partial \theta^2} + \frac{2}{r^3}\frac{\partial^3}{\partial r \partial \theta^2} + \frac{2}{r^2}\frac{\partial^4}{\partial r^2 \partial \theta^2} + \frac{2}{r}\frac{\partial^3}{\partial r^3}$$

Substituting Eqs.(14)-(17) into Eq.(10) gives

$$\begin{aligned}
B_z(r,\theta,z) &= B_z(r,\theta,z=0) + z\left[\frac{\partial B_z(r,\theta,z)}{\partial z}\right]_{z=0} - \frac{z^2}{2}\left[\Gamma^2 B_z(r,\theta,z)\right]_{z=0} \\
&- \frac{z^3}{6}\Gamma^2\left[\frac{\partial B_z(r,\theta,z)}{\partial z}\right]_{z=0} + \frac{z^4}{24}\left[\Gamma^4 B_z(r,\theta,z)\right]_{z=0} + ...
\end{aligned} \tag{21}$$

Note that we only consider the terms up to the order of $z^4$, and **all the partial derivatives are taken in the geometrical median plane where $z = 0$ (the same in the following for the $B_r$ and $B_\theta$ components)**.

## 1.2 $B_r$ component

Inserting Eq.(4) into Eq.(8) gives

$$B_r(r,\theta,z) = -\frac{\partial\Phi_0}{\partial r} - z\frac{\partial\Phi_1}{\partial r} - z^2\frac{\partial\Phi_2}{\partial r} - z^3\frac{\partial\Phi_3}{\partial r} - z^4\frac{\partial\Phi_4}{\partial r} + ...$$

$$= -\left(\frac{\partial\Phi}{\partial r}\right)_{z=0} - z\left[\frac{\partial}{\partial r}\left(\frac{\partial\Phi}{\partial z}\right)\right]_{z=0} - z^2\left[\frac{\partial}{\partial r}\left(\frac{1}{2}\frac{\partial^2\Phi}{\partial z^2}\right)\right]_{z=0}$$

$$-z^3\left[\frac{\partial}{\partial r}\left(\frac{1}{6}\frac{\partial^3\Phi}{\partial z^3}\right)\right]_{z=0} - z^4\left[\frac{\partial}{\partial r}\left(\frac{1}{24}\frac{\partial^4\Phi}{\partial z^4}\right)\right]_{z=0} + ... \tag{22}$$

Utilizing Eq.(8), we have

$$\left(\frac{\partial\Phi}{\partial r}\right)_{z=0} = -B_r(r,\theta,z=0)\ , \tag{23}$$

Utilizing Eq.(7), we have

$$\left(\frac{\partial\Phi}{\partial z}\right)_{z=0} = -B_z(r,\theta,z=0)\ , \tag{24}$$

and

$$\left(\frac{\partial^2\Phi}{\partial z^2}\right)_{z=0} = \left[\frac{\partial}{\partial z}\left(\frac{\partial\Phi}{\partial z}\right)\right]_{z=0} = -\left[\frac{\partial B_z(r,\theta,z)}{\partial z}\right]_{z=0} \tag{25}$$

Utilizing Eq.(15) and Eq.(16), we have

$$\left(\frac{\partial^3\Phi}{\partial z^3}\right)_{z=0} = \Gamma^2 B_z(r,\theta,z=0) \tag{26}$$

and

$$\left(\frac{\partial^4\Phi}{\partial z^4}\right)_{z=0} = \Gamma^2\left[\frac{\partial B_z(r,\theta,z)}{\partial z}\right]_{z=0} \tag{27}$$

Substituting Eqs.(23)-(27) into Eq.(22), we arrive at

$$B_r(r,\theta,z) = B_r(r,\theta,z=0) + z\left[\frac{\partial B_z(r,\theta,z=0)}{\partial r}\right] + \frac{z^2}{2}\left\{\frac{\partial}{\partial r}\left[\frac{\partial B_z(r,\theta,z)}{\partial z}\right]_{z=0}\right\}$$

$$-\frac{z^3}{6}\left[\frac{\partial}{\partial r}\left(\Gamma^2 B_z(r,\theta,z=0)\right)\right] - \frac{z^4}{24}\left(\frac{\partial}{\partial r}\left\{\Gamma^2\left[\frac{\partial B_z(r,\theta,z)}{\partial z}\right]_{z=0}\right\}\right) + ... \tag{28}$$

Further, we write

$$\Gamma^3 \equiv \frac{\partial}{\partial r}\Gamma^2 = \frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right) = \frac{\partial^3}{\partial r^3} - \frac{1}{r^2}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial^2}{\partial r^2} - \frac{2}{r^3}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2}\frac{\partial^3}{\partial r\partial\theta^2} \tag{29}$$

We finally have

$$B_r(r,\theta,z) = B_r(r,\theta,z=0) + z\left[\frac{\partial B_z(r,\theta,z=0)}{\partial r}\right] + \frac{z^2}{2}\left\{\frac{\partial}{\partial r}\left[\frac{\partial B_z(r,\theta,z)}{\partial z}\right]_{z=0}\right\}$$

$$-\frac{z^3}{6}\left[\Gamma^3 B_z(r,\theta,z=0)\right] - \frac{z^4}{24}\left\{\Gamma^3\left[\frac{\partial B_z(r,\theta,z)}{\partial z}\right]_{z=0}\right\} + ... \tag{30}$$

## 1.3 $B_\theta$ component

Inserting Eq.(4) into Eq.(9) gives

$$B_\theta(r,\theta,z) = -\frac{1}{r}\left(\frac{\partial\Phi_0}{\partial\theta} + z\frac{\partial\Phi_1}{\partial\theta} + z^2\frac{\partial\Phi_2}{\partial\theta} + z^3\frac{\partial\Phi_3}{\partial\theta} + z^4\frac{\partial\Phi_4}{\partial\theta} + ...\right)$$

$$= -\frac{1}{r}\left(\frac{\partial\Phi}{\partial\theta}\right)_{z=0} - \frac{z}{r}\left[\frac{\partial}{\partial\theta}\left(\frac{\partial\Phi}{\partial z}\right)\right]_{z=0} - \frac{z^2}{r}\left[\frac{\partial}{\partial\theta}\left(\frac{1}{2}\frac{\partial^2\Phi}{\partial z^2}\right)\right]_{z=0}$$

$$-\frac{z^3}{r}\left[\frac{\partial}{\partial\theta}\left(\frac{1}{6}\frac{\partial^3\Phi}{\partial z^3}\right)\right]_{z=0} - \frac{z^4}{r}\left[\frac{\partial}{\partial\theta}\left(\frac{1}{24}\frac{\partial^4\Phi}{\partial z^4}\right)\right]_{z=0} + ... \tag{31}$$

Utilizing Eq.(9), we have

$$-\frac{1}{r}\left(\frac{\partial\Phi}{\partial\theta}\right)_{z=0} = B_\theta(r,\theta,z=0) \tag{32}$$

Inserting Eq.(32) and Eqs.(23)-(27) into Eq.(31), we get

$$B_\theta(r,\theta,z) = B_\theta(r,\theta,z=0) + \frac{z}{r}\left[\frac{\partial B_z(r,\theta,z=0)}{\partial\theta}\right] + \frac{z^2}{2r}\left\{\frac{\partial}{\partial\theta}\left[\frac{\partial B_z(r,\theta,z)}{\partial z}\right]_{z=0}\right\}$$

$$-\frac{z^3}{6r}\left[\frac{\partial}{\partial\theta}\left(\Gamma^2 B_z(r,\theta,z=0)\right)\right] - \frac{z^4}{24r}\left(\frac{\partial}{\partial\theta}\left\{\Gamma^2\left[\frac{\partial B_z(r,\theta,z)}{\partial z}\right]_{z=0}\right\}\right) + ... \tag{33}$$

Further, we write

$$\Lambda^3 \equiv \frac{\partial}{\partial\theta}\Gamma^2 = \frac{\partial}{\partial\theta}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right) = \frac{\partial^3}{\partial r^2\partial\theta} + \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial^3}{\partial\theta^3} \tag{34}$$

We eventually have

$$B_\theta(r,\theta,z) = B_\theta(r,\theta,z=0) + \frac{z}{r}\left[\frac{\partial B_z(r,\theta,z=0)}{\partial\theta}\right] + \frac{z^2}{2r}\left\{\frac{\partial}{\partial\theta}\left[\frac{\partial B_z(r,\theta,z)}{\partial z}\right]_{z=0}\right\}$$

$$-\frac{z^3}{6r}\left[\Lambda^3 B_z(r,\theta,z=0)\right] - \frac{z^4}{24r}\left\{\Lambda^3\left[\frac{\partial B_z(r,\theta,z)}{\partial z}\right]_{z=0}\right\} + ... \tag{35}$$

## 2 Discussion

If it's assumed that the B-field is perfectly symmetrical w.r.t. the geometrical median plane (GMP) where $z = 0$, then in the GMP we have

$$B_r(r,\theta,z=0) = 0\,, \quad B_\theta(r,\theta,z=0) = 0\,, \quad \text{and} \quad \left[\frac{\partial B_z(r,\theta,z)}{\partial z}\right]_{z=0} = 0$$

In this case, the above Eqs.(21),(30) and (35) become

$$B_z(r,\theta,z) = B_z(r,\theta,z=0) - \frac{z^2}{2}\left[\Gamma^2 B_z(r,\theta,z)\right]_{z=0} + \frac{z^4}{24}\left[\Gamma^4 B_z(r,\theta,z)\right]_{z=0} + \ldots \tag{36}$$

$$B_r(r,\theta,z) = z\left[\frac{\partial B_z(r,\theta,z=0)}{\partial r}\right] - \frac{z^3}{6}\left[\Gamma^3 B_z(r,\theta,z=0)\right] + \ldots \tag{37}$$

$$B_\theta(r,\theta,z) = \frac{z}{r}\left[\frac{\partial B_z(r,\theta,z=0)}{\partial \theta}\right] - \frac{z^3}{6r}\left[\Lambda^3 B_z(r,\theta,z=0)\right] + \ldots \tag{38}$$

Still, these are up to the order of $z^4$. Clearly, the $B_z(r,\theta,z)$ component is an even function of $z$; while the other two components are odd functions of $z$, meaning reversed directions. **Again, should be emphasized that all the partial derivates are taken in the GMP, using the median plane field map** $B_z(r,\theta,z=0)$. But, since $\Gamma^2$ and $\Gamma^4$ involve, respectively, second and fourth derivatives of the median plane field, these derivative functions (especially $\Gamma^4$ ) will always display much more noise than the original field data. However, according to Gordon [1], this noise usually looks more troublesome than it actually is, because the process of orbit integration acts like an averaging process in filtering out the high frequency components of the noise. Nevertheless, some caution is required.

In the design stage of a cyclotron, people often assume a perfect symmetry w.r.t. the GMP. On the contrary, if the B-field does have non-zero asymmetrical components ($B_r, B_\theta, \frac{\partial B_z}{\partial z}$) available in the GMP (for example in TRIUMF 500 MeV cyclotron), then one must explicitly give these components by taking a complete field survey.

In TRIUMF 500 MeV cyclotron, the B-field is expanded to the order of $z^2$ at maximum and applied in the computer codes `CYCLOPS` and `GOBLIN`. This is good enough because the beam size of ∼0.5 inch (2rms) vertically is much smaller than the magnet's pole gap of 20.8 inch. Even if the vertical oscillation of ±1.0 inch is taken into account for the equilibrium orbit throughout the entire energy range, still the magnetic field can hardly assume a strong nonlinearity within a vertical extent of ±1.5 inch from the GMP.

However, for a compact cyclotron like superconducting machine, the magnet pole gap (between hills) is relatively small, in the order of 5 cm. In this case, the B-field has to be expanded to a higher order of $z$ (e.g. $z^4$) to account for the strong non-linear effect [1].

# 3 Motion Equations

In the cylindrical coordinate system, the complete and general form of equations of motion of particle in the magnetic field is

$$\frac{dr}{d\theta} = r\,\frac{p_r}{p_\theta} \tag{39}$$

$$\frac{dp_r}{d\theta} = p_\theta + |q|\left(-rB_z + r\frac{p_z}{p_\theta}B_\theta\right) \tag{40}$$

$$\frac{dz}{d\theta} = r\,\frac{p_z}{p_\theta} \tag{41}$$

$$\frac{dp_z}{d\theta} = |q|\left(-r\frac{p_r}{p_\theta}B_\theta + rB_r\right) \tag{42}$$

$$p_\theta = \sqrt{p^2 - p_r^2 - p_z^2} \tag{43}$$

where

$$B_z = B_z(r,\theta,z), \quad B_r = B_r(r,\theta,z), \quad B_\theta = B_\theta(r,\theta,z),$$

$p$ is the total momentum of particle with charge $q$. **Should be pointed out that in these motion equations, there is NOT any approximation made, and every single component of the magnetic field $(B_r, B_\theta, B_z)$ has been properly reversed with sign to account for the fact that the B-field given must produce a centripetal instead of centrifugal force on the particle, no matter the particle is positively charged $(q > 0)$ or negatively charged $(q < 0)$.** Specifically, the $B_z$ component in the GMP must be given in an opposite direction of $z$-axis when the particle is positively charged; conversely, for negatively charged particle, $B_z$ component in the GMP must be given in the same direction as $z$-axis. Also, **should be noted that the order of $z$ in the motion equations is exactly identical to that in the B-field representation.**

Besides, keep in mind that the cylindrical coordinate $r$-$\theta$-$z$ is (must be) right-handed, and is, by definition, such that the $r$-axis is pointing radially outward, $\theta$-axis is in particle's travelling direction no matter it is positively or negatively charged, while $z$-axis is pointing upward (NOT downward).

# References


[1] M.M. GORDON, V. TAIVASSALO, *THE $Z^4$ ORBIT CODE AND THE FOCUSING BAR FIELDS USED IN BEAM EXTRACTION CALCULATIONS FOR SUPERCONDUCTING CYCLOTRONS* , Nuclear Instruments and Methods in Physics Research A247 (1986) 423-430.