Poisson equation solver with symmetry conditions in the radial direction

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Dear Eduard,

I would like to follow your advice and simulate the effect of neighboring turns by applying symmetry conditions in the radial direction. In this brief note, I describe one possible way to compute the self field taking into account this additional boundary condition. I followed the approach described in your dissertation (P. 25–28), and I tried to apply it to this slightly different case.

In your dissertation you start by calculating the potential created by charges on a single vertical meshline. In the present case, I think one must start by calculating the potential created by charges on a 2D slice of the beam, with "metallic" boundary conditions on two edges of the slice, and "symmetry" boundary conditions on the two other edges (see Fig. 1).

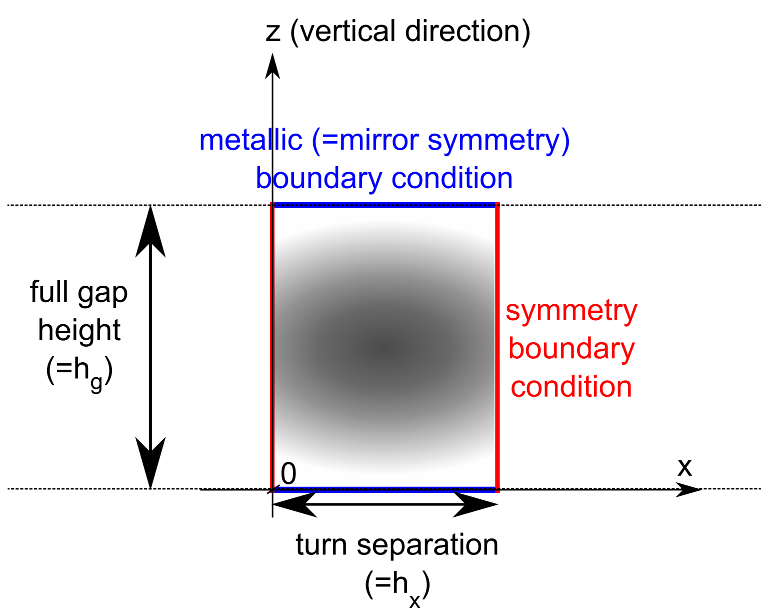

Figure 1: 2D beam "slice", with two types of boundary conditions on the edges.

With these boundary conditions, the charge distribution is periodic in both x and z directions, and is odd in the z direction. The charge density can be expressed as a sum of Fourier harmonics:

$$\rho(x,z) = \sum_{l,m} \lambda_{l,m} \exp(i2\pi x \frac{l}{h_x}) \sin(2\pi z \frac{m}{h_z}), \tag{1}$$

with $h_z = 2h_g$. To simplify the notations let's define from now:

$$\begin{aligned} \omega_l &= 2\pi \frac{l}{h_x} && (2) \\ \omega_m &= 2\pi \frac{m}{h_z} && (3) \\ & && (4) \end{aligned}$$

$\rho(x,z)$ becomes:

$$\rho(x,z) = \sum_{l,m} \lambda_{l,m} \exp(i\omega_l x) \sin(\omega_m z). \tag{5}$$

As you did in your dissertation, let's now assume that the potential produced by the (l,m)–th harmonic can be written in the form:

$$\phi_{l,m}(x,z,y) = \exp(i\omega_l x) \sin(\omega_m z) \cdot f_{l,m}(y) + const. \tag{6}$$

We chose $const. = 0$ (as if to say, we choose the potential along the metalic boundaried $= 0$).

To find $f_{l,m}(y)$, let's solve the Laplace equation (in Cartesian coordinates):

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2} \right) \phi = 0, \tag{7}$$

which yields for the (l,m)–th harmonic to:

$$\frac{\mathrm{d}^2 f_{l,m}}{\mathrm{d}y^2} - \left(\omega_l^2 + \omega_m^2\right) f_{l,m} = 0. \tag{8}$$

The general solution of this equation can be written as:

$$f_{l,m} = C_1 \exp(\omega_{l,m} y) + C_2 \exp(-\omega_{l,m} y), \tag{9}$$

with C_1 and C_2 two real numbers, and $\omega_{l,m} = \sqrt{\omega_l^2 + \omega_m^2}$.

Now, let's bring back some physics in this matter: $f_{m,l}$ must tend to 0 far from the plane containing the charges. This means that the solution we are looking for can be written as:

$$f_{l,m} = C_{l,m} \exp(-\omega_{l,m}|y|). \tag{10}$$

To find the value of the constant $C_{l,m}$, let's calculate the electric potential using the Coulomb law at $x = y = 0$, and $z = -\dfrac{\pi}{2\omega_m}$ ($\omega_m \neq 0$):

$$\begin{aligned}
\phi_{l,m}(0,0,-\frac{\pi}{2\omega_m}) &= C_{l,m} &(11)\\
&= \iint_{-\infty}^{+\infty} \frac{\lambda_{l,m}\exp(i\omega_l x)\cos(\omega_m z)}{\sqrt{x^2+z^2}}\mathrm{d}x\mathrm{d}z &(12)\\
&= 2\pi\frac{\lambda_{l,m}}{\sqrt{\omega_l^2+\omega_m^2}} &(13)\\
&= 2\pi\frac{\lambda_{l,m}}{\omega_{l,m}}. &(14)
\end{aligned}$$

Mathematica found the integral for me...

The potential created by one slice of the beam is the sum of the contribution of each harmonic:

$$\phi(x,y,z) = \sum_{l,m} \exp(i\omega_l x)\sin(\omega_m z)\cdot C_{l,m}\exp(-\omega_{l,m}|y|). \tag{15}$$

The potential created by the whole beam is the sum of the potential created by each slice:

$$\phi_{tot}(x,y,z) = \sum_{l,m} D_{l,m}\exp(i\omega_l x)\sin(\omega_m z), \tag{16}$$

with:

$$D_{l,m} = \sum_{n} C_{l,m,n}\exp(-\omega_{l,m}|y-y_n|), \tag{17}$$

if we call y_n the y coordinate of the n^{th} slice. One now gets ϕ_{tot} in each slice from the inverse fourier transform of $D_{l,m}$.