Courant-Snyder-like Space-Charge Hamiltonian

Thomas Planche*

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Contents

1 Units

To simplify equation, we will use a natural system of units where $\mathrm{c} = \epsilon_0 = \mu_0 = 1$ (respectively: the speed of light, vacuum permitivity and permeability).

2 Low's Hamiltonian under a single integral

My starting point the relativistic Low's Hamiltonian [1, 2]:

$$H = \int \sqrt{\tilde{m}^2 + (\boldsymbol{\mathcal{P}} - \tilde{q}\mathbf{A}(\mathbf{X}))^2}\,\mathrm{d}^3\!v_1\,\mathrm{d}^3\!x_1 + \int \frac{\boldsymbol{\mathcal{E}}^2}{2} + \frac{(\nabla \times \mathbf{A})^2}{2}\mathrm{d}^3\!x_1\,, \tag{1}$$

with $\tilde{m} = f\,m$, $\tilde{q} = f\,q$, where m is the particle mass, q its charge and $f(\mathbf{x}_1, \mathbf{y}_1, t_1)$ is the plasma density function. $\boldsymbol{\mathcal{P}}$ is a momentum density. $\boldsymbol{\mathcal{P}}$ and $\mathbf{X}$ are function of all dummy variables $\mathbf{x}_1$, $\mathbf{y}_1$, and t_1 while $\mathbf{A}$ and $\boldsymbol{\mathcal{E}}$ are functions of only $\mathbf{x}_1$, and t_1.

*tplanche@triumf.ca

I propose to re-write it like this:

$$H = \int \sqrt{\tilde{m}^2 + (\boldsymbol{\mathcal{P}} - \tilde{q}\mathbf{A}(\mathbf{X}))^2} + \frac{3}{8\pi}\left(\boldsymbol{\mathcal{E}}^2 + (\nabla \times \mathbf{A})^2\right) \,\mathrm{d}^3 v_1 \mathrm{d}^3 x_1 \mathrm{d}t_1 \,. \tag{2}$$

given that $\int \mathrm{d}^3 v_1 = \frac{4\pi}{3}$. Canonical pairs are: $(\mathbf{X}, \boldsymbol{\mathcal{P}})$, $(\mathbf{A}, -\boldsymbol{\mathcal{E}})$, $(t, -\mathcal{H})$ where $\mathcal{H}$ is the Hamiltonian density:

$$H = \int \mathcal{H} \,\mathrm{d}^3 v_1 \mathrm{d}^3 x_1 \mathrm{d}t_1 \,. \tag{3}$$

The canonical Poisson bracket writes:

$$\{F, G\} := \int \left(\sum_i \frac{\delta F}{\delta q_i}\frac{\delta G}{\delta p_i} - \frac{\delta F}{\delta p_i}\frac{\delta G}{\delta q_i} \right) \mathrm{d}^3 v_1 \mathrm{d}^3 x_1 \mathrm{d}t_1 \tag{4}$$

Equations of motions that follow from it are:

$$\begin{aligned}
\dot{\mathbf{X}} &= \{\mathbf{X}, H\} = \frac{\boldsymbol{\mathcal{P}} - \tilde{q}\mathbf{A}(\mathbf{X})}{\sqrt{\tilde{m}^2 + (\boldsymbol{\mathcal{P}} - \tilde{q}\mathbf{A}(\mathbf{X}))^2}} \\
\dot{\boldsymbol{\mathcal{P}}} &= \{\boldsymbol{\mathcal{P}}, H\} = \tilde{q}\nabla(\dot{\mathbf{X}} \cdot \mathbf{A}) \\
\dot{\mathbf{A}} &= \{\mathbf{A}, H\} = -\boldsymbol{\mathcal{E}} \\
\dot{\boldsymbol{\mathcal{E}}} &= \{\boldsymbol{\mathcal{E}}, H\} = \nabla \times \nabla \times \mathbf{A} - \int \tilde{q}\dot{\mathbf{X}} \,\mathrm{d}^3 v_1 \\
\dot{\mathcal{H}} &= \{\mathcal{H}, H\} = 0 \,.
\end{aligned} \tag{5}$$

The last equation is not completely trivial: it emphasizes the fact that we have not yet included external fields (in particular time-varying external fields) into our picture.

3 Transformation to Frenet-Serret coordinates

The Frenet-Serret reference orbit is given by $\mathbf{X} = \mathbf{R}_0(s)$, see Section 3. We also would like the time to be measured w.r.t. the reference particle time $t = t_0(s)$. The new coordinates are (x, y, s, Δt) and we wish to transform to this system:

$$\begin{aligned}
\mathbf{X}(x, y, s) &= x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + \mathbf{R}_0(s) \\
t(\Delta t, s) &= t_0(s) + \Delta t \\
\mathbf{A}(A_x, A_y, A_s) &= A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_s\hat{\mathbf{s}} \,.
\end{aligned} \tag{6}$$

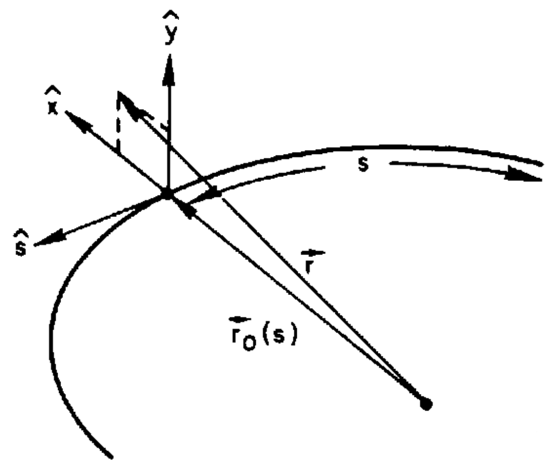

Figure 1: Frenet-Serret coodinates (x, y, s)

We also know that:

$$\begin{aligned}\hat{\mathbf{s}} =& \frac{\mathrm{d}\mathbf{X}_0(s)}{\mathrm{d}s} \\ \frac{\mathrm{d}\hat{\mathbf{x}}}{\mathrm{d}s} =& \frac{\hat{\mathbf{s}}}{\rho} \\ \frac{\mathrm{d}\hat{\mathbf{y}}}{\mathrm{d}s} =& \mathbf{0} \\ \frac{\mathrm{d}\hat{\mathbf{s}}}{\mathrm{d}s} =& -\frac{\hat{\mathbf{x}}}{\rho} \\ \frac{\mathrm{d}t_0}{\mathrm{d}s} =& \frac{1}{\beta_0}\,,\end{aligned} \tag{7}$$

where $\rho(s)$ is the local curvature of the reference orbit; β_0 is the velocity of the reference particle (remember $c = 1$). To change to the Frenet-Serret system let's use as generating function:

$$F_3 = \boldsymbol{\mathcal{P}} \cdot \mathbf{X} - \mathcal{H} \cdot t - \boldsymbol{\mathcal{E}} \cdot \mathbf{A} \tag{8}$$

It is a generating function of the type "new positions old momenta", so requires that derivatives w.r.t. old momenta are the old positions. And it does. Now derivatives w.r.t. new positions give the new momenta:

$$\begin{aligned}
\mathcal{P}_x &= \frac{\partial F_3}{\partial x} = \boldsymbol{\mathcal{P}} \cdot \hat{\mathbf{x}} \\
\mathcal{P}_y &= \frac{\partial F_3}{\partial y} = \boldsymbol{\mathcal{P}} \cdot \hat{\mathbf{y}} \\
\mathcal{P}_s &= \frac{\partial F_3}{\partial s} = (1 + \frac{x}{\rho})\boldsymbol{\mathcal{P}} \cdot \hat{\mathbf{s}} - \frac{\mathcal{H}}{\beta_0} \\
\mathcal{H} &= \frac{\partial F_3}{\partial \Delta t} = \mathcal{H} \\
\mathcal{E}_x &= \frac{\partial F_3}{\partial \mathcal{A}_x} = \boldsymbol{\mathcal{E}} \cdot \hat{\mathbf{x}} \\
\mathcal{E}_y &= \frac{\partial F_3}{\partial \mathcal{A}_y} = \boldsymbol{\mathcal{E}} \cdot \hat{\mathbf{y}} \\
\mathcal{E}_s &= \frac{\partial F_3}{\partial \mathcal{A}_s} = \boldsymbol{\mathcal{E}} \cdot \hat{\mathbf{s}}
\end{aligned} \tag{9}$$

The new Hamiltonian density becomes:

$$\mathcal{H} = \frac{3}{8\pi}\left(\boldsymbol{\mathcal{E}}^2 + (\nabla \times \mathbf{A})^2\right) + \sqrt{\tilde{m}^2 + (\mathcal{P}_x - \tilde{q}\mathcal{A}_x(\mathbf{x}))^2 + (\mathcal{P}_y - \tilde{q}\mathcal{A}_y(\mathbf{x}))^2 + (\frac{\mathcal{P}_s + \frac{\mathcal{H}}{\beta_0}}{1 + \frac{x}{\rho}} - \tilde{q}\mathcal{A}_s(\mathbf{x}))^2}\,. \tag{10}$$

4 Change of independent variable

Eq. (10) can be written as:

$$-\mathcal{P}_s = \frac{\mathcal{H}}{\beta_0} - (1 + \frac{x}{\rho})\left(\tilde{q}\mathcal{A}_s(\mathbf{x}) + \sqrt{\left(\mathcal{H} - \frac{3}{8\pi}(\boldsymbol{\mathcal{E}}^2 + (\nabla \times \mathbf{A})^2)\right)^2 - \tilde{m}^2 - (\mathcal{P}_x - \tilde{q}\mathcal{A}_x(\mathbf{x}))^2 - (\mathcal{P}_y - \tilde{q}\mathcal{A}_y(\mathbf{x}))^2}\right)\,. \tag{11}$$

The Hamiltonian with s as independent variable becomes:

$$H_s = \int -\mathcal{P}_s \,\mathrm{d}^3 v_1 \mathrm{d}^3 x_1 \mathrm{d}t_1\,. \tag{12}$$

5 Equations of motion

$$
\begin{aligned}
x' &= \{x, -\mathcal{P}_s\} = (1 + \frac{x}{\rho})\frac{\mathcal{P}_x - \tilde{q}\mathcal{A}_x(\mathbf{x})}{\sqrt{\cdots}} \\
y' &= \{y, -\mathcal{P}_s\} = (1 + \frac{x}{\rho})\frac{\mathcal{P}_y - \tilde{q}\mathcal{A}_y(\mathbf{x})}{\sqrt{\cdots}} \\
\Delta t' &= \{\Delta t, -\mathcal{P}_s\} = (1 + \frac{x}{\rho})\frac{\mathcal{H}_b}{\sqrt{\cdots}} - \frac{1}{\beta_0} \\
\boldsymbol{\mathcal{P}}' &= \{\boldsymbol{\mathcal{P}}, -\mathcal{P}_s\} = \tilde{q}\nabla(\mathbf{v}\cdot\mathbf{A}) + \sqrt{\cdots}\,\nabla(1 + \frac{x}{\rho}) \\
\mathbf{A}' &= \{\mathbf{A}, -\mathcal{P}_s\} = -\frac{3}{4\pi}(1 + \frac{x}{\rho})\boldsymbol{\mathcal{E}}\int \frac{\mathcal{H}_b}{\sqrt{\cdots}}\,\mathrm{d}^3 v_1 \\
\boldsymbol{\mathcal{E}}' &= \{\boldsymbol{\mathcal{E}}, -\mathcal{P}_s\} = \frac{3}{4\pi}\nabla\times\nabla\times\mathbf{A}\int \frac{\mathcal{H}_b}{\sqrt{\cdots}}\,\mathrm{d}^3 v_1 - \int \tilde{q}\mathbf{v}\,\mathrm{d}^3 v_1 \\
\mathcal{H}' &= \{\mathcal{H}, -\mathcal{P}_s\} = 0
\end{aligned}
\tag{13}
$$

where

$$
\mathbf{v} = \begin{bmatrix} x' \\ y' \\ 1 + \frac{x}{\rho} \end{bmatrix}, \tag{14}
$$

with

$$
\mathcal{H}_b = \mathcal{H} - \frac{3}{8\pi}\left(\boldsymbol{\mathcal{E}}^2 + (\nabla\times\mathbf{A})^2\right), \tag{15}
$$

and

$$
\sqrt{\cdots} = \sqrt{\left(\mathcal{H} - \frac{3}{8\pi}\left(\boldsymbol{\mathcal{E}}^2 + (\nabla\times\mathbf{A})^2\right)\right)^2 - \tilde{m}^2 - (\mathcal{P}_x - \tilde{q}\mathcal{A}_x(\mathbf{x}))^2 - (\mathcal{P}_y - \tilde{q}\mathcal{A}_y(\mathbf{x}))^2}\,. \tag{16}
$$

References

[1] F. E. Low, A lagrangian formulation of the boltzmann-vlasov equation for plasmas, Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences 248 (1253) (1958) 282–287. doi:10.1098/rspa.1958.0244.

[2] T. Planche, P. M. Jung, Relativistic Lagrangian and Hamiltonian description of a beam with space-charge (2016). arXiv:1603.02976.