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Subject Solenoids as Beam Twisters		

1. Introduction

In recent years beam twisters have come into use in various laboratories. A beam twister provides a mechanism for the interchange (or non-interchange) of horizontal and vertical phase spaces. It is, therefore, ideal to matching a beam line to a spectrometer.

Another report [1] presents the design of a beam twister for beam line 4-B at TRIUMF. The twister consists of a five-quadrupole array; the axes of the quadrupoles are rotated 45° with respect to the orientation of the axes of other quadrupoles of the beam line. It has been pointed out [2] that two solenoids could also be used to interchange phase-space coordinates.

The purpose of this report is to present the results of a study of solenoid twisters and of their usefulness at TRIUMF.

General Properties of Solenoids

A brief discussion of the effects of a solenoid on phase space is given in ref [1]. A somewhat more detailed discussion will be given in this section.

2.1 Transfer Matrix

The transfer matrix of a solenoid can be shown to be [3]

$$\boldsymbol{M_s(k)} = \begin{bmatrix} c^2 & \dfrac{s\,c}{k} & s\,c & \dfrac{s^2}{k} \\ -k\,s\,c & c^2 & -k\,s^2 & s\,c \\ -s\,c & -\dfrac{s^2}{k} & c^2 & \dfrac{s\,c}{k} \\ k\,s^2 & -s\,c & -k\,s\,c & c^2 \end{bmatrix}$$

in which:

$$\begin{aligned} c &= \cos kL \\ s &= \sin kL \\ k &= B/[2\,(B\,\rho)_0] \\ L &= \text{the effective length of the solenoid.} \end{aligned}$$

The quantities B and $(B\,\rho)_0$ are, respectively, the axial field of the solenoid and the magnetic rigidity of the particles that define the central trajectory of the transport system.

2.2 Effect of a Solenoid on Phase Space

If the angle θ is defined by the relation

$$\theta = kL = \frac{B\,L}{2\,(B\,\rho)_0}$$

the expression for $\boldsymbol{M_s}$ may be rewritten as

$$\boldsymbol{M_s(k)} = \boldsymbol{M}\,\boldsymbol{R}$$

where $\boldsymbol{M}$ and $\boldsymbol{R}$ are matrices that are defined as follows:

$$\boldsymbol{M} = \begin{bmatrix} c & \frac{s}{k} & 0 & 0 \\ -k\,s & c & 0 & 0 \\ 0 & 0 & c & \frac{s}{k} \\ 0 & 0 & -k\,s & c \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{R} = \begin{bmatrix} c & 0 & s & 0 \\ 0 & c & 0 & s \\ -s & 0 & c & 0 \\ 0 & -s & 0 & c \end{bmatrix}$$

again with $c = \cos\theta$ and $s = \sin\theta$. The matrix $\boldsymbol{R}$ represents a rotation of the x-y axes through an angle θ about the z-axis. Thus the action of a solenoid may be regarded as a rotation of phase space about the beam axis followed by the action of the matrix $\boldsymbol{M}$ in the rotated system. Note that the matrix $\boldsymbol{M}$ that acts in the rotated system does *not* couple horizontal and vertical motion.

2.3 Focal Properties of a Solenoid

The matrix $\boldsymbol{M}$ has two non-zero submatrices. These may be decomposed as follows.

$$\begin{bmatrix} c & \frac{s}{k} \\ -k\,s & c \end{bmatrix} = \begin{bmatrix} 1 & z_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & z_1 \\ 0 & 1 \end{bmatrix}$$

with

$$f = \frac{1}{k \sin\theta} = \frac{1}{k \sin kL}$$

and

$$z = z_1 = z_2 = f(1 - \cos\theta) = \frac{1 - \cos\theta}{k \sin\theta}$$

Thus each submatrix represents a thin lens of focal length f with a drift of length z on either side. Because the product $k \sin kL$ is always positive, f is always positive. Therefore a solenoid is *always* focussing and focusses *equally* in the horizontal and vertical planes.

2.4 Effect of a Solenoid on the Polarization of a Particle

Banford[3)] shows that in a solenoid particle spin is rotated through an angle ψ defined by

$$\psi = \mu\,\phi$$

with

$$\phi = \frac{B\,L}{(B\,\rho)_0} = 2(kL) = 2\,\theta$$

and μ = the magnetic moment of the particle. Consequently, one can write

$$\psi = 2\mu\,\theta$$

where θ is the angle through which phase space is rotated. It is seen that solenoids rotate both particle phase space and polarization. If, then, a solenoid is used in a beam line for polarized particles, the user must be aware of each of these effects.

3. Matrix for a Solenoid and Drift Spaces

Consider the configuration of a drift of length l_1 preceding a solenoid and one of length l_2 following it. The solenoid has an effective length L_1 and k-value k_1. The transfer matrix for this system is

$$\boldsymbol{T_{211}} \equiv \boldsymbol{T_{l_2 k_1 l_1}} = \boldsymbol{l_2}\, \boldsymbol{M_s(k_1)}\, \boldsymbol{l_1}$$

with $\boldsymbol{l_i}$ being the transfer matrix for a drift of length l_i and $\boldsymbol{M_s}$ is the matrix of section 2.1. Explicitly,

$$\boldsymbol{T_{211}} = \begin{bmatrix} 1 & l_2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} & & & \\ & & & \\ & \boldsymbol{M_s(k_1)} & & \\ & & & \end{bmatrix} \begin{bmatrix} 1 & l_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} A_{12}c_1 & B_{112}c_1 & A_{12}s_1 & B_{112}s_1 \\ -k_1 s_1 c_1 & A_{11}c_1 & -k_1 s_1^2 & A_{11}s_1 \\ -A_{12}s_1 & -B_{112}s_1 & A_{12}c_1 & B_{112}c_1 \\ k_1 s_1^2 & -A_{11}s_1 & -k_1 s_1 c_1 & A_{11}c_1 \end{bmatrix}$$

In the above matrix the following quantities have been defined.

$$c_i = \cos k_i L_i = \cos\theta_i$$

$$s_i = \sin k_i L_i = \sin\theta_i$$

$$A_{ij} = c_i - r_{ij} s_i$$

$$B_{ijk} = \frac{1}{k_i}\left[s_i(1 - r_{ij} r_{ik}) + c_i(r_{ij} + r_{ik})\right]$$

and $r_{ij} = k_i L_j$. The matrix for a system with a drift of length l_3 preceding a solenoid of k-value k_2 and a following drift of length l_4 will be $\boldsymbol{T_{423}}$ but, of course, with different matrix elements than $\boldsymbol{T_{211}}$.

4. Transfer Matrix of a Two Solenoid System

The configuration of interest in this report is that of two solenoids—that is, a system described by the matrix $\boldsymbol{T_{211}}$ followed by one described by the matrix $\boldsymbol{T_{423}}$. The overall transfer matrix of the system, $\boldsymbol{T}$, is given by

$$\boldsymbol{T} = \boldsymbol{T_{423}}\, \boldsymbol{T_{211}}$$

With a bit of manipulation, the transfer matrix $\boldsymbol{T}$ can be written as

$$\boldsymbol{T} = \begin{bmatrix} A\cos\theta & B\cos\theta & A\sin\theta & B\sin\theta \\ -C\cos\theta & D\cos\theta & -C\sin\theta & D\sin\theta \\ -A\sin\theta & -B\sin\theta & A\cos\theta & B\cos\theta \\ C\sin\theta & -D\sin\theta & -C\cos\theta & D\cos\theta \end{bmatrix}$$

in which

$$\begin{aligned}
A &= A_{12}\,A_{24} - k_1\,s_1\,B_{234} \\
&= (c_2 - r_{24}\,s_2)(c_1 - r_{12}\,s_1) - \frac{k_1}{k_2}\,s_1[r_{24}(c_2 - r_{23}\,s_2) + (s_2 + r_{23}\,c_2)] \ , \\
B &= A_{24}\,B_{112} + A_{11}\,B_{234} \\
&= (c_1 - r_{11}\,s_1)[c_2(l_3 + l_4) + \frac{s_2}{k_2}(1 - r_{23}\,r_{24})] + (c_2 - r_{24}\,s_2)[c_1(l_1 + l_2) + \frac{s_2}{k_2}(1 - r_{11}\,r_{12})] \ , \\
C &= k_2\,s_2\,A_{12} + k_1\,s_1\,A_{23} \\
&= k_2\,s_2(c_1 - r_{12}\,s_1) + k_1\,s_1(c_2 - r_{23}\,s_2) \ , \\
D &= A_{11}\,A_{23} - k_2\,s_2\,B_{112} \\
&= (c_1 - r_{11}\,s_1)(c_2 - r_{23}\,s_2) - \frac{k_2}{k_1}\,s_2[r_{11}(c_1 - r_{12}\,s_1) + (s_1 + r_{12}\,c_1)] \ .
\end{aligned}$$

and $\theta = \theta_1 + \theta_2$. The matrix $\boldsymbol{T}$ can then be written as

$$\begin{aligned}
\boldsymbol{T} &= \begin{bmatrix} A & B & 0 & 0 \\ -C & D & 0 & 0 \\ 0 & 0 & A & B \\ 0 & 0 & -C & D \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} A & B & 0 & 0 \\ -C & D & 0 & 0 \\ 0 & 0 & A & B \\ 0 & 0 & -C & D \end{bmatrix} \boldsymbol{R}(\theta)
\end{aligned}$$

where $\boldsymbol{R}(\theta)$ is the matrix for rotation about the z-axis by an angle $\theta = \theta_1 + \theta_2$. From either of these expressions for $\boldsymbol{T}$ one may make the following statements.

1. If $\theta_2 = -\theta_1$, then $\theta = 0$ and

$$\boldsymbol{T} = \begin{bmatrix} A & B & 0 & 0 \\ -C & D & 0 & 0 \\ 0 & 0 & A & B \\ 0 & 0 & -C & D \end{bmatrix} .$$

   Thus, if the solenoids are of equal but opposite polarities such that $\theta_2 = -\theta_1$ (or $k_2\,L_2 = -k_1\,L_1$), then vertical and horizontal phase spaces are *not* interchanged. There is also *no* coupling of the phase spaces.

2. If $\theta = \theta_1 + \theta_2 = (2\,n + 1)\pi/2$ with $(n = 0, 1, \ldots)$, then $\boldsymbol{T}$ becomes

$$\boldsymbol{T} = \begin{bmatrix} 0 & 0 & \alpha A & \alpha B \\ 0 & 0 & -\alpha C & \alpha D \\ -\alpha A & -\alpha B & 0 & 0 \\ \alpha C & -\alpha D & 0 & 0 \end{bmatrix}$$

   where $\alpha = +1$ if $\theta = \pi/2 + 2\,n\,\pi$ and $\alpha = -1$ if $\theta = 3\,\pi/2 + 2\,n\,\pi$.

   In this case, interchange of vertical and horizontal phase spaces occurs. This occurs, of course, because phase space is rotated by an odd multiple of $\pi/2$.

5. Special Cases with C = 0 and B = 0

For optimum use in beam transport one would like to have the off-diagonal elements B and C equal to zero. In this case transport through the system would be with spatial and angular magnifications with magnitudes A and $d(= 1/A)$ respectively. In the preceding section the matrix elements C and B were given as

$$\begin{aligned} B &= A_{11}\,B_{234} + A_{24}\,B_{112} \\ &= (c_1 - r_{11}\,s_1)[c_2(l_3 + l_4) + \frac{s_2}{k_2}(1 - r_{23}\,r_{24})] + (c_2 - r_{24}\,s_2)[c_1(l_1 + l_2) + \frac{s_2}{k_2}(1 - r_{11}\,r_{12})] \ , \end{aligned}$$

$$\begin{aligned} C &= k_2\,s_2\,A_{12} + k_1\,s_1\,A_{23} \\ &= k_2\,s_2(c_1 - r_{12}\,s_1) + k_1\,s_1(c_2 - r_{23}\,s_2) \ . \end{aligned}$$

Clearly, $C = 0$ if

$$c_1 = r_{12}\,s_1 \qquad \text{and} \qquad c_2 = r_{23}\,s_2$$

or

$$\tan\theta_1 = \frac{1}{k_1\,l_2} \qquad \text{and} \qquad \tan\theta_2 = \frac{1}{k_2\,l_3}$$

Similarly, $B = 0$ if

$$\tan\theta_1 = \frac{1}{k_1\,l_1} \qquad \text{and} \qquad \tan\theta_2 = \frac{1}{k_2\,l_4}$$

Consequently, B and C will be zero simultaneously if

$$\tan\theta_1 = \frac{1}{k_1\,l_1} = \frac{1}{k_1\,l_2} \qquad \text{and} \qquad \tan\theta_2 = \frac{1}{k_2\,l_3} = \frac{1}{k_2\,l_4}$$

This requires that $l_1 = l_2$ and $l_3 = l_4$. Under these conditions the values of A and D are

$$A = -\frac{k_1}{k_2}\frac{\sin\theta_1}{\sin\theta_2}$$

$$D = -\frac{k_2}{k_1}\frac{\sin\theta_2}{\sin\theta_1}$$

(It is noted in passing that $A\,D = 1$ as it should be because of Liouville's Theorem.)

It is convenient to use the θ_i and the L_i as independent variables. Then the k_i are defined by

$$k_i = \frac{\theta_i}{L_i}$$

If N is defined by $L_2 = N\,L_1$, then

$$A = -\frac{k_1}{k_2}\frac{\sin\theta_1}{\sin\theta_2} = -N\frac{\theta_1}{\theta_2}\frac{\sin\theta_1}{\sin\theta_2} \ .$$

Figure 1 is a plot of θ_2 versus θ_1 for various values of $-A/N$. In this plot θ_1 and θ_2 have been restricted to the ranges of $0 < \theta_1 \leq \pi/4$ and $0 < \theta_2 \leq \pi/2$.

6. The Unity Transfer Matrix

The expression given above for A, viz:

$$A = -N\frac{\theta_1}{\theta_2}\frac{\sin\theta_1}{\sin\theta_2} \ ,$$

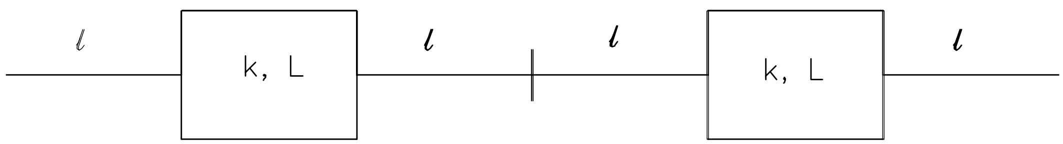
shows immediately that one may obtain a unit transfer matrix for the system. Choosing $N = 1$—that is, choosing the effective lengths of the solenoids to be equal (as would probably be the case in practice)—and choosing $|\theta_2| = \theta_1$, one has

$$A = -1 \qquad \text{and} \qquad D = 1/A = -1\ .$$

The condition that $|\theta_2| = \theta_1$ requires that

$$|k_2 l_3| = k_1 l_1\ .$$

Note: Throughout this section it is assumed that k_1 is positive. But, because $k_i = \theta_i/L_i$, $|\theta_2| = \theta_1$ and $L_2 = L_1$, then $|k_2| = k_1$. Thus one must have $l_3 = l_1$. Setting $L = L_1 = L_2$, $k = k_1 = k_2$ and $l = l_1 = l_2 = l_3 = l_4$, a condition for a unit transfer matrix can be given pictorially as shown below.



Throughout the last two sections it has been implicitly assumed that the vertical and horizontal phase spaces are decoupled. In §4 this was shown to occur if

a) $\theta_2 = -\theta_1$ for non-interchange of axes, or

b) $\theta_1 = \theta_2 = (2n+1)\pi/2$ for interchange of axes.

Because $|\theta_2| = \theta_1$, condition a) is true for all values of θ_1. However, for condition b) to be simultaneously met, one must have $|\theta_2| = \theta_1 = \pi/4$ where it is assumed that the θ_i are kept in the range $0 < \theta_i \le \pi/2$.

The configuration $|\theta_2| = \theta_1 = \pi/4$ is particularly useful because if $\theta_2 = \theta_1 = \pi/4$, then the overall matrix $\boldsymbol{T}$ becomes

$$\boldsymbol{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

whereas, if $-\theta_2 = \theta_1 = \pi/4$, $\boldsymbol{T}$ becomes

$$\boldsymbol{T} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

that is, one can obtain an interchange or a non-interchange of phase spaces by running the solenoids in opposition or in parallel.

7. The effect of two solenoids on polarization

In §2.4 it was shown that in passing through a solenoid, the spin of a particle is precessed by an angle Φ given by

$$\Phi \;=\; 2\,\mu\,\theta \;=\; 2\,\mu\,(\theta_1+\theta_2)$$

If x- and y-phase spaces are *not* interchanged, then $\theta_2=-\theta_1$ and $\Phi=0$. In this case, there is no rotation of the spin of the particle. On the other hand, if the interchange of vertical and horizontal phase spaces is required, then

$$\Phi \;=\; 2\,\mu\,(2n+1)\pi/2 \qquad (n=0,1,\ldots)\ .$$

Thus, unless $\mu=1/2$, it is seen that there cannot be simultaneous spin and phase-space rotations of an odd multiple of $\pi/2$.

8. Example of a solenoid twister

A five-quadrupole twister is to be installed on beam line 4B at TRIUMF. As an example, a solenoid twister will be substituted for the five-quadrupole array.

The twister is to be installed between targets 4BT1 and 4BT2. Table 1 lists TRANSPORT input for the interchange and non-interchange of the horizontal and vertical phase spaces. For convenience, each solenoid was divided into two halves. Table 2 gives the transfer matrices from 4BT1 to 4BT2 for each case. It is seen from the listings that the effective length and field of each solenoid have been taken as 1.4338 m and 39.83448 kG, respectively. Then

$$k \;=\; \frac{B}{2(B\rho)_0} \;=\; \frac{39.83448}{2(33.356\times 1.09007)} \;=\; 0.54777\ \mathrm{m}^{-1}$$

and

$$\theta \;=\; k\,L \;=\; 0.78540\ \text{radian} \;=\; 45^\circ.$$

In order that a unit, uncoupled transfer matrix be obtained, it is also required that $\tan\theta = k\,l$ or

$$l \;=\; \frac{1}{k\tan\theta} \;=\; \frac{1}{k} \;=\; 1.82557\ \mathrm{m},$$

the drift distances either side of the solenoid that are shown in the listings.

Table 3 presents the listing for the beam line with the solenoid twister installed. The case is for vertical dispersion at the 4BT2 target. By simply changing the sign of the field of the second solenoid one can obtain horizontal dispersion at that target. The overall transfer matrices to 4BT1 and 4BT2 targets are given in table 4.

The usefulness of a solenoid twister on beam line 4B is questionable for two reasons. First and foremost is the fact that this beam line operates with polarized beam and, as has been pointed out, a solenoid twister would rotate the beam polarization. A second reason is that to obtain a unit transfer matrix between the two target locations, the distance from 4BT1 to the solenoid downstream is required to be 1.83 m. Existing experimental apparatus at 4BT1 necessitates that this distance be more like 3 m.

9. Discussion

Data presented in this report shows that a beam twister can be designed using solenoids. It has been shown that by adjusting the strengths and separations of the solenoids, varying spatial and angular magnifications may be attained. Specifically, two identical solenoids can be used to give a unit transfer matrix with either an interchange or non-interchange of horizontal and vertical phase spaces. (Note: Some of the configurations indicated in figure 1 require impractical solenoid fields and/or separations.)

On the other hand, it has been shown impossible to obtain a phase-space rotation of 90° and a spin rotation of 90° simultaneously. This makes the system impractical for use with polarized beam. (A quadrupole twister can be rotated to compensate for a solenoid placed upstream (see reference 1). The symmetry of a two-solenoid twister precludes such rotation.)

The two solenoid twister would appear, then, to be a simple two-knob system appropriate to a beam line in which polarized beam is not used or in which the rotation of the beam polarization is of no consequence.

References

1. D. A. Hutcheon and G. M. Stinson, *A twister configuration for beam line 4B*, TRI-DNA-81-2, TRIUMF, 1981.

2. J. Nolen, *Workshop on High Resolution, Large-Acceptance Spectrometers*, Argonne National Laboratory, September 8, 1981.

3. A. P. Banford, *The Transport of Charged Particle Beams*, Spon, 1966.

Table 1(a)

TRANSPORT input for the solenoid twister
for interchange of horizontal and vertical phase spaces

```
'82/02/01 -- 500 MeV -- UNIT TRANSFER MATRIX -- X--Y INTERCHANGE'
0
1.000000 'B ' 0.12700 1.60000 0.66900 0.55600 0.0 0.10000 1.09007;
17. '2ND' ;
3. 1.825573 'DS1A' ;
19. 0.71690 39.83448 'S1A' ;
19. 0.71690 39.83448 'S1B' ;
3. 1.825573 'DS1B' ;
3. 1.825573 'DS2A' ;
19. 0.71690 39.83448 'S2A' ;
19. 0.71690 39.83448 'S2B' ;
3. 1.825573 'DS2B' ;
3. 0.00001 '4BT2' ;
SENTINEL ;
SENTINEL ;
```

Table 1(b)

TRANSPORT input for the solenoid twister
for non-interchange of horizontal and vertical phase spaces

```
'82/02/01 -- 500 MeV -- UNIT TRANSFER MATRIX -- NO X--Y INTERCHANGE'
0
1.000000 'B ' 0.12700 1.60000 0.66900 0.55600 0.0 0.10000 1.09007;
17. '2ND' ;
3. 1.825573 'DS1A' ;
19. 0.71690 39.83448 'S1A' ;
19. 0.71690 39.83448 'S1B' ;
3. 1.825573 'DS1B' ;
3. 1.825573 'DS2A' ;
19. 0.71690 -39.83448 'S2A' ;
19. 0.71690 -39.83448 'S2B' ;
3. 1.825573 'DS2B' ;
3. 0.00001 '4BT2' ;
SENTINEL ;
SENTINEL ;
```

Table 2(a)

Transfer matrix of the solenoid twister
for interchange of horizontal and vertical phase spaces

0.0000	0.0000	−1.0000	0.0000	0.0	0.0
0.0000	0.0000	−0.0002	−1.0000	0.0	0.0
1.0000	0.0000	0.0000	0.0000	0.0	0.0
0.0002	1.0000	0.0000	0.0000	0.0	0.0
0.0	0.0	0.0	0.0	1.0000	0.0
0.0	0.0	0.0	0.0	0.0	1.0000

Table 2(b)

Transfer matrix of the solenoid twister
for non-interchange of horizontal and vertical phase spaces

−1.0000	0.0000	0.0000	0.0000	0.0	0.0
−0.0002	−1.0000	0.0000	0.0000	0.0	0.0
0.0000	0.0000	−1.0000	0.0000	0.0	0.0
0.0000	0.0000	−0.0002	−1.0000	0.0	0.0
0.0	0.0	0.0	0.0	1.0000	0.0
0.0	0.0	0.0	0.0	0.0	1.0000

Table 3

Solenoid twister on beam line 4B
Phase space interchange giving vertical dispersion at 4BT2

```
'82/02/01 -- 500 MeV -- SOLENOID TWISTER -- VERTICAL DISPERSION AT 4BT2'
0
1. 'BEAM' 0.12700 1.60000 0.66900 0.55600 0.0 0.10000 1.09007 ;
17. '2ND ' ;
12. 'CORR' 0.0 0.0 0.0 0.0 0.0 -0.963 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 ;
1. 'FOIL' 0.0 0.659 0.0 0.659 0.0 0.0 0.0 0.0 ;
14. 'R1 ' -0.06689 0.34860 0.0 0.0 0.0 1.27800 1.  ;
14. 'R2 ' -2.94700 0.26320 0.0 0.0 0.0 1.57300 2.  ;
14. 'R3 ' 0.0 0.0 1.06900 0.62430 0.0 0.0 3.  ;
14. 'R3 ' 0.0 0.0 0.25800 1.08600 0.0 0.0 4.  ;
3. '4VM1' 0.31440 ;
3. '  '  0.17060 ;
5. '4VQ1' 0.40640 4.70617 5.08000 ;
3. '  '  0.63020 ;
5. '4VQ2' 0.40640 -3.72192 5.08000 ;
3. '  '  0.69900 ;
3. '  '  0.75750 ;
3. '  '  0.42950 ;
5. '4VQ3' 0.40640 1.66014 5.08000 ;
3. 'SX1I' 0.50000 ;
18. 'SEX1' 0.25000 0.0 10.16000 ;
3. '4VM2' 0.44330 ;
3. '  '  1.08230 ;
20. '  '  180.00000 ;
2. 'A1 ' 17.50000 ;
4. 'M35 ' 1.56290 14.21160 0.0 ;
2. 'A2 ' 17.50000 ;
20. '4VB1' -180.00000 ;
3. 'MID ' 0.27130 ;
3. 'M2IN' 0.27130 ;
20. '  '  180.00000 ;
2. 'A3 ' 12.50000 ;
4. 'M25 ' 1.56520 10.13620 0.0 ;
2. 'A4 ' 12.50000 ;
20. '4VB2' -180.00000 ;
3. '4BM3' 2.75130 ;
3. 'SX2I' 0.50000 ;
18. 'SEX2' 0.25000 0.0 10.16000 ;
3. 'Q4IN' 0.42567 ;
5. '4BQ4' 0.40640 2.40435 5.08000 ;
3. '  '  0.34780 ;
5. '4BQ5' 0.40640 -2.57730 5.08000 ;
3. '4BM4' 0.40890 ;
3. 'SX3I' 0.50000 ;
18. 'SEX3' 0.25000 0.0 10.16000 ;
3. '4BW1' 4.00480 ;
3. '4BM5' 2.92300 ;
3. '4BT1' 0.20000 ;
```

Table 3 (Continued)

```
3. 'DS1A' 1.825573 ;
19. 'S1A ' 0.71690 39.83448 ;
19. 'S1B ' 0.71690 39.83448 ;
3. 'DS1B' 1.825573 ;
3. 'DS2A' 1.825573 ;
19. 'S2A ' 0.71690 39.83448 ;
19. 'S2B ' 0.71690 39.83448 ;
3. 'DS2B' 1.825573 ;
3. '4BT1' 0.00001 ;
3. 'W2T2' 0.660000 ;
3. 'WXIM' 2.500000 ;
3. '4BM8' 7.27810 ;
5. 'BQ11' 0.50000 −4.49766 10.16 ;
3. '  '  0.40000 ;
5. 'BQ12' 0.50000 4.80349 10.16 ;
3. 'WALL' 3.27360 ;
3. 'DUMP' 2.22570 ;
SENTINEL ;
SENTINEL ;
```

Table 4

Beam sizes and transfer matrices at 4VT1 and 4BT2
Interchange of horizontal and vertical phase spaces

Z = 23.541 M ; LABEL = '4BT1'

X =	1.008 CM					
THETA=	1.802 MR	0.146				
Y =	0.493 CM	0.000	0.000			
PHI =	1.090 MR	0.000	0.000	−0.539		
L =	1.792 CM	0.224	−0.931	0.000	0.000	
DELTA=	0.100 PC	−0.992	−0.135	0.000	0.000	−0.234

0.9836	0.0000	0.0000	0.0000	0.0	−10.0001
1.4270	1.0266	0.0000	0.0000	0.0	−2.4250
0.0000	0.0000	−0.6013	−0.7288	0.0	0.0
0.0000	0.0000	2.0622	0.8369	0.0	0.0
−0.8224	−1.0054	0.0	0.0	1.0000	−4.1860
0.0	0.0	0.0	0.0	0.0	1.0000

Z = 33.712 M ; LABEL = '4BT2'

X =	0.493 CM					
THETA=	1.090 MR	−0.539				
Y =	1.008 CM	0.000	0.000			
PHI =	1.802 MR	0.000	0.000	0.146		
L =	1.793 CM	0.000	0.000	0.225	−0.931	
DELTA=	0.100 PC	0.000	0.000	−0.992	−0.135	−0.234

0.0000	0.0000	0.6013	0.7288	0.0	0.0002
0.0000	0.0000	−2.0622	−0.8369	0.0	0.0001
0.9836	0.0000	0.0000	0.0000	0.0	−10.0000
1.4270	1.0266	0.0000	0.0000	0.0	−2.4268
−0.8224	−1.0054	0.0	0.0	1.0000	−4.1860
0.0	0.0	0.0	0.0	0.0	1.0000

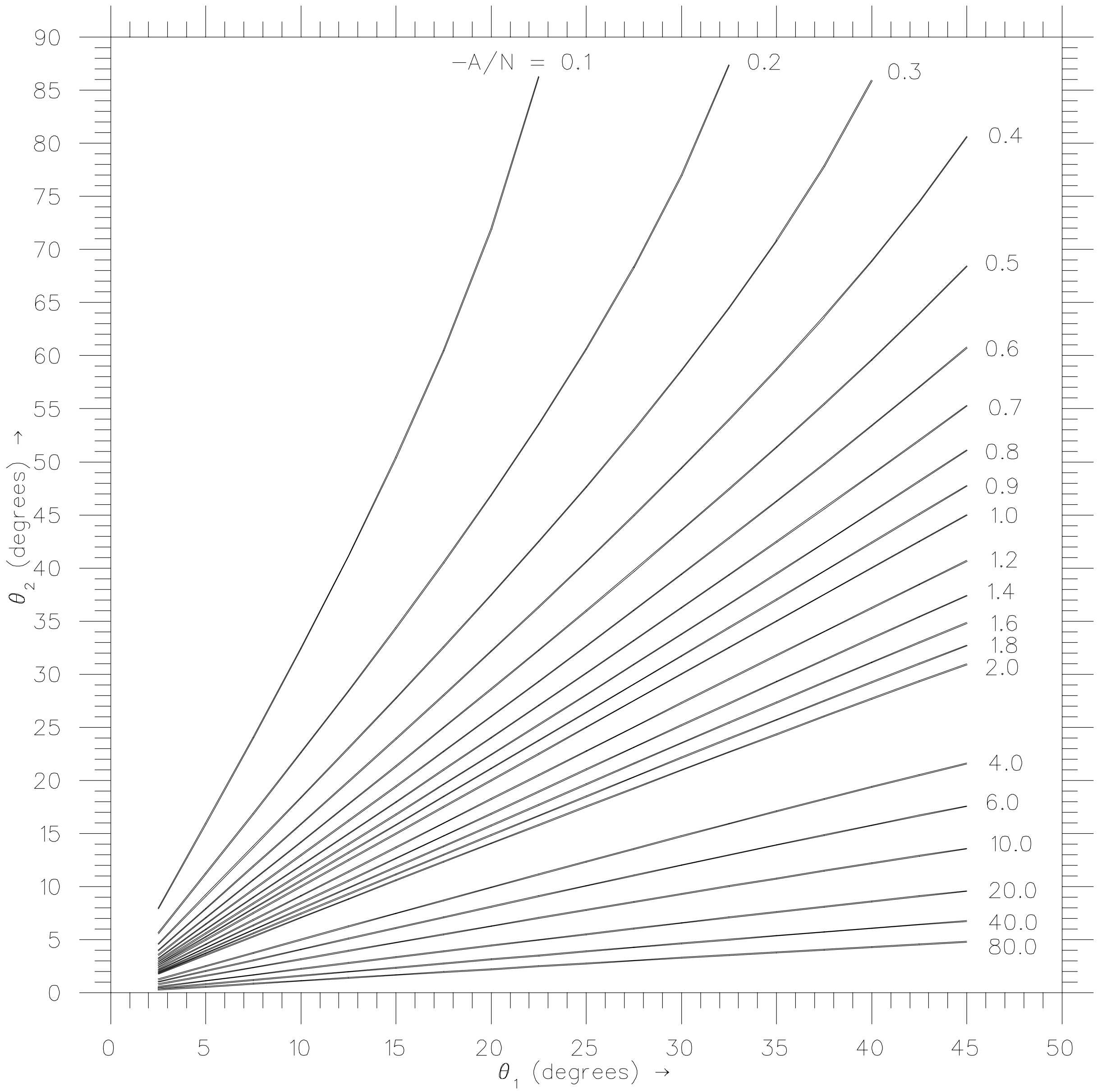


Fig. 1. The variation of θ_2 with θ_1 required to achieve $C = 0$ and $D = 0$.