Fourier expansion of Long Range Beam-Beam Hamiltonian, DRAFT Triumf Note

Dobrin Kaltchev

April 10, 2018

Contents

1 Introduction 1
2 2D Hamiltonian 3
3 Coefficients via two-dimensional Bessel 5
4 Footprint 7
5 Tracking tests (footprint) 8
  5.1 Single Head-on collision 8
  5.2 Single Long-range collision 9
6 Appendix 10

1 Introduction

The two-dimensional coefficients $c_{m,k}$ (resonance basis) in the Fourier expansion of the long-range beam-beam Hamiltonian are expressed through generalized modified Bessel functions depending on two variables, $u$ and $v$, see e.g. [1] and [2]:

$$I_m(u, v) \equiv \sum_{k=-\infty}^{\infty} I_{m-2k}(u)I_k(v).$$

(1)
These $I_m(u, v)$ have properties very similar to the ordinary Bessel functions $I_m(u)$, but are much less familiar. Below, formulas for the generating function, recursion and derivatives are used, derived by simply transforming results from [2] – a paper devoted the very similar two-dimensional Bessel $J$ functions.

To our knowledge, analytic formulas utilizing $I_m(u)$ for a long-range beam-beam potential were first derived by the authors of [3] in one-dimension, i.e. coefficients $c_m$. The same basis was independently used in [4] to construct approximate 1D long-range invariant (the exhibition below follows closely the notations in this paper). Formulas for the tune shift away from resonances, i.e. $c_0$ only, had been previously derived in [8].

In two dimensions, for a round-beam collision point (a.k.a. IP) the expression derived in this paper is:

$$c_{m,k} = \int_0^1 \frac{dt}{t} \left[ \delta(m)\delta(k) - i^{m+k} \prod_z e^{u_2^{(z)} + u_3^{(z)}}  I_j(u_1^{(z)}, u_2^{(z)}) \right].$$

(2)

Here $u_1^{(z)} = ta_z d_z, u_2^{(z)} = -t^2 a_z^2, u_3^{(z)} = -t^2 d_z^2$; $a_z, d_z$ are the normalized particle amplitude and separation. Under the product sign $z = x$ or $y$ and if $z = x$, then $j = m$; if $z = y$, then $j = k$.

The form (2) indicates that one might attempt to find $c_{m,k}$ recursively using the relation

$$u_1 [I_{m-1} - I_{m+1}] + 2u_2 [I_{m-2} - I_{m+2}] = 2m I_m,$$

(3)

where arguments $(u_1^{(z)}, u_2^{(z)})$ of all $I$ are assumed. Sample Mathematica script to test (3) for some randomly chosen $u_1, u_2$ is given in the Appendix.

In case of success, i.e. in case that all $c_{m,k}$ can be expressed through low order ones, there is an interesting consequence. Assume “multipole” is defined in the usual way, but in this new resonance basis. Since $I_m$ for all $m$ can be found given only the first four such functions: $I_m$ for $m = 0, 1, 2, 3$, it appears that the high-order resonance coefficients are not independent. They can be all expressed through the ones of order up to and including order $m=3$ (the redefined “octupole”).

Imagine that as a result of some lumped correction, local, i.e. at this IP, compensation of all terms to order $m = 3$ has taken place. This would mean that all resonance terms are also canceled.

On the other hand, the two derivative properties

$$\frac{\partial}{\partial u_1} I_m = 1/2 [I_{m-1} + I_{m+1}] ; \quad \frac{\partial}{\partial u_2} I_m = 1/2 [I_{m-2} + I_{m+2}]$$

(4)

allow to compute the two-dimensional tune-shifts (footprint) by knowing $I_m$ for $m=0,1$ and 2 (see below).

---

1Remark: the $I_m(u, v)$ also obey a differential equation

2valid also for flat-beam IP, see barred variables below
One may then conclude that, if terms up to order 2 ("sextupole") are locally minimized, then the footprint is reduced. If in addition the \( m = 3 \) term is minimized, then this leads to all resonance terms being small. Such observations have been reported in [5].

## 2 2D Hamiltonian

The Hamiltonian describing a long-range beam-beam kick (or -potential) divided by \( \frac{N_{b}r_{0}}{\gamma} \) can be written as:

\[
H(x, y) = \int_{0}^{\infty} \frac{dq}{\sqrt{q + 2\sigma_{x}^{2}} \sqrt{q + 2\sigma_{y}^{2}}} \left[ 1 - \exp \left( \frac{-(x + D_{x})^{2}}{q + 2\sigma_{x}^{2}} - \frac{(y + D_{y})^{2}}{q + 2\sigma_{y}^{2}} \right) \right],
\]

which, with \( q = 2 \left( -1 + \frac{1}{t} \right) \sigma_{x}^{2}, dq dt = -2\sigma_{x}^{2} t^{2} \) and \( r \equiv \sigma_{y} / \sigma_{x} \), is transformed to

\[
H(x, y) = \int_{0}^{1} \frac{dt}{t \sqrt{1 + (-1 + r^{2}) t}} \left[ 1 - \exp \left( -\frac{t(x + D_{x})^{2}}{2\sigma_{x}^{2}} - \frac{t(y + D_{y})^{2}}{2\sigma_{y}^{2}} \left[ 1 + (-1 + r^{2}) t \right] \right) \right].
\]

Here \( D_{x,y} \) is the real space offset of the collision point in \( x \) or \( y \) direction and \( \sigma_{x} \) and \( \sigma_{y} \) are the ones of the strong beam. This Hamiltonian, when differentiated over \( x \) or \( y \) can be shown to produce the Bassetti-Erskine kick [9].

Let \( d_{x,y} = D_{x,y} / \sigma_{x,y} \) be the normalized offsets – relative separations between the orbits of the colliding bunches. With \( r \equiv \sigma_{y} / \sigma_{x} \), (6) becomes

\[
H(x, y) = \int_{0}^{1} \frac{1 - e^{-\frac{1}{2t} \left[ \frac{2}{\sigma_{x}^{2}}(x_{x} + d_{x})^{2} + r^{2} \frac{2}{\sigma_{y}^{2}}(y_{y} + d_{y})^{2} \right]}}{t \sqrt{1 + (-1 + r^{2}) t}} dt.
\]

For the LHC, in case of a round-beam IR optics, we have the relations \( \sigma_{x}^{w} = \sigma_{y}, \sigma_{y}^{w} = \sigma_{x} ; \sigma_{x,y}^{w} = \sqrt{\epsilon \beta_{x,y}^{w}}, \) with \( \beta_{x,y}^{w} \) being betas at the IP for the weak beam and \( \epsilon \) the emittance, assumed to be the same for weak and strong beam. Introduce action-angle variables \( (J_{x,y}, \phi_{x,y}) \), or alternatively \( (I_{x,y}, \phi_{x,y}) \), with

\[
x = \sqrt{2\beta_{x}^{w}J_{x}} \sin \phi_{x} = \sigma_{x}^{w} \sqrt{2I_{x}} \sin \phi_{x} = \sigma_{x} a_{x} \sin \phi_{x} = r \sigma_{x} a_{x} \sin \phi_{x}
\]

(since, from the definition of \( r, \sigma_{y} = r \sigma_{x} \)). In the same way:

\[
y = \sqrt{2\beta_{y}^{w}J_{y}} \sin \phi_{y} = \sigma_{y}^{w} \sqrt{2I_{y}} \sin \phi_{y} = \sigma_{y} a_{y} \sin \phi_{y} = \frac{\sigma_{y}}{r} a_{y} \sin \phi_{y}
\]
The “$n$-sigma” variable, particle amplitude measured in units of beam size, is $a_z = \sqrt{2J_0/\epsilon} = \sqrt{2T_z}$, ($z=x,y$).

The Hamiltonian (7) is formally expanded in Fourier series:

$$H(x, y) = H(a_x\sigma_x \sin \phi_x, a_y\sigma_y \sin \phi_y) = \int_0^1 \frac{1 - e^{-t(P_x + P_y)}}{t\sqrt{1 + (-1 + r^2) t}} \, dt = \sum_{m=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} c_{m,k} e^{im\phi_x + ik\phi_y},$$

where:

$$P_x = \frac{1}{2} \left( d_x + ra_x \sin \phi_x \right)^2 = \frac{1}{2} \left( \bar{d}_x + \bar{a}_x \sin \phi_x \right)^2$$

$$P_y = \frac{1}{2} \left( d_y + \frac{a_y}{r} \sin \phi_y \right)^2 \frac{r^2}{[1 + (-1 + r^2) t]} = \frac{1}{2} \left( \bar{d}_y + \bar{a}_y \sin \phi_y \right)^2$$

with $g(t) = \sqrt{1 + (r^2 - 1) t}, \bar{d}_x = d_x, \bar{a}_x = ra_x, \bar{d}_y = d_y \frac{r}{g(t)}, \bar{a}_y = a_y \frac{1}{g(t)}$.

We need to find the coefficients:

$$c_{m,k} = \int_0^1 \frac{dt}{t \, g(t)} \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} e^{-im\phi_x - ik\phi_y} \int_0^1 \frac{1 - e^{-t(P_x + P_y)}}{t \, g(t)} \, dt \, d\phi_x d\phi_y$$

with $m,k = \text{integer}$. For this, change the order of integration and re-group

$$c_{m,k} = \int_0^1 \frac{dt}{t \, g(t)} \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} e^{-im\phi_x - ik\phi_y} \left( 1 - e^{-t(P_x + P_y)} \right) d\phi_x d\phi_y.$$  \hspace{1cm} (13)

The unity produces a product of delta-functions that does not contribute to coefficients other than $c_{0,0}$ (for $c_{0,0}$ it turns into 1) $^3$.

$$c_{m,k} = \int_0^1 \frac{dt}{t \, g(t)} \left[ \delta(m)\delta(k) - \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} e^{-im\phi_x - ik\phi_y} e^{-t(P_x + P_y)} \, d\phi_x d\phi_y \right].$$ \hspace{1cm} (14)

with $m,k = 0,1,2,\ldots$

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-im\phi} \, d\phi = \delta(m) \equiv \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m \neq 0. \end{cases}$$
3 Coefficients via two-dimensional Bessel

Rewrite (14) as:

\[
c_{m,k} = \int_0^1 \frac{dt}{t g(t)} \left[ \delta(m) \delta(k) - Q_{m}^{(x)} Q_{k}^{(y)} \right], \tag{15}
\]

Under the integral

\[
Q_{m}^{(x)} = \frac{1}{2\pi} \int_0^{2\pi} e^{-im\phi_x} e^{-tP_x} d\phi_x, \quad Q_{k}^{(y)} = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\phi_y} e^{-tP_y} d\phi_x. \tag{16}
\]

These can be represented as Bessel series [3],[4]. In case of \( Q_{m}^{(x)} \), rewrite the factor of the exponent as

\[
-t P_x = -u_1^{(x)} \sin \phi_x + 2u_2^{(x)} \sin^2 \phi_x + u_3^{(x)}, \tag{17}
\]

\[
-t P_y = -u_1^{(y)} \sin \phi_y + 2u_2^{(y)} \sin^2 \phi_y + u_3^{(y)}, \tag{18}
\]

where

\[
u_1^{(x)} = t\bar{a}_x \bar{d}_x \quad u_1^{(y)} = t\bar{a}_y \bar{d}_y
\]

\[
u_2^{(x)} = -\frac{t}{4} \bar{a}_x^2 \quad u_2^{(y)} = -\frac{t}{4} \bar{a}_y^2
\]

\[
u_3^{(x)} = -\frac{t}{2} \bar{d}_x \quad u_3^{(y)} = -\frac{t}{2} \bar{d}_y
\]

and use the integral representation of the two-dimensional modified Bessel function. In the Appendix it is shown that the result is:

\[
Q_{m}^{(x)}(a_x, d_x, t) = \imath^m e^{-\frac{1}{4}t(2d_x^2+a_x^2)} \sum_{q=-\infty}^{\infty} I_{m-2q}(ta_x d_x) I_q(-\frac{t}{4} a_x^2) =
\]

\[
= \imath^m e^{-\frac{1}{4}t(2d_x^2+a_x^2)} I_m(ta_x d_x, -\frac{t}{4} a_x^2) =
\]

\[
= \imath^m e^{u_2^{(x)}+u_3^{(x)}} I_m(u_1^{(x)}, u_2^{(x)})
\]

The above can be expressed via generalized Bessel \( \Lambda_n(u) = e^{-u}I_n(u) \) in two dimensions \( \Lambda_n(u_1, u_2) = e^{-u_1-u_2}I_n(u_1, u_2) \). For this, notice that

\[
u_2 + u_3 = -t/2(a_x - d_x)^2 - u_1 - u_2,
\]

which follows from

\[-t/2(a_x - d_x)^2 = -t/2a_x^2 + ta_x d_x - t/2d_x^2 = u_1 + 2u_2 + u_3
\]
to rewrite (20) as

\[
Q_m^{(x)}(a_x, d_x, t) = i^m e^{u_2^{(x)} + u_3^{(x)}} I_m(u_1^{(x)}, u_2^{(x)}) = \\
i^m e^{-\frac{t}{2} (a_x - d_x)^2} e^{-u_1 - u_2} I_m(u_1^{(x)}, u_2^{(x)}) = \\
i^m e^{-\frac{t}{2} (a_x - d_x)^2} \Lambda_m(u_1^{(x)}, u_2^{(x)}).
\]

(21)

The expressions for \(y\)

\[
Q_y^{(x)} = Q_m^{(x)}(d_x \to \bar{d}_y, a_x \to \bar{a}_y, m \to k)
\]

are identical to (20) with index \(x\) replaced by \(y\) and \(a_x\) replaced by \(a_y\) in (19).

We finally get the Eqn. 2, already advertised in the Introduction:

\[
c_{m,k} = \int_0^1 \frac{dt}{t g(t)} \left[ \delta(m) \delta(k) - i^{m+k} e^{u_3^{(x)} + u_3^{(y)}} e^{u_2^{(x)} + u_2^{(y)}} I_m(u_1^{(x)}, u_2^{(x)}) I_k(u_1^{(y)}, u_2^{(y)}) \right].
\]

(22)

We used \(\Lambda\) instead of \(I\) then \(a_x = d_x\) means: traverses the strong B core

For round beam \(r = 1\), this expression (14) is symmetric \((x \leftrightarrow y)\) and only one \(Q\) need be computed. We see that in case of a round beam IP, in order to find some coefficient, the Bessel expansion (20) is multiplied by an identical one, the result is substituted in (15), i.e. it is divided by \(t\) and integrated over \(t\) from 0 to 1. It’s easy to see that under the integral there is no singularity at \(t = 0\) for any \(m, k\) (by using the asymptotic of \(I\) for small argument: \(t \to 0\)).

**Check: Coefficients for single-plane head-on as in [7]**

For collision point without offset, \(d_x = 0\) in (20), from the delta property for zero argument \(I_{m-2q}(0) = \delta(m - 2q)\) it follows that only even \(m\) remain. Alternatively, one can use \(I_m(0, u_2) = I_{m/2}(u_2)\).

We get

\[
Q_m^{(x)}(a_x, 0, t) = \begin{cases} 
- e^{-\frac{t}{4} a_x^2} I_0(-\frac{t}{4} a_x^2), & \text{if } m=0 \\
- i^m e^{-\frac{t}{4} a_x^2} I_{m/2}(-\frac{t}{4} a_x^2), & \text{if } m=\text{even} \neq 0 \\
0 & \text{otherwise.}
\end{cases}
\]

(23)

The corresponding single-plane expansion coefficient is

\[
c_{m,0} = \begin{cases} 
\int_0^1 \frac{dt}{t} \left( 1 - e^{-\frac{t}{4} a_x^2} I_0(-\frac{t}{4} a_x^2) \right), & \text{if } m=0 \\
-i^m \int_0^1 \frac{dt}{t} e^{-\frac{t}{4} a_x^2} I_{m/2}(-\frac{t}{4} a_x^2), & \text{if } m=\text{even} \neq 0 \\
0 & \text{otherwise.}
\end{cases}
\]

(24)
and may be found for example in the Lectures of A. Chao [7].

**Coefficients for single-plane long-range**

By setting now \( d_x \neq 0 \) in (20) and using for the y-plane the fact that \( I_0(0, 0) = 1 \), the single-plane long-range expansion coefficient (this time in terms of \( I_0 \)) is

\[
c_{m,0} = \begin{cases} 
  \int_0^1 \frac{dt}{t} (1 - K), & \text{if } m=0 \\
  \int_0^1 \frac{dt}{t} (-K), & \text{if } m \neq 0
\end{cases}
\] (25)

where \( K = i^m e^{-\frac{1}{4}a_x^2}e^{-\frac{1}{2}d_x^2}I_{m,0}(ta_xd_x, -\frac{1}{4}a_x^2). \)

### 4 Footprint

For the footprint we need the partial derivative of \( c_{0,0} \) over the actions, e.g. \( \Delta Q_x = -\frac{1}{2\pi} \frac{\partial}{\partial x} c_{0,0} \), which is expressed through the derivative over amplitude by using \( \frac{d a_x}{d x} = -\frac{1}{\epsilon a_x}. \)

By setting \( m = k = 0 \) in (22) and also replacing the delta-product with unity, we get

\[
c_{0,0} = \int_0^1 dt \left[ 1 - Q_0(a_x, d_x, t) Q_0(a_y, d_y, t) \right];
\]

\[
\frac{\partial}{\partial a_x} c_{0,0} = -\int_0^1 dt \frac{\partial Q_0(a_x, d_x, t)}{\partial a_x} Q_0(a_y, d_y, t);\]

\[
\frac{\partial}{\partial a_y} c_{0,0} = -\int_0^1 dt \frac{\partial Q_0(a_x, d_x, t)}{\partial a_y} Q_0(a_y, d_y, t),
\] (26)

where (all arguments should be with bars for flat)

\[
Q_0(a_x, d_x, t) = e^{u_2^{(x)} + u_3^{(x)}} I_0(u_1^{(x)}, u_2^{(x)}) = e^{-\frac{1}{2}(a_x - d_x)^2} a_0(u_1^{(x)}, u_2^{(x)}),
\] (27)

Recall that \( c_0 \) is in units \( \frac{N_b r_0}{\gamma} \) and denote \( \xi \equiv \frac{N_b r_0}{4\pi\gamma \epsilon} \). The tune-shifts are

\[
\Delta Q_x = 2\xi \frac{1}{a_x} \frac{\partial}{\partial a_x} c_{0,0}; \quad \Delta Q_y = 2\xi \frac{1}{a_y} \frac{\partial}{\partial a_y} c_{0,0}.
\] (28)

The expressions (26), (27), (28) have been programmed in Mathematica and used to compute the footprint (with the derivative \( \frac{\partial Q_0}{\partial a_x} \) in (26) taken numerically).

---

4 To compare with [7] use \( \int_0^\beta (1 - e^{-\alpha}) \frac{d\alpha}{\alpha} = \int_0^1 (1 - e^{-t\beta}) \frac{dt}{t} \)
5 Tracking tests (footprint)

The same derivative follows from the recursive property (4), which allows also to see which indices \( m \) participate in the footprint. The relations (4) become very simple for \( m = 0 \). Because of the symmetry property \( I_k = I_{-k} \) for any integer \( k \), we have \( \frac{\partial}{\partial u_1} I_0 = I_1 \) and \( \frac{\partial}{\partial u_2} I_0 = I_2 \). Then it is easily shown that the derivative in (26) depends on the first three coefficients (as advertised in the Introduction):

5 Tracking tests (footprint)

The expressions (28) have been compared with MadX (dynaptune) using HL-LHC settings with normalized emittance \( \epsilon_{\text{norm}} = 2.5 \) and a number of particles per bunch \( N_b = 1 \times 10^{11} \) (tune-shift per IP \( \xi = 0.00488 \)). The \( \beta^* \) is 15 cm. Tracking is on-momentum with sextupoles OFF.

5.1 Single Head-on collision

Figure 1 shows the comparison with MadX tracking for a single head-on collision in IR5. Footprint parameters are: mtot=10, Amin=1, Amax=10, dA=1, where A is normalized amplitude.

![Figure 1: Green = analytic, Blue = MadX for the case: single Head-on at IP5. The IP5 X-ing angle is set to zero. ; Bessel series taken up to \( q_{\text{max}}=15 \).](image-url)
5.2 Single Long-range collision

Figure 2 shows the comparison with MadX tracking for a single long-range collision. Footprint parameters: \( \text{mtot}=15, \text{Amin}=1, \text{Amax}=10, \text{dA}=1 \)

![Diagram showing comparison between analytic and MadX tracking for single long-range collision.](image)

Figure 2: Green = analytic, Blue = MadX. A single LR in IR5, IP5 X-ing angle = 295/2, giving separation in \( x \) \( d_x=6.23 \) and round beam at this point \( \sigma_x = \sigma_y = 0.1769 \) mm; Bessel series up to \( q_{\text{max}}=25 \)

For the last case the particle initial coordinates are shown on Fig. 3.
6 Appendix

We use
\[ e^{-u_1 \sin \phi} = \sum_{q=-\infty}^{\infty} i^q e^{iq\phi} I_q(u_1), \]
\[ e^{2u_2 \sin^2 \phi} = e^{u_2} \sum_{k=-\infty}^{\infty} (-1)^k e^{2ik\phi} I_k(u_2). \]

For (16) we have:
\[
\frac{1}{2\pi} \int_0^{2\pi} e^{-im\phi} e^{-u_1 \sin \phi} e^{2u_2 \sin^2 \phi + u_3} = \\
= e^{u_2+u_3} \sum_{q,k=-\infty}^{\infty} (-1)^k i^q I_q(u_1) I_k(u_2) \delta(2k + q - m) = \\
= i^m e^{u_2+u_3} \sum_{k=-\infty}^{\infty} I_{m-2k}(u_1) I_k(u_2) = \\
= i^m e^{u_2+u_3} I_m(u_1, u_2). \tag{29}
\]

Alternatively, (16) follows from the generating function:
\[ e^{-u_1 \sin \phi + u_2(2\sin^2 \phi - 1)} = \sum_{k=-\infty}^{\infty} i^k I_k(u_1, u_2) e^{ik\phi}. \]
Code *Mathematica* 1: Recursive calculation of generalized Bessel function of two variables \(I_{two}\) using Eq. (3)

\[\begin{align*}
\text{\textit{(* Define } I_{two} \text{*)}} \\
I_{two}[u1_, u2_, m_] := \\
\text{Sum[} \text{BesselI[m - 2 k, u1]} \text{BesselI[k, u2]}, \{k, -kmax, kmax}\text{]} \\
\text{\textit{(* Choose arbitrary parameters*)}} \\
kmax = 10 ; \\
u1 = 5.678 ; \\
u2 = -3.456 ; \\
\text{\textit{(* Solve (3) to arbitrary order } m \text{ (here 10) for given *)}} \\
\text{\textit{(* the first four } I_{two} 0, 1, 2, 3 \text{ *)}} \\
\text{\textit{(* and print the result in Table a *)}} \\
\text{RecurrenceTable[} \\
\{-2 \text{ u2 a[m + 2] - u1 a[m + 1] ==} \\
\quad 2 \text{ m a[m] - 2 u2 a[m - 2] - u1 a[m - 1]}, \\
\quad a[0] == I_{two}[u1, u2, 0] , \\
\quad a[-1] == I_{two}[u1, u2, 1], \\
\quad a[-2] == I_{two}[u1, u2, 2] , \\
\quad a[-3] == I_{two}[u1, u2, 3] \\
\}, \\
a, \\
\{m, 0, 10\}\text{]} \\
\text{\textit{(* Test some coefficients of order } m>3 \text{*})} \\
I_{two}[u1, u2, 4] \\
I_{two}[u1, u2, 5] \\
I_{two}[u1, u2, 10] \\
\end{align*}\]
\[-0.974148, -1.70009, -0.231208, 0.445237\] 
\[-2.32314\] 
6.23567  
0.445237

References


