Iterative Learning Control Revisited

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This work builds on the results reported in TRI-BN-16-17 and TRI-BN-17-02 for one-pole and two-pole closed-loop transfer functions, respectively.

http://lin12.triumf.ca/text/design_notes

One-pole closed loop transfer function \( \frac{A}{z - B} \)

Two-pole closed loop transfer function \( \frac{A(z - 1)}{(z - B1)(z - B2)} \)

Z-domain conditions for stability under iteration of 1, 2 & 3 term learning functions used with the two-pole plant were reported in TRI-BN-17-02.

For the conditions used therein, and the 3-term ILC, the domain of stability in the space of \((A,B1,B2)\) was disappointing: very narrow for the look-ahead ILC and non-existent for the look-back ILC.

In this report, the conditions on the learning functions are revisited with a view to increasing the domain of stability (a.k.a. monotonic convergence).

In addition we introduce learning functions that are infinite series and find their domain of stability for one-pole & two-pole plant.
Closed-loop transfer function without ILC: \( P(z) = \frac{G(z)}{1 + C(z)G(z)} \)

Transfer function with ILC: \( T(z) = Q(z)[1 - \nu L(z)zP(z)] \)

Stability analysis in z-domain proceeds by substituting \( z = \text{Exp}[i\Theta] \) into \( T \), and then forming the locus of \( T \) in the complex plane as \( \Theta \) varies from 0 to \( \pi \).

Let \( R = |T(z)| \) and \( \tan \phi = \text{Im}[T(\Theta)]/\text{Re}[T(\Theta)] \).

System is stable if \( T \) remains within the unit circle \( R \leq 1 \) for all \( \Theta \).

If \( R = 1 \), then \( dR/d\phi \) must be identically zero; and \( d^2R/d\phi^2 < 0 \).

[For comparison, on the unit circle all derivatives of \( R \) w.r.t. \( \phi \) must be zero]

Because \( |T(\phi)| < 1 \) is (generally) not a unit circle, so it follows that when \( R(\phi) = 1 \) odd derivatives w.r.t. \( \phi \) must be zero & even derivatives < 0.

Typically \( T(\Theta) \) is less complicated than \( T(\phi) \).

Fortunately, we can often work with \( T(\Theta) \) because:

\[
\frac{dR}{d\phi} = (\frac{dR}{d\Theta})(\frac{d\Theta}{d\phi}) = 0 \text{ if } \frac{dR}{d\Theta} = 0 \text{ OR } \frac{d\Theta}{d\phi} = 0
\]

\[
\frac{d^2R}{d\phi^2} = (\frac{d^2R}{d\Theta^2})(\frac{d\Theta}{d\phi})^2 + (\frac{d^2\Theta}{d\phi^2}) \left(\frac{dR}{d\Theta}\right) < 0 \text{ if } \frac{dR}{d\Theta} = 0 \text{ AND } (\frac{d^2R}{d\Theta^2}) < 0.
\]
For the system described in TRI-DN-13-23, analysis is simplified if we transform from variables $U = aT$ & $Kp$ to $A$ & $B$

\[
A = 1 - e^{-U} \geq 0
\]

\[
B = e^{-U} (1 + Kp) - Kp
\]

sampling rate $\frac{1}{T}$, the cavity time constant $\frac{1}{a}$

\[
B_1 = \frac{1}{2} \left( b - \sqrt{b^2 - 4B} \right), \quad B_2 = \frac{1}{2} \left( b + \sqrt{b^2 - 4B} \right)
\]

\[
b = 1 + e^{-U} - AKp(1 + U)
\]

Some useful properties:

\[
B_1 + B_2 = b
\]

\[
B_2 - B_1 = \sqrt{b^2 - 4B}
\]

\[
B_1 \times B_2 = B
\]

Always $v>0$, so roots always pure real
PI Control & $Ki=a.Kp$

Closed Loop Gain Function

$$P(z)=\frac{(-1+e^U)(-1+z)}{1+Kp-e^U Kp-z-e^U z-Kpz+e^U Kpz-KpUz+e^U KpUz+e^U z^2}$$

Can be written:

$$\frac{A(z-1)}{(z-B1)(z-B2)}$$

**Stability conditions:**

$B1 > 0$ implies $U > 0$ & $Kp \leq \frac{1}{-1+e^U}$

$B2 < 1$ implies $U > 0$ & $Kp \geq 0$

$B = B1 = 0$ & $B2 = 1-e^{-U}U$ when $Kp = \frac{1}{-1+e^U}$

Sampling rate = $1/T$; cavity time constant = $1/a$

$U = aT$; $\nu$ = learning gain
STABILITY OF THE 2-POLE SYSTEM WITH ITERATIVE LEARNING CONTROL
3-Term, “Look Back 2 steps”; Ki=a.Kp

$L = \nu(1 + 1/z + c/z^2); \text{ causal = use information from present and 2 previous time steps}$

$C=1 \rightarrow \text{TRI-BN-17-02} \rightarrow$

$$T = 1 - \frac{A\nu e^{i\theta} (-1 + e^{i\theta})(1 + e^{-i\theta} + e^{-2i\theta})}{(-B1 + e^{i\theta})(-B2 + e^{i\theta})}$$

$R = |T| = 1$ when $\Theta = 0$ or $2\pi/3$.

$dR/d\Theta = 0$ when $\Theta = 0$ or $\pi$.

The condition for $dR/d\Theta = 0$ when $\Theta = 2\pi/3$ is the split $B2 = -\frac{B1}{1 + B1}$.

Introducing this leads to the sufficient condition $A\nu \leq 1$.

For the particular relation between $B1$ and $B2$, the split implies $B = -b < -1$.

This note: there are no useful solutions for $C \geq 2/3$

$C = 1/3 \rightarrow$ this note

$R = |T| = 1$ when $\Theta = 0$. $dR/d\Theta = 0$ when $\Theta = 0$ or $\pi$. $d^2R/d\Theta^2 < 0$ implies

$$A\nu < -\frac{3}{49}(-17 + 3B1 + 3B2 + 11B1B2) \& \& -\frac{1}{3} < B1 < \frac{1}{5} \& \& -\frac{1}{3} < B2 < \frac{1}{5} \& \& c = \frac{1}{3}$$

$$A\nu < \frac{1}{49}(51 - 9b - 33B)$$

$$0 < U < 1 \& \& 0 < A < 1 \& \& Kp > 0 \& \& \nu < \frac{1}{49}(42 + 42Kp + 9KpU)$$
3-Term, “Look Back 2 steps”; Ki=a.Kp

\[ L = v(1 + \frac{1}{z} + \frac{c}{z^2}) ; \]

\[ C = \frac{1}{3} \rightarrow \text{this note} \]

\[ Av \leq \frac{53}{49} \]
3-Term, “Look Back 2 steps”; \( Ki = a.Kp \)

\[
L = v(1 + 1/z + c/z^2);
\]

\[
C = \frac{1}{4} \rightarrow {\text{this note}}
\]

\[
Av < -\frac{4}{27}(-7 + B1 + B2 + 5B1B2) \land -\frac{2}{5} < B1 < \frac{1}{5} \land -\frac{2}{5} < B2 < \frac{1}{5} \land c = \frac{1}{4}
\]

\[
Av < -\frac{4}{27}(-7 + b + 5B)
\]

\[
0 < U < 1 \land 0 < A < 1 \land Kp > 0 \land v < \frac{1}{27}(24 + 24Kp + 4KpU)
\]
$L = v(1 + 1/z + c/z^2)$;

$C = 1/4 \rightarrow$ this note

$A_v \leq 28/27$

$A_v (B1,B2)$
3-Term “Look Ahead 2 steps”; Ki=a.Kp

\[ L = \nu(1+z+c z^2); \text{ use information from present and two next time steps} \]

C=1 \rightarrow \text{TRI-BN-17-02} \rightarrow

\[ T = 1 - \frac{Ave^{i\theta}(-1 + e^{3i\theta})}{(-B1 + e^{i\theta})(-B2 + e^{i\theta})} \]

R = |T| = 1 when \( \Theta = 0 \) or \( 2\pi/3 \).

dR/d\Theta = 0 when \( \Theta = 0 \) or \( \pi \).

dR/d\Theta \text{ NOT } = 0 \text{ at } \Theta = 2\pi/3 \text{ unless } B1 \times B2 = 1

Introducing, this condition

\[ \begin{cases} 
Av < -2 + \frac{1}{B1} + B1 & \text{& } 0 < B1 < 1 \\
0 < B1 < 1 & \text{& } 0 < B2 < 1 
\end{cases} \]

Av < -2 + b \quad \nu < -1 - Kp(1 + U)

C = 1/4 \rightarrow \text{this note}

Av < -\frac{4}{27}(1 - 7B1 - 7B2 + 13B1B2) \text{&& } \frac{1}{7} < B1 < 1 \text{&& } \frac{1}{7} < B2 < 1 \text{&& } c = \frac{1}{4}

Av < \frac{1}{27}(-4 + 28b - 52B)

0 < U < 1 \&& 0 < A < 1 \&& Kp > 0 \&& \nu < \frac{4}{27}(6 + 6Kp - 7KpU)
3-Term “Look Ahead 2 steps”; $Ki = a.Kp$

$L = v(1 + z + c z^2);$  

$C = 1/4 \rightarrow$ this note

\[ Av \leq 16/147 \quad Av \ (B1,B2) \]
\[ L = \frac{\nu}{1 + z^c} = \text{look ahead} \]

\[ \frac{1}{1+z} = 1 - z + z^2 - z^3 + z^4 - z^5 + z^6 + ... \]

\[ T(z) = 1 + \frac{\nu (1 - z) z}{(B1 - z)(B2 - z)(1 + cz)} \]

\[ T(\Theta) = 1 - \frac{\text{Ave}^i \Theta (-1 + e^{i\Theta})}{(-B1 + e^{i\Theta})(-B2 + e^{i\Theta})(1 + ce^{i\Theta})} \]

\[ \frac{dT}{d\Theta} = 0 \text{ when } \Theta = 0 \text{ or } \pi \]

The conditions \( R = |T| \leq 1 \) and \( d^2R/d\Theta^2 < 0 \) at \( \Theta = 0 \), and \( R \leq 1 \) at \( \Theta = \pi \), lead to

\[ -1 < B2 < 1 \& \& \ -1 < B1 \leq 1 \& \& \ \frac{1 + B1 + B2 - 3B1B2}{-3 + B1 + B2 + B1B2} < c < B1B2 \& \& \ -1 < c < 1 \]

\[ \nu < 1 + B1 + B2 - 3B1B2 - (-3 + B1 + B2 + B1B2)c \]

\[ \nu < 1 + b - 3B + 3c - bc - Bc \]

\[ 0 < U < 1 \& \& \ 0 < A < 1 \& \& \ Kp > 0 \& \& \nu < 2(1 + c + Kp) + Kp(2c - U + cU) \]
STABILITY OF THE 1-POLE SYSTEM WITH ITERATIVE LEARNING CONTROL USING INFINITE SERIES
Proportional Control
Closed Loop Gain Function

\[
P(z) = \frac{-1 + e^U}{-1 - Kp + e^U Kp + e^U z}
\]

Can be written:
\[
\frac{A}{z - B}
\]

U > 0

Stability conditions:
- B > -1 implies
- B > 0 implies
- B < 1 implies
- U = 0 implies

Kp < \frac{1 + e^U}{-1 + e^U}

Kp ≥ -1 & U > 0

B = 1 for all Kp

Numerical Example:
U = 0.0025

B(U, Kp)
L = v/(1+z c) = look ahead

\[ \frac{1}{1+z} = 1 - z + z^2 - z^3 + z^4 - z^5 + z^6 + \ldots \]

\[ T(z) = 1 - \frac{Avz}{(-B + z)(1 + cz)} \]

\[ T(\theta) = 1 - \frac{Ave^{i\theta}}{(-B + e^{i\theta})(1 + ce^{i\theta})} \]

dT/d\theta = 0 when \( \theta = 0 \) or \( \pi \)

The conditions \( R = |T| \leq 1 \) and \( d^2R/d\theta^2 < 0 \) at \( \theta = 0 \) or \( \pi \) lead to two viable ranges of the parameter \( c \).

Sol1: \(-1 < c < 1 && c \leq B < 1\)

- \( Av < 2(1 - B)(1 + c) \)
- \( v < 2(1 + c)(1 + Kp) \)

Sol2: \(-1 < c < 1 && -1 < B \leq c\)

- \( Av < 2(1 + B)(1 - c) \)
- \( v < 2(1 - c)(2/A - 1 - Kp) \)
The largest possible value of $A\nu$ for given $c$ is the solution of

$$\frac{2 - A\nu - 2c}{-2 + 2c} = \frac{2 - A\nu + 2c}{2 + 2c}$$

$$\text{Max}[A\nu] = 2(1 - c)(1 + c)$$

Note that when $c \rightarrow 0$, this reduces (correctly) to the case $L = \nu$
\( L = \nu \text{Exp}[z c] = \text{look ahead} \)

The exponential is expanded as an infinite series in integer powers of \( z \)

\[
e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + ...\]

\[
T(z) = 1 - \frac{\text{Ave}^{cz}z}{-B + z} \quad \quad T(\theta) = 1 - \frac{\text{Ave}^{ce^{i\theta} + i\theta}}{-B + e^{i\theta}}
\]

\[
dT/d\theta = 0 \text{ when } \theta = 0 \text{ or } \pi
\]

The conditions \( R = |T| \leq 1 \) and \( d^2R/d\theta^2 < 0 \) at \( \theta = 0 \) or \( \pi \) lead to three viable ranges of the parameter \( c \).

\[
B = -\frac{c(1 + c)}{1 - c + c^2}
\]

\[
B = \frac{(-1 + c)c}{1 + c + c^2}
\]

If \( c > +1/2 \), then only solution 1 applies alone
If \( c < -1/2 \), then only solution 2 applies alone
If \(-1/2 < c < +1/2\), then both solutions apply; so it becomes complicated
\[ L = v \text{Exp}[z \ c] = \text{look ahead} \]

Sol1: \[ \frac{1}{2} \leq c \leq \text{Cmax} \text{ && } \frac{(-1+c)c}{1+c+c^2} < B < 1 \text{ && } Av < 2(1 - B)e^{-c} \]

\[ Av < 2Ae^{-c}(1 + Kp) \]
\[ v < 2e^{-c}(1 + Kp) \]

Sol2: \[ -\text{Cmax} \leq c \leq -\frac{1}{2} \text{ && } -1 < B < -\frac{c(1+c)}{1-c+c^2} \text{ && } Av < 2(1 + B)e^c \]

\[ Av < -2e^c(-2 + A + AKp) \]
\[ v < 2e^c(2/A - 1 - Kp) \]

Cmax is the solution of \((-6 + c^2 + 2c^3)=0\)
\[ \text{Cmax} = \frac{1}{6} \left(-1 + (323 - 18\sqrt{322})^{1/3} + (323 + 18\sqrt{322})^{1/3}\right) \]
\[ \text{Cmax} \approx 1.2933763343221676 \approx 53/41 \]
\[ L = \nu \text{Exp}[z \cdot c] = \text{look ahead} \]

\[
\text{Sol3: } -\frac{1}{2} < c < \frac{1}{2} \&\& \\
\left( -1 < B < -\frac{c(1 + c)}{1 - c + c^2} \&\& A_v < 2(1 + B)e^c \right) \text{ OR } \left( \frac{(-1 + c)c}{1 + c + c^2} < B < 1 \&\& A_v < 2(1 - B)e^{-c} \right)
\]

Depending on the ranges of \( B \) and \( c \), the limitation on \( A_v \) appears as \text{Sol1} or \text{Sol2}. 
The largest possible value of $A\nu$ for given $c$ is the solution of

$$1 - \frac{A\nu e^c}{2} = -1 + \frac{A\nu e^{-c}}{2}$$

$$\text{Max}[A\nu] = \frac{4e^c}{1 + e^{2c}}$$

Note that when $c \to 0$, this reduces (correctly) to the case $L = \nu$.