

Poisson equation solver with symmetry conditions in the radial direction

Thomas Planche

August 3, 2012

Dear Eduard,

I would like to follow your advice and simulate the effect of neighboring turns by applying symmetry conditions in the radial direction. In this brief note, I describe one possible way to compute the self field taking into account this additional boundary condition. I followed the approach described in your dissertation (P. 25–28), and I tried to apply it to this slightly different case.

In your dissertation you start by calculating the potential created by charges on a single vertical meshline. In the present case, I think one must start by calculating the potential created by charges on a 2D slice of the beam, with “metallic” boundary conditions on two edges of the slice, and “symmetry” boundary conditions on the two other edges (see Fig. 1).

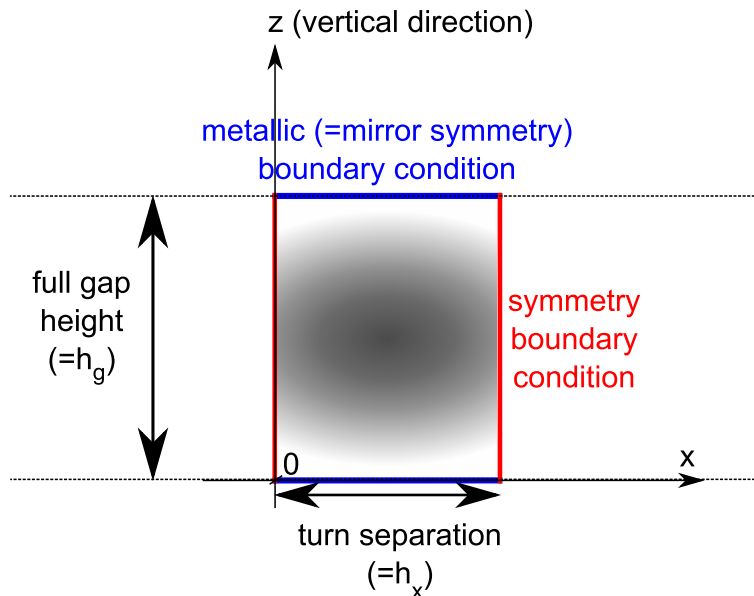


Figure 1: 2D beam “slice”, with two types of boundary conditions on the edges.

With these boundary conditions, the charge distribution is periodic in both x and z directions, and is odd in the z direction. The charge density can be expressed as a sum of Fourier harmonics:

$$\rho(x, z) = \sum_{l,m} \lambda_{l,m} \exp(i2\pi x \frac{l}{h_x}) \sin(2\pi z \frac{m}{h_z}), \quad (1)$$

with $h_z = 2h_g$. To simplify the notations let's define from now:

$$\omega_l = 2\pi \frac{l}{h_x} \quad (2)$$

$$\omega_m = 2\pi \frac{m}{h_z} \quad (3)$$

$$(4)$$

$\rho(x, z)$ becomes:

$$\rho(x, z) = \sum_{l,m} \lambda_{l,m} \exp(i\omega_l x) \sin(\omega_m z). \quad (5)$$

As you did in your dissertation, let's now assume that the potential produced by the (l, m) -th harmonic can be written in the form:

$$\phi_{l,m}(x, z, y) = \exp(i\omega_l x) \sin(\omega_m z) \cdot f_{l,m}(y) + \text{const.} \quad (6)$$

We chose $\text{const.} = 0$ (as if to say, we choose the potential along the metallic boundary = 0).

To find $f_{l,m}(y)$, let's solve the Laplace equation (in Cartesian coordinates):

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2} \right) \phi = 0, \quad (7)$$

which yields for the (l, m) -th harmonic to:

$$\frac{d^2 f_{l,m}}{dy^2} - (\omega_l^2 + \omega_m^2) f_{l,m} = 0. \quad (8)$$

The general solution of this equation can be written as:

$$f_{l,m} = C_1 \exp(\omega_{l,m} y) + C_2 \exp(-\omega_{l,m} y), \quad (9)$$

with C_1 and C_2 two real numbers, and $\omega_{l,m} = \sqrt{\omega_l^2 + \omega_m^2}$.

Now, let's bring back some physics in this matter: $f_{l,m}$ must tend to 0 far from the plane containing the charges. This means that the solution we are looking for can be written as:

$$f_{l,m} = C_{l,m} \exp(-\omega_{l,m} |y|). \quad (10)$$

To find the value of the constant $C_{l,m}$, let's calculate the electric potential using the Coulomb law at $x = y = 0$, and $z = -\frac{\pi}{2\omega_m}$ ($\omega_m \neq 0$):

$$\phi_{l,m}(0, 0, -\frac{\pi}{2\omega_m}) = C_{l,m} \quad (11)$$

$$= \iint_{-\infty}^{+\infty} \frac{\lambda_{l,m} \exp(i\omega_l x) \cos(\omega_m z)}{\sqrt{x^2 + z^2}} dx dz \quad (12)$$

$$= 2\pi \frac{\lambda_{l,m}}{\sqrt{\omega_l^2 + \omega_m^2}} \quad (13)$$

$$= 2\pi \frac{\lambda_{l,m}}{\omega_{l,m}}. \quad (14)$$

Mathematica found the integral for me...

The potential created by one slice of the beam is the sum of the contribution of each harmonic:

$$\phi(x, y, z) = \sum_{l,m} \exp(i\omega_l x) \sin(\omega_m z) \cdot C_{l,m} \exp(-\omega_{l,m}|y|). \quad (15)$$

The potential created by the whole beam is the sum of the potential created by each slice:

$$\phi_{tot}(x, y, z) = \sum_{l,m} D_{l,m} \exp(i\omega_l x) \sin(\omega_m z), \quad (16)$$

with:

$$D_{l,m} = \sum_n C_{l,m,n} \exp(-\omega_{l,m}|y - y_n|), \quad (17)$$

if we call y_n the y coordinate of the n^{th} slice. One now gets ϕ_{tot} in each slice from the inverse fourier transform of $D_{l,m}$.