Iterative Learning Control Revisited

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This work builds on the results reported in TRI-BN-16-17 and TRI-BN-17-02 for one-pole and two-pole closed-loop transfer functions, respectively.
http://lin12.triumf.ca/text/design_notes

One-pole closed loop transfer function $\dfrac{A}{z-B}$

Two-pole closed loop transfer function $\dfrac{A(z-1)}{(z-B1)(z-B2)}$

Z-domain conditions for stability under iteration of 1, 2 & 3 term learning functions used with the two-pole plant were reported in TRI-BN-17-02.
For the conditions used therein, and the 3-term ILC, the domain of stability in the space of (A,B1,B2) was disappointing: very narrow for the look-ahead ILC and non-existent for the look-back ILC.
In this report, the conditions on the learning functions are revisited with a view to increasing the domain of stability (a.k.a. monotonic convergence).
In addition we introduce learning functions that are infinite series and find their domain of stability for one-pole & two-pole plant.

Closed-loop transfer function without ILC: $P(z)=\frac{G(z)}{1+C(z)G(z)}$

Transfer function with ILC: $T(z)= Q(z)[1 - \nu L(z).zP(z)]$

Stability analysis in z-domain proceeds by substituting z=Exp[iϴ] into T, and then forming the locus of T in the complex plane as ϴ varies from 0 to π.

Let R=|T(z)| and Tanϕ = Im[T(ϴ)]/Re[T(ϴ)].

System is stable if T remains within the unit circle R ≤ 1 for all ϴ.

If R=1, then dR/dϕ must be identically zero; and d^2R/dϕ^2 <0.

[For comparison, on the unit circle all derivatives of R w.r.t. ϕ must be zero]

Because |T(ϕ)| <1 is (generally) not a unit circle, so it follows that when R(ϕ)=1 odd derivatives w.r.t. ϕ must be zero & even derivatives < 0.

Typically T(ϴ) is less complicated than T(ϕ).

Fortunately, we can often work with T(ϴ) because:

dR/dϕ = (dR/dϴ)(dϴ/dϕ) = 0 if dR/dϴ=0 OR dϴ/dϕ=0

d^2R/dϕ^2 = (d^2R/dϴ^2)(dϴ/dϕ)^2 + (d^2ϴ/dϕ^2) (dR/dϴ) < 0 if dR/dϴ=0 AND (d^2R/dϴ^2) < 0.

For the system described in TRI-DN-13-23, analysis is simplified if we transform from variables U=aT & Kp to A & B

$$\mathrm{A} = 1 - \mathrm{e}^{-U} \geq 0$$

$$\mathrm{B} = \mathrm{e}^{-U}(1 + \mathrm{Kp}) - \mathrm{Kp}$$

$$\mathrm{B} = \mathrm{e}^{-U} - A\mathrm{Kp}$$

$$\mathrm{B} = 1 - A(1 + \mathrm{Kp})$$

sampling rate $\frac{1}{T}$, the cavity time constant $\frac{1}{a}$

Always √>0, so roots always pure real

$$\left\{\left\{\mathrm{B1} = \frac{1}{2}\left(b - \sqrt{b^2 - 4B}\right)\right\}, \left\{\mathrm{B2} = \frac{1}{2}\left(b + \sqrt{b^2 - 4B}\right)\right\}\right\}$$

$$b = 1 + \mathrm{e}^{-U} - A\mathrm{Kp}(1 + U)$$

Some useful properties:

$$\mathrm{B1} + \mathrm{B2} = b$$

$$\mathrm{B2} - \mathrm{B1} = \sqrt{b^2 - 4B}$$

$$\mathrm{B1} \times \mathrm{B2} = B$$

PI Control & Ki=a.Kp
Closed Loop Gain Function

Sampling rate = 1/T; cavity time constant = 1/a

U=aT; ν = learning gain

$$P(z)= \frac{(-1+e^{U})(-1+z)}{1+Kp-e^{U}Kp-z-e^{U}z-Kpz+e^{U}Kpz-KpUz+e^{U}KpUz+e^{U}z^{2}}$$

Can be written: $$\frac{A(z-1)}{(z-B1)(z-B2)}$$

Stability conditions:

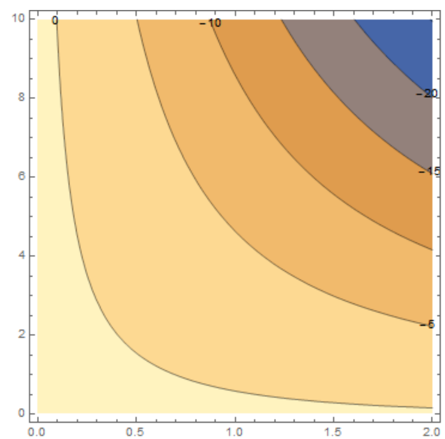
B1 >0 implies

$$U>0\ \&\ Kp\le\frac{1}{-1+e^{U}}$$

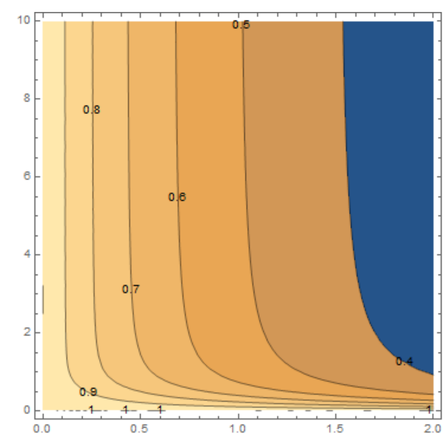
B2 <1 implies

$$U>0\ \&\ Kp\ge 0$$

B=B1=0 &
B2= 1-e^{-U}U when $Kp=\frac{1}{-1+e^{U}}$



B1(U,Kp)



B2(U,Kp)

STABILITY OF THE 2-POLE SYSTEM WITH ITERATIVE LEARNING CONTROL

3-Term, "Look Back 2 steps"; Ki=a.Kp

L = v(1+ 1/z+ c/z^2); causal = use information from present and 2 previous time steps

C=1 → TRI-BN-17-02 →

$$T = 1 - \frac{Av e^{i\theta}\left(-1 + e^{i\theta}\right)\left(1 + e^{-i\theta} + e^{-2i\theta}\right)}{\left(-B1 + e^{i\theta}\right)\left(-B2 + e^{i\theta}\right)}$$

R=|T|=1 when Θ =0 or 2π/3.
dR/dΘ =0 when Θ =0 or π.

The condition for dR/dΘ =0 when Θ = 2π/3 is the split $B2 = -\frac{B1}{1 + B1}$

Introducing this leads to the sufficient condition $Av \leq 1$

For the particular relation between B1 and B2, the split implies $B = -b < -1$

This note: there are no useful solutions for C ≥ 2/3

C= 1/3 → this note

R=|T|=1 when Θ =0. dR/dΘ =0 when Θ =0 or π. d^2R/dΘ^2<0 implies

$$Av < -\frac{3}{49}(-17 + 3B1 + 3B2 + 11B1B2) \&\& -\frac{1}{3} < B1 < \frac{1}{5} \&\& -\frac{1}{3} < B2 < \frac{1}{5} \&\& c == \frac{1}{3}$$

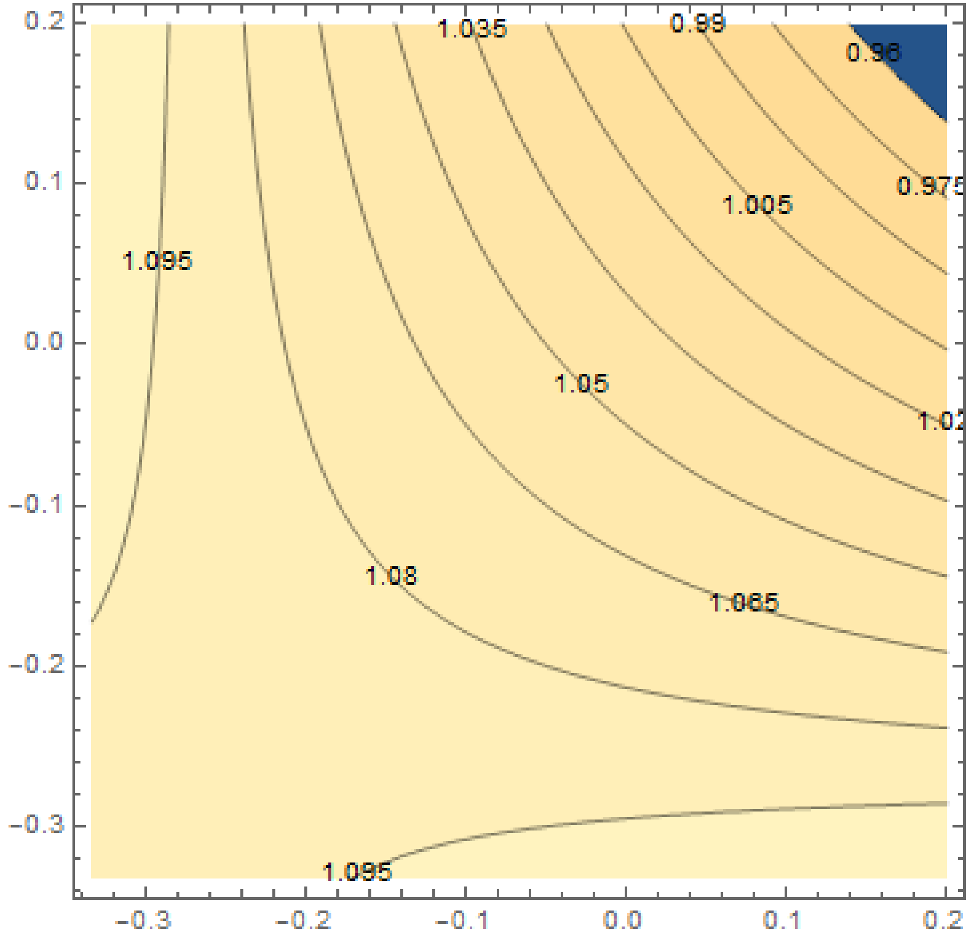
$$Av < \frac{1}{49}(51 - 9b - 33B)$$

$$0 < U < 1 \;\&\&\; 0 < A < 1 \;\&\&\; \mathrm{Kp} > 0 \;\&\&\; v < \frac{1}{49}(42 + 42\mathrm{Kp} + 9\mathrm{Kp}U)$$

3-Term, “Look Back 2 steps”; Ki=a.Kp

$L = v(1+ 1/z+ c/z^2)$;

C= 1/3 → this note



Av ≤ 53/49

Av (B1,B2)

3-Term, "Look Back 2 steps"; Ki=a.Kp

L = v(1+ 1/z+ c/z²);

C= 1/4 → this note

$$\mathrm{A}v < -\frac{4}{27}(-7 + \mathrm{B1} + \mathrm{B2} + 5\mathrm{B1B2})\&\& -\frac{2}{5} < \mathrm{B1} < \frac{1}{5}\&\& -\frac{2}{5} < \mathrm{B2} < \frac{1}{5}\&\&c == \frac{1}{4}$$

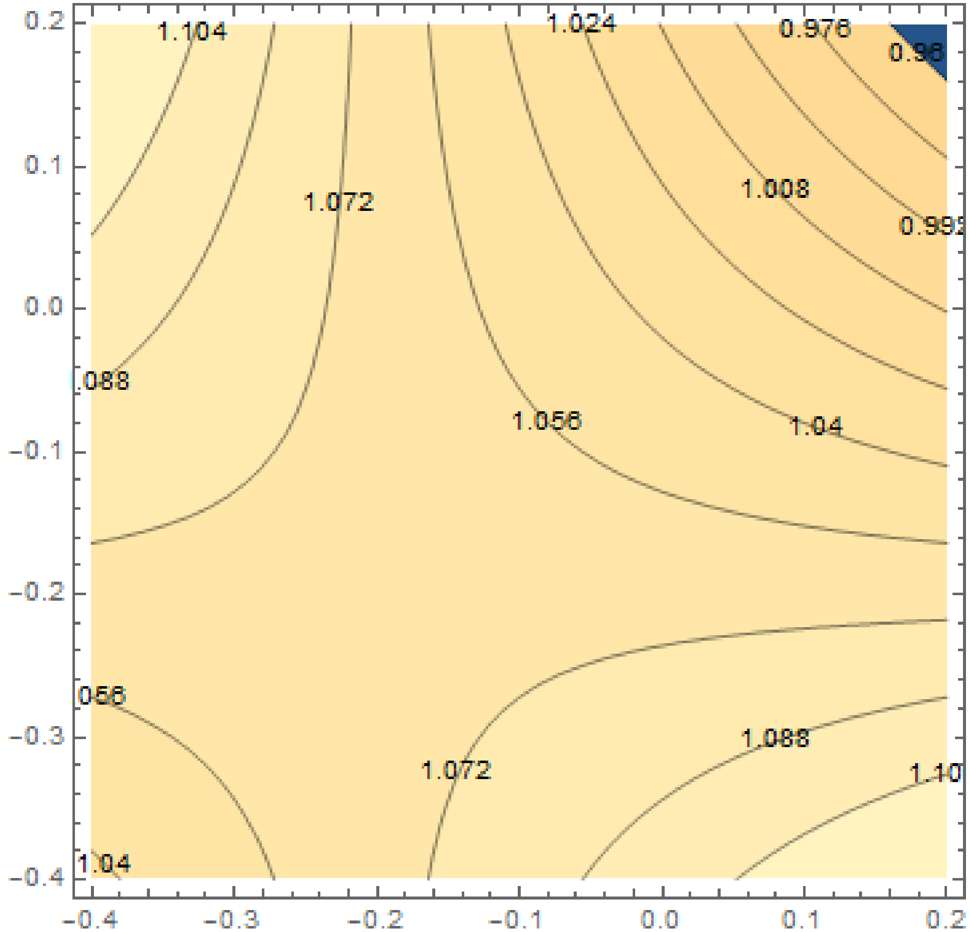
$$\mathrm{A}v < -\frac{4}{27}(-7 + b + 5B)$$

$$0 < U < 1 \;\&\&\; 0 < A < 1 \;\&\&\; \mathrm{Kp} > 0\&\&\; v < \frac{1}{27}(24 + 24\mathrm{Kp} + 4\mathrm{Kp}U)$$

3-Term, "Look Back 2 steps"; Ki=a.Kp

$L = v(1+ 1/z+ c/z^2)$;

C= 1/4 → this note



Av ≤ 28/27

Av (B1,B2)

3-Term "Look Ahead 2 steps"; Ki=a.Kp

L = v(1+z+c z²); use information from present and two next time steps

C=1 → TRI-BN-17-02 →

$$T = 1 - \frac{Av e^{i\theta}\left(-1 + e^{3i\theta}\right)}{\left(-B1 + e^{i\theta}\right)\left(-B2 + e^{i\theta}\right)}$$

R=|T|=1 when Θ= 0 or 2π/3.
dR/dΘ=0 when Θ= 0 or π.

dR/dΘ NOT =0 at Θ= 2π/3 unless B1×B2=1 Introducing, this condition

$$\left\{Av < -2 + \frac{1}{B1} + B1\right\} \& \, 0 < B1 < 1 \qquad 0 < B1 < 1 \,\&\, 0 < B2 < 1$$

$$Av < -2 + b \qquad v < -1 - Kp(1 + U)$$

C= 1/4 → this note

$$Av < -\frac{4}{27}(1 - 7B1 - 7B2 + 13B1B2) \,\&\&\, \frac{1}{7} < B1 < 1 \,\&\&\, \frac{1}{7} < B2 < 1 \,\&\& c == \frac{1}{4}$$

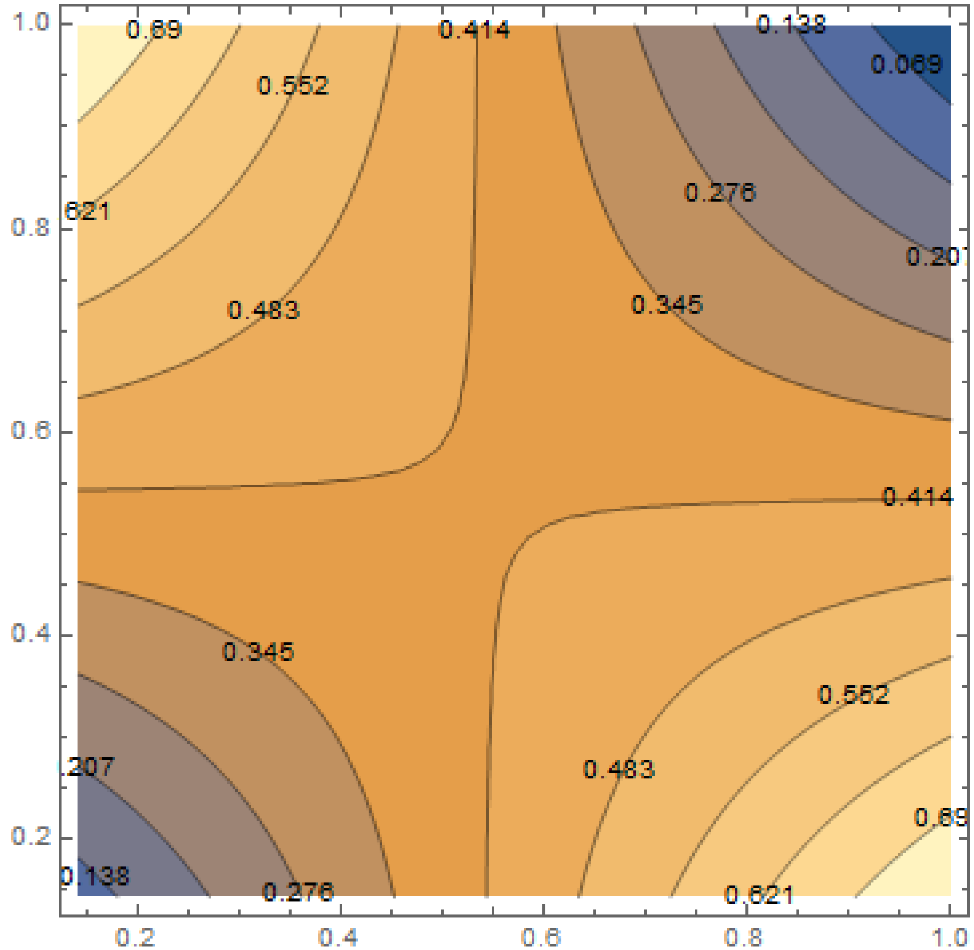
$$Av < \frac{1}{27}(-4 + 28b - 52B)$$

$$0 < U < 1 \&\& 0 < A < 1 \,\&\&\, Kp > 0 \,\&\&\, v < \frac{4}{27}(6 + 6Kp - 7KpU)$$

3-Term “Look Ahead 2 steps”; Ki=a.Kp

L = v(1+z+c z^2);

C= 1/4 → this note



Av ≤ 16/147

Av (B1,B2)

L = $v/(1+z\,c)$ = look ahead

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + z^4 - z^5 + z^6 + \ldots$$

$$\mathrm{T(z)} = 1 + \frac{\mathrm{A}v(1-z)z}{(\mathrm{B1}-z)(\mathrm{B2}-z)(1+cz)}$$

$$\mathrm{T}(\Theta) = 1 - \frac{\mathrm{A}v\mathrm{e}^{\mathrm{i}\theta}\left(-1+\mathrm{e}^{\mathrm{i}\theta}\right)}{(-\mathrm{B1}+\mathrm{e}^{\mathrm{i}\theta})(-\mathrm{B2}+\mathrm{e}^{\mathrm{i}\theta})(1+c\mathrm{e}^{\mathrm{i}\theta})}$$

dT/dΘ = 0 when Θ =0 or π

The conditions R=|T|≤ 1 and d^2R/dΘ^2 <0 at Θ =0, and R ≤ 1 at Θ = π, lead to

$$-1 < \mathrm{B2} < 1\&\& - 1 < \mathrm{B1} \leq 1\&\& \frac{1+\mathrm{B1}+\mathrm{B2}-3\mathrm{B1B2}}{-3+\mathrm{B1}+\mathrm{B2}+\mathrm{B1B2}} < c < \mathrm{B1B2}\&\& - 1 < c < 1$$

$$\mathrm{A}v < 1 + \mathrm{B1} + \mathrm{B2} - 3\mathrm{B1B2} - (-3 + \mathrm{B1} + \mathrm{B2} + \mathrm{B1B2})c$$

$$\mathrm{A}v < 1 + b - 3B + 3c - bc - Bc$$

$$0 < U < 1\&\ 0 < A < 1\ \&\ \mathrm{Kp} > 0\ \&\ v < 2(1 + c + \mathrm{Kp}) + \mathrm{Kp}(2c - U + cU)$$

STABILITY OF THE 1-POLE SYSTEM WITH ITERATIVE LEARNING CONTROL USING INFINITE SERIES

Proportional Control
Closed Loop Gain Function

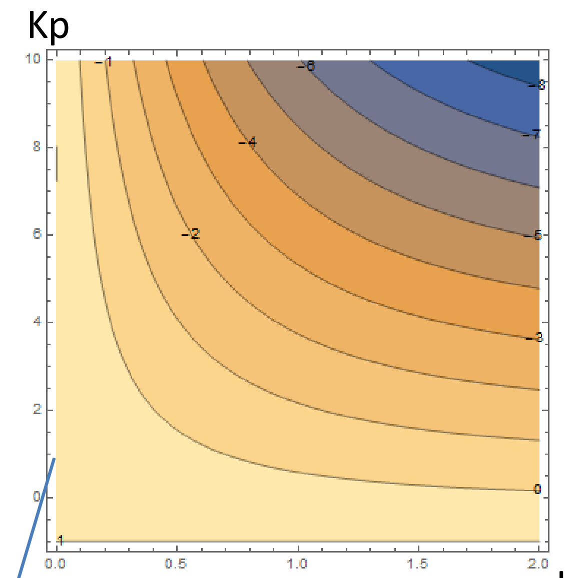
$$P(z)=\frac{-1+e^{U}}{-1-Kp+e^{U}Kp+e^{U}z}$$

Can be written: $\frac{A}{z-B}$

$U > 0$

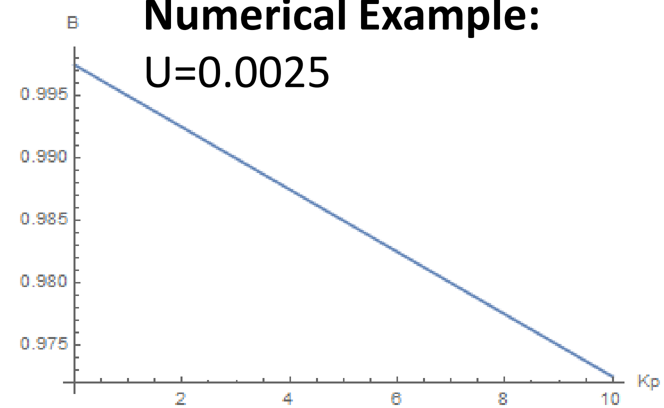
Stability conditions:

- B >-1 implies $Kp < \frac{1+e^{U}}{-1+e^{U}}$
- B >0 implies $Kp < \frac{1}{-1+e^{U}}$
- B <1 implies $Kp \geq -1$ & U>0
- U=0 implies B=1 for all Kp



B(U,Kp)

Numerical Example:
U=0.0025



L = v/(1+z c) = look ahead

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + z^4 - z^5 + z^6 + \ldots$$

$$\mathrm{T(z)} = 1 - \frac{\mathrm{A}vz}{(-B+z)(1+cz)} \qquad \mathrm{T}(\Theta) = 1 - \frac{\mathrm{A}v\mathrm{e}^{\mathrm{i}\theta}}{(-B+\mathrm{e}^{\mathrm{i}\theta})(1+c\mathrm{e}^{\mathrm{i}\theta})}$$

dT/dΘ = 0 when Θ =0 or π

The conditions R=|T|≤ 1 and d^2R/dΘ^2 <0 at Θ =0 or π lead to two viable ranges of the parameter c.

Sol1: $-1 < c < 1 \&\& \ c \le B < 1$

$$\mathrm{A}v < 2(1-B)(1+c)$$

$$v < 2(1+c)(1+\mathrm{Kp})$$

Sol2: $-1 < c < 1 \&\& -1 < B \le c$

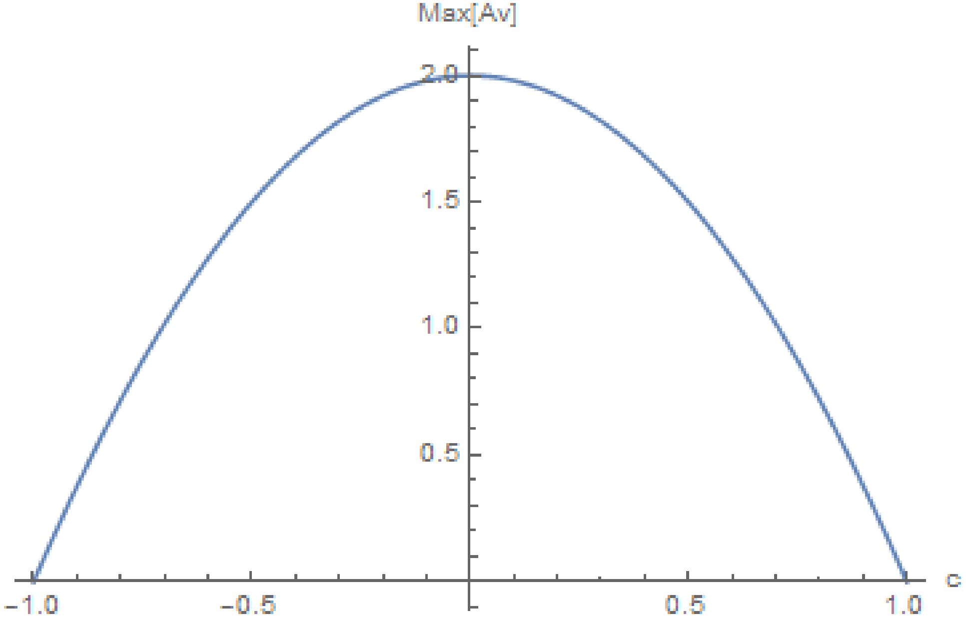
$$\mathrm{A}v < 2(1+B)(1-c)$$

$$v < 2(1-c)(2/A - 1 - \mathrm{Kp})$$

$L = v/(1+z\,c)$ = look ahead

The largest possible value of A*v* for given *c* is the solution of

$$\frac{2 - \mathrm{A}v - 2c}{-2 + 2c} = \frac{2 - \mathrm{A}v + 2c}{2 + 2c} \qquad \mathrm{Max}[\mathrm{A}v] = 2(1-c)(1+c)$$



Note that when c→0, this reduces (correctly) to the case L=*v*

L = νExp[z c] = look ahead

The exponential is expanded as an infinite series in integer powers of z

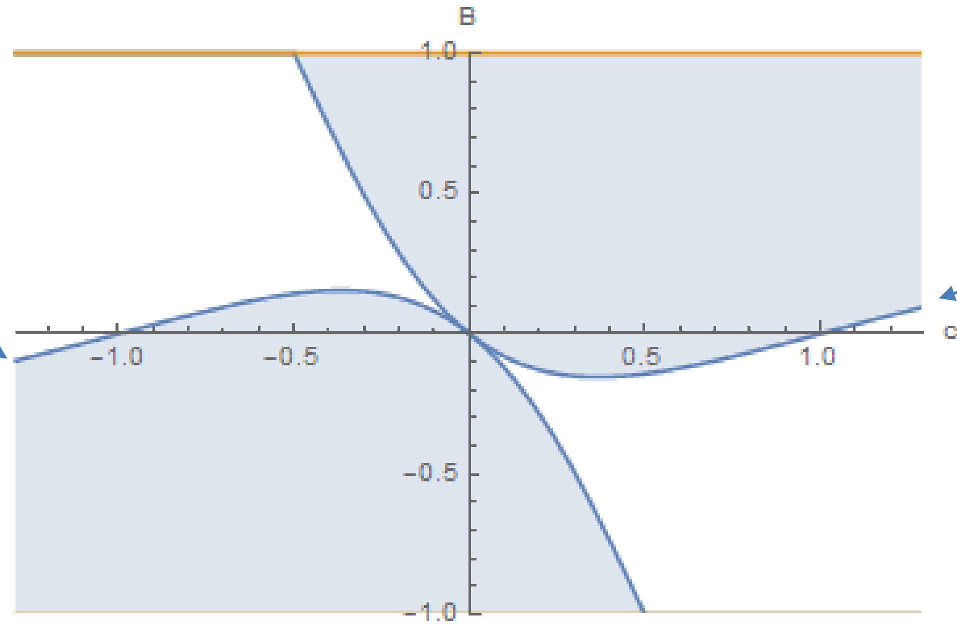
$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \ldots$$

$$\mathrm{T(z)} = 1 - \frac{\mathrm{Ave}^{cz} z}{-B + z} \qquad \mathrm{T}(\Theta) = 1 - \frac{\mathrm{Ave}^{c e^{i\theta} + i\theta}}{-B + e^{i\theta}}$$

dT/dϴ = 0 when ϴ =0 or π

The conditions R=|T|≤ 1 and d^2R/dϴ^2 <0 at ϴ =0 or π lead to three viable ranges of the parameter c.

$$B = -\frac{c(1+c)}{1-c+c^2}$$



$$B = \frac{(-1+c)c}{1+c+c^2}$$

If c > +1/2, then only solution1 applies alone

If c < -1/2, then only solution2 applies alone

If - 1/2 < c < +1/2, then both solutions apply; so it becomes complicated

L = vExp[z c] = look ahead

Sol1: $\frac{1}{2} \leq c \leq \text{Cmax}\ \&\&\ \frac{(-1+c)c}{1+c+c^2} < B < 1\ \&\&\ \text{A}v < 2(1-B)\text{e}^{-c}$

$Av < 2A\text{e}^{-c}(1+\text{Kp})$

$v < 2\text{e}^{-c}(1+\text{Kp})$

Sol2: $-\text{Cmax} \leq c \leq -\frac{1}{2}\ \&\& -1 < B < -\frac{c(1+c)}{1-c+c^2}\ \&\&\ \text{A}v < 2(1+B)\text{e}^{c}$

$Av < -2\text{e}^{c}(-2+A+A\text{Kp})$

$v < 2\text{e}^{c}(2/A-1-\text{Kp})$

Cmax is the solution of $(-6+c^2+2c^3)$=0

$$\text{Cmax} = \frac{1}{6}\left(-1+\left(323-18\sqrt{322}\right)^{1/3}+\left(323+18\sqrt{322}\right)^{1/3}\right)$$

Cmax $\approx 1.293376334322 1676 \approx 53/41$

$L = v \,\mathrm{Exp}[z\, c] =$ look ahead

Sol3: $-\frac{1}{2} < c < \frac{1}{2}$&&

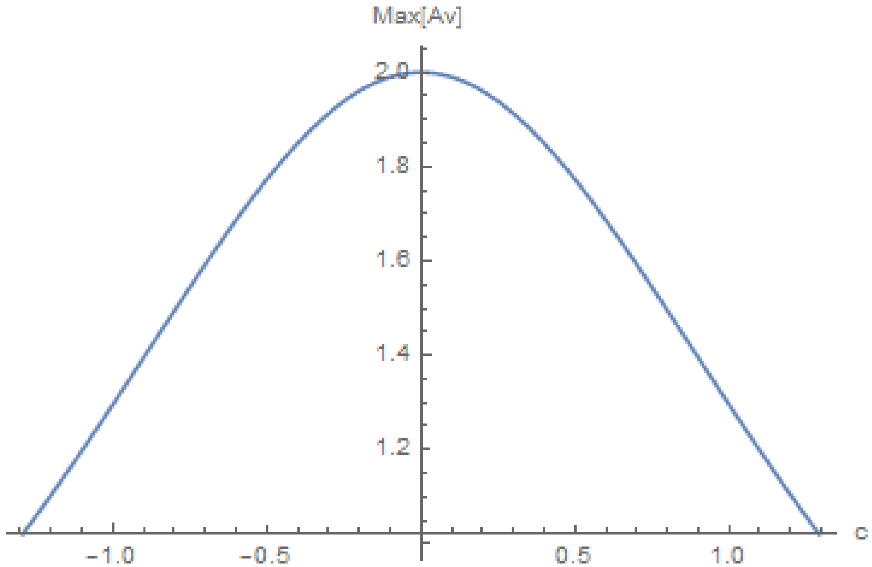
$$\left(\left(-1 < B < -\frac{c(1+c)}{1-c+c^2} \&\& \mathrm{A}v < 2(1+B)\mathrm{e}^{c}\right) \mathrm{OR} \left(\frac{(-1+c)c}{1+c+c^2} < B < 1 \&\& \mathrm{A}v < 2(1-B)\mathrm{e}^{-c}\right)\right)$$

Depending on the ranges of *B* and *c*, the limitation on A*v* appears as Sol1 or Sol2.

$L = v\,\mathrm{Exp}[z\,c]$ = look ahead

The largest possible value of A*v* for given *c* is the solution of

$$1 - \frac{Av e^{c}}{2} = -1 + \frac{Av e^{-c}}{2} \qquad \mathrm{Max}[Av] = \frac{4e^{c}}{1 + e^{2c}}$$



Note that when c→0, this reduces (correctly) to the case L=*v*