1. Introduction

In order that personnel involved with beam production at TRIUMF be able to be fully aware of what is happening, it is necessary that they have some knowledge of beam transport theory. Toward this end a series of discussions were held in early 1983 with senior operators. In these discussions the basics of beam transport were outlined. This report is a summary of the information given in those sessions.

The author makes no pretense that the contents of this note are original. In fact, much of what is contained here follows the treatment of the subject by Brown[1]. Other material has been taken from the books of Banford[2], Steffen[3], Septier[4], Carey[5], Wollnik[6] and others listed in the references.

Treatment of the subject of beam transport in this report is done in the matrix formalism. For readers who are not familiar with matrices the information contained in appendix A will be of use. Those familiar with matrices may proceed directly to the next section.

2. The thin lens in matrix notation

In the study of geometrical optics one becomes familiar with diagrams that show image formation by thin lenses. The diagram below is one such diagram for a single focusing lens.

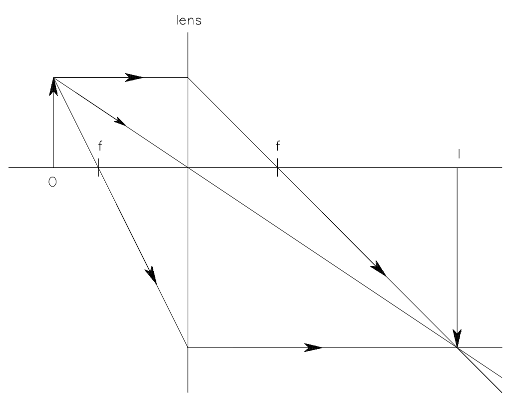

Fig. 1. A conventional ray diagram for focusing by a thin lens.

By definition, a lens is 'thin' if it acts only to change the slope of an incoming ray. It is assumed that the 'height' of the ray immediately before lens action and that immediately after lens action are equal. This assumption is equivalent to ignoring displacement of the ray because of refraction at the entry and exit surfaces. All rays initially parallel to the axis of the lens are bent such that they all intersect the axis at a *focal point*. There are two focal points: one to the right of the lens that corresponds to rays coming from an infinite distance to its left, and one to the left of the lens that corresponds to rays coming from an infinite distance to its right. The distance from the lens' center to the focal point is called the *focal length* and is denoted by f. If an object is located a distance p to the left of a lens of focal length f, then the object position is located from the well-known thin lens equation

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f} .$$

By convention, p is considered positive if the object lies to the left of the lens and q is positive if the image lies to its right. The focal length f is positive if the lens is focusing.

For a matrix treatment of the problem we proceed as follows. Let (x_0, θ_0) be the coordinates of a ray at the object a distance p upstream of a lens of focal length f and $(x_1, \theta_1 = \theta_0)$ be those at the lens before lens action. Following lens action its coordinates are (x_2, θ_2). A distance q downstream of the lens the coordinates of the ray are $(x_3, \theta_3 = \theta_2)$. We wish to find the relationship between (x_3, θ_3) and (x_0, θ_0) and, in particular, that relation at a focus. The situation is illustrated below, noting that by convention, angles are considered positive when measured *counterclockwise* from the axis of the lens.

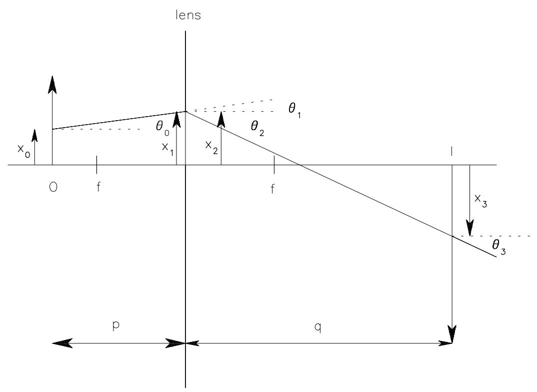

Fig. 2. Definition of coordinates used for a matrix treatment.

2.1 The small angle approximation

In what follows we shall be using the *small angle* or *paraxial ray* approximation. Trigonometric and hyperbolic functions may be expanded in a power series of their arguments. The trigonometric expansions are

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots$$

and, for future reference, the hyperbolic expansions are

$$\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \qquad \cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots$$

where the angle x is expressed in *radians.*

The small angle approximation consists of replacing the function with the first term of these expansions. Thus, for example, we set

$$\sin x = x \qquad \cos x = 1 \qquad \tan x = x$$

where, again, the angle x is expressed in *radians.*

To show that these approximations are not frivolous, consider the evaluation of the trigonometric functions for $x = 0.20$ radian $= 11.459°$. We find

$$\frac{x}{\sin x} = 1.006698 \qquad \frac{1}{\cos x} = 1.020339 \qquad \frac{x}{\tan x} = 0.986631 \ ,$$

that is, the approximation for an angle as large as 11.5° is good to 2% or better. For $x = 0.005$ r $= 5$ mr—a typical divergence in a beam line—we find agreement to better than 0.001%;

$$\frac{x}{\sin x} = 1.00000417 \qquad \frac{1}{\cos x} = 1.00001250 \qquad \frac{x}{\tan x} = 0.99999167 \ .$$

2.2 The transfer matrix for a thin lens

Consider now the transformation from the point (x_1, θ_1) to (x_2, θ_2) in figure 2. By the definition of a thin lens, the height of the ray does not change during the action of the lens. Therefore

$$x_2 = x_1 \tag{1}$$

Also, for small θ_0, θ_1, and θ_2 we have

$$\begin{aligned} \tan\theta_0 \ = \ \tan\theta_1 \ &\approx \ \theta_1 \ = \ x_1/p \\ \tan\theta_2 \ &\approx \ \theta_2 \ = \ -\,x_2/q \ , \end{aligned}$$

so that

$$\theta_2 \ = \ -\,\frac{x_2}{q} = -\,x_2\left[\frac{1}{f} - \frac{1}{p}\right] = -x_1\left[\frac{1}{f}\right] + \frac{x_1}{p} \tag{2}$$

because $x_2 = x_1$. Thus

$$\theta_2 \ = \ -\,\frac{x_1}{f} + \theta_1 \ .$$

In matrix notation, equations 1 and 2 take the form

$$\begin{bmatrix} x_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix} \tag{3}$$

that we write as

$$\boldsymbol{x_2} \ = \ \boldsymbol{R}\cdot\boldsymbol{x_2}$$

The matrix

$$\boldsymbol{R} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \tag{4}$$

is called the *transfer matrix* of the lens and relates the coordinates (x_2, θ_2) immediately to the right of the lens—after lens action—to those immediately to the left of the lens—before lens action.

2.3 The transfer matrix for a drift space

Now consider a region in which there exists neither lenses nor electric or magnetic fields. Such a region is shown in figure 3 on the next page.

In such a region a particle will travel in a straight line. If a particle has coordinates (x_0, θ_0) at the point A and coordinates (x_1, θ_2) at the point B a distance L downstream of A, then we may find the coordinates at B from those at A as follows.

Clearly, the angle of the ray does not change between the two points. Therefore

$$\theta_1 \ = \ \theta_0 \ ,$$

and, from the geometry of the situation, we have

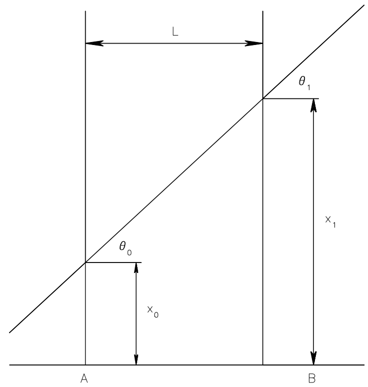

Fig. 3. Ray trajectory in a drift space.

$$\begin{aligned} x_1 &= x_0 + L\tan\theta_1 \\ &= x_0 + L\tan\theta_0 \\ &= x_0 + L|,\theta_0 \end{aligned}$$

where we have again used the small angle approximation. We again rewrite these equations in matrix form to obtain

$$\begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \end{bmatrix} \tag{5}$$

Thus the transfer matrix for a drift space is

$$\boldsymbol{R} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \tag{6}$$

2.4 Image formation by a thin lens – Matrix-wise

Let us return to figure 2 in which an object is placed a distance p upstream of a lens of focal length f. We wish to find the distance q downstream of the lens where the image is located. Thus we require the relationship between (x_3, θ_3) and (x_0, θ_0) and, in particular, that relation at a focus. We trace back from the image point as follows.

$$\begin{aligned} \begin{bmatrix} x_3 \\ \theta_3 \end{bmatrix} &= \begin{bmatrix} \text{Drift} \\ \text{length} \\ q \end{bmatrix} \begin{bmatrix} x_2 \\ \theta_2 \end{bmatrix} \\ &= \begin{bmatrix} \text{Drift} \\ \text{length} \\ q \end{bmatrix} \begin{bmatrix} \text{Lens} \\ \\ \text{action} \end{bmatrix} \begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix} \\ &= \begin{bmatrix} \text{Drift} \\ \text{length} \\ q \end{bmatrix} \begin{bmatrix} \text{Lens} \\ \\ \text{action} \end{bmatrix} \begin{bmatrix} \text{Drift} \\ \text{length} \\ p \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \end{bmatrix} \end{aligned} \tag{7}$$

where the appropriate matrices have been inserted into equation 7. Doing the matrix multiplication yields

$$\begin{bmatrix} x_3 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & p \\ -\frac{1}{f} & 1-\frac{p}{f} \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\frac{q}{f} & q+p\left(1-\frac{q}{f}\right) \\ -\frac{1}{f} & 1-\frac{p}{f} \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \end{bmatrix} \tag{8}$$

or

$$\begin{bmatrix} x_3 \\ \theta_3 \end{bmatrix} = \boldsymbol{R} \begin{bmatrix} x_0 \\ \theta_0 \end{bmatrix} \tag{9}$$

where $\boldsymbol{R}$ is the overall transfer matrix from the object to the image.

Now we ask "What does a focus mean?" At a focus all rays emanating from any one point on the object are refocused to the corresponding point of the image. That is, the *final position* must be *independent* of the *initial divergence*. This requires that the R_{12} matrix element be zero. Thus

$$q+p\left(1-\frac{q}{f}\right) = 0$$

which can be rewritten as

$$\frac{1}{p}+\frac{1}{q} = \frac{1}{f}\,, \tag{10}$$

that is, the (standard) thin lens formula. Using this relationship, the matrix transformation between the object and the image becomes

$$\begin{bmatrix} x_3 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} -\frac{q}{p} & 0 \\ -\frac{1}{f} & 1-\frac{p}{q} \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \end{bmatrix}.$$

This shows that at a focus

$$\frac{x_3}{x_0} = -\frac{q}{p}$$

which is the well-known magnification from geometrical optics. It follows that in a *focus-to-focus* situation the magnification is given by the R_{11} matrix element.

3. Thick lenses and principal planes

The above has been based on the assumption that the lens was thin—that is, its thickness could be neglected and refraction at its entry and exit surfaces could be neglected. If this is not the case then the prescription used above cannot be used.

It is not the purpose of this note to give a detailed account of geometric optics, but we will give an overview of the treatment of thick lenses in what follows and then show its application to beam optics.

The upper portion of figure 4 shows the treatment of a thick lens when each surface is treated independently.

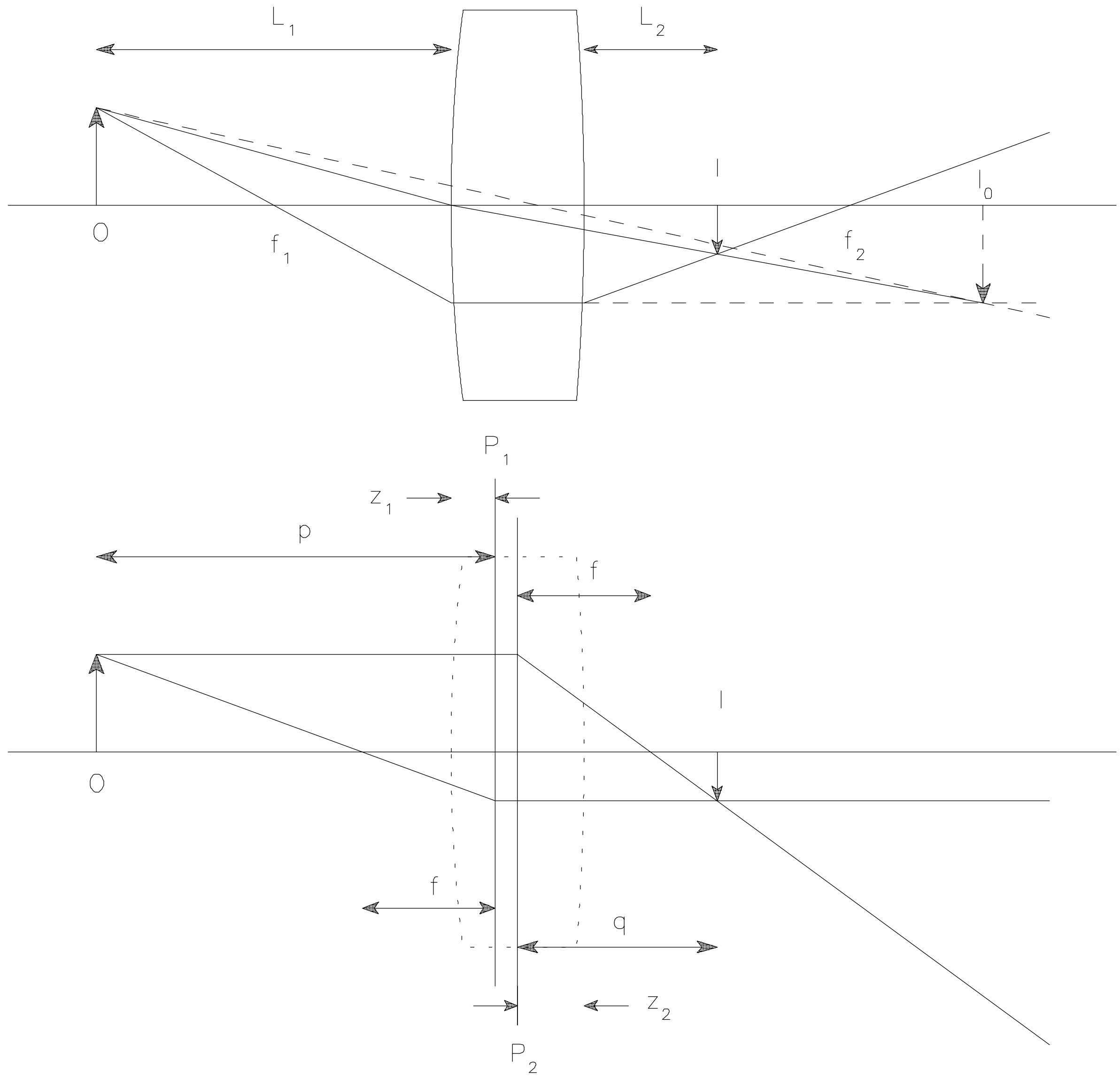

Fig. 4. Ray tracing for a thick lens: each surface treated independently (top) and the principal plane method (bottom).

The entry face of the lens has a radius of curvature of R_1 and that of the exit face is R_2. Both R_1 and R_2 are taken as positive if their respective faces are *convex* to incident light. Thus, R_1 is positive and R_2 is negative in figure 4.

Recall that the focal lengths of the two surfaces are determined by their radii of curvature. If n is the index of refraction of the lens relative to that of the surrounding medium, the the focal lengths of the entry surface f_1 and that of the exit surface f_2 are given by

$$f_1 = \frac{R_1}{n-1} \qquad \text{and} \qquad f_2 = \frac{R_2}{1-n} \; .$$

In figure 4, the upper dashed line is a ray that leaves an object a distance L_1 from the entrance surface and passes through the center of curvature of that surface. Because it enters the lens normal to its surface this ray is not refracted. If the lens extended to the right this ray would trace the trajectory shown. The lower solid ray is one that passes through f_1 and, consequently, is refracted to be parallel to the lens' axis. Again, if the lens path. At the intersection of these two rays an image would be formed at the point labeled I_0. This image acts as the object for the exit face of the lens.

The upper solid ray of figure 4 leaves the object and passes through the center of curvature of the exit face of the lens (as drawn, this corresponds to the point of intersection of the entrance face and the lens' axis). Again, this ray is normal to the exit face and is not refracted there; it continues as shown. However, as far as the exit face is concerned, the ray that passed through the focal point f_1 is now parallel to the axis. Consequently, this ray is refracted to pass through the downstream focal point f_2—as indicated by the solid line. At the point of intersection of these two rays the final image is formed at the point I, a distance L_2 downstream of the exit face. In general, we find that

$$\frac{1}{L_1} + \frac{1}{L_2} \neq \frac{1}{f}$$

where f is neither f_1 nor f_2.

The lower portion of figure 4 shows how we would like to treat this problem. Our wish is to find two planes, the *principal planes*, P_1 and P_2 and an *equivalent* focal length f such that if we measure with respect to the principal planes we can treat the problem using the thin-lens formula. As indicated in figure 4, suppose that P_1 is located a distance z_1 from the entrance face and P_2 is located a distance z_2 from the exit face of the lens. We take z_1 to be *positive* if it lies to the *right* of the entrance face and z_2 to be *positive* if it lies to the *left* of the exit face. We find that the *second* principal plane is positioned at the intersection of the incoming and outgoing ray such that the outgoing ray intersects the lens' axis at the downstream focal point. Similarly, an incident ray through the upstream focal point intersects the *first* principal plane and exits parallel to the axis of the lens. We want to find the quantities z_1, z_2 and f such that if we write

$$p \; = \; L_1 + z_1 \qquad \text{and} \qquad q \; = \; L_2 + z_2 \; ,$$

we can also write

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f} \; .$$

We must now develop a prescription by which these parameters may be determined. Let the matrix $\boldsymbol{R}$ be the transformation from immediately outside the entrance face of the lens to immediately outside its exit face. We wish to replace this matrix with two drift lengths and a thin lens—that is, we wish to write

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} 1 & z_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(1/f) & 1 \end{bmatrix} \begin{bmatrix} 1 & z_1 \\ 0 & 1 \end{bmatrix} \tag{11}$$

Doing the matrix multiplication and equating individual matrix elements yields

$$z_1 \; = \; \frac{R_{22} - 1}{R_{21}} \qquad\qquad z_2 \; = \; \frac{R_{11} - 1}{R_{21}} \tag{12}$$

and

$$\frac{1}{f} \; = \; -R_{21} \tag{13}$$

In the above the matrix $\boldsymbol{R}$ was arbitrary. Consequently, if we know the transfer matrix for a system, equations (12) and (13) give a method for finding the principal planes and equivalent thin lens with which that system may be replaced.

Rather than explicitly indicate the principal plane formalism for a thick lens, an example of its use for a combination of two thin lenses will be given.

Example:
An object is placed 30 units to the left of a lens of focal length +15 units. A second lens of focal length 20 units is placed 10 units beyond the first. Determine the position and magnification of the final image.

This problem will be solved with the three methods given so far—that is,

a) the thin lens equations,
b) the matrix method,
c) the principal plane technique.

In what follows the subscripts '1' and '2' refer to properties associated with the first and second lenses respectively.

a) Thin lens approach

We first find the position of the image produced by the first lens. With $p_1 = 30$ and $f_1 = 15$ the thin-lens equation gives

$$\frac{1}{q_1} = \frac{1}{f_1} - \frac{1}{p_1} = \frac{1}{15} - \frac{1}{30} = \frac{1}{30}$$

or $q_1 = 30$ units. This image becomes the object for the second lens and

$$p_2 = \text{lens separation} - q_1 = 10 - 30 = -20 \text{ units} .$$

so that

$$\frac{1}{q_2} = \frac{1}{f_2} - \frac{1}{p_2} = \frac{1}{20} - \frac{1}{-20} = \frac{1}{10}$$

Thus the final image is located 10 units downstream of the second lens. The overall magnification of the system is found from

$$\begin{aligned} M &= (\text{magnification of first lens})(\text{magnification of second lens}) \\ &= \frac{-q_1}{p_1} \cdot \frac{-q_2}{p_2} \\ &= \frac{-30}{30} \cdot \frac{-10}{-20} \\ &= -\frac{1}{2} \end{aligned}$$

b) Matrix method

Here we note that the problem consists of a drift length p_1, a lens of focal length f_1, a drift length d_1 corresponding to the separation of the lenses, another lens of focal length f_2, and a final drift of length q_2 to the image. In matrix notation, writing $F_1 = 1/f_1$ and $F_2 = 1/f_2$, this configuration is written as

$$\begin{aligned} \begin{bmatrix} x_2 \\ \theta_2 \end{bmatrix} &= \begin{bmatrix} 1 & q_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -F_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -F_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & p_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & q_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 - d_1F_1 & d_1 \\ -f_1 - F_2(1 - d_1F_1) & 1 - d_1F_2 \end{bmatrix} \begin{bmatrix} 1 & p_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \end{bmatrix} \\ &= \begin{bmatrix} (1 - d_1F_1)(1 - q_2F_2) - q_2F_1 & p_1 + d_1 + q_2 - p_1f_1(d_1 + q_2) - q_2F_2(p_1 + d_1) + d_1p_1q_2F_1F_2 \\ -F_1 - F_2(1 - d_1f_1 & (1 - d_1F_2)(1 - p_1F_1) - p_1F_2 \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \end{bmatrix} \\ &= \begin{bmatrix} (1/3) - (q_2/12) & 30 - 3q_2 \\ -(1/12) & -2 \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \end{bmatrix} \end{aligned}$$

where we have inserted the appropriate numerical values.

In order that we have a focus it is necessary that x_2 be independent of θ_0. Thus $R_{12} = 30 - 3q_2 = 0$ or

$$q_2 = 10 \text{ units} .$$

With this value of q_2 then

$$R_{11} = (1/3) - (10/12) = -(1/2)$$

so that

$$x_2 = \quad = -\frac{1}{2}x_0 .$$

Thus the final image lies 10 units downstream of the second lens and is magnified by a factor of −0.5—as determined using the thin-lens equations.

c) Principal plane approach

Here we consider the two lenses and separating drift spaces as a thick lens. The transfer matrix for this system was found in b) above to be

$$\begin{bmatrix} 1 & 0 \\ -F_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -F_1 & 1 \end{bmatrix} = \begin{bmatrix} 1 - d_1F_1 & d_1 \\ -f_1 - F_2(1 - d_1F_1) & 1 - d_1F_2 \end{bmatrix} .$$

Using the definitions given in equations (12) and (13) we find

$$-\frac{1}{f} = -\frac{1}{12} , \qquad z_1 = \frac{[1-(10/20)]-1}{(-1/12)} = 6 , \qquad z_2 = \frac{[1-(10/15)]-1}{(-1/12)} = 8 .$$

Then

$$p = L_1 + z_1 = 30 + 6 = 36$$

and

$$\frac{1}{q} = \frac{1}{f} - \frac{1}{p} = \frac{1}{12} - \frac{1}{36} = \frac{1}{18} .$$

Thus the image is $q = 18$ units to the right of the second principal plane. To calculate its location relative to the second lens we have

$$L_2 = q - z_2 = 18 - 8 = 10 \text{ units,}$$

the same location as before. The magnification is obtained from

$$M = -\frac{q}{p} = -\frac{18}{36} = -2 ,$$

that, again, is in agreement with that obtained using the other approaches. It is left as an exercise to draw ray diagrams for each of the thin-lens and principal plane approaches.

Exercise: Repeat the above for the following problem.
Two lenses, each of focal length 2 units, are placed 10 units apart. An object is placed 5 units in front of the first lens. Find the position and magnification of the image. Draw a ray diagram for the thin lens and the principal plane approaches.

4. Quadrupoles and quadrupole arrays

It is not the purpose of this report to derive formulae for the calculation of particle trajectories through magnetic elements. However, in Appendix Q the (first-order) transfer matrix through a quadrupole is derived from the trajectory equations. This has been done to indicate the process involved. This section will take the quadrupole transfer matrix and examine it in order to show the similarities between it and the geometric optics that have been discussed above.

Appendix Q gives the transfer matrix of a horizontally-focusing quadrupole as

$$\boldsymbol{R}_Q = \begin{bmatrix} \cos\theta & \dfrac{\sin\theta}{k} & 0 & 0 \\ -k\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cosh\theta & \dfrac{\sinh\theta}{k} \\ 0 & 0 & k\sinh\theta & \cosh\theta \end{bmatrix} \tag{14}$$

in which

$$\begin{aligned} L &= \text{effective length in m of the quadrupole,} \\ \theta &= kL, \\ k^2 &= (B_0/a)/(B\rho)_0, \\ (B\rho)_0 &= \text{the magnetic rigidity of the particle,} \\ B_0 &= \text{the pole-tip field of the quadrupole,} \\ a &= \text{the radius in m of the quadrupole aperture.} \end{aligned}$$

The transfer matrix of a vertically-focusing quadrupole has the two non-zero sub-matrices of equation (14) interchanged.

This 4×4 matrix transforms an initial coordinate $\boldsymbol{x_0} = (x_0, \theta_0, y_0, \phi_0)$ into the coordinate $\boldsymbol{x_1} = (x_1, \theta_1, y_1, \phi_1)$. Notice that the (x, θ) and the (y, ϕ) coordinates are completely decoupled in the matrix equation $\boldsymbol{x}_1 = \boldsymbol{R}_Q\, \boldsymbol{x}_0$. It is also important to notice that unlike an optical thin lens, a quadrupole focuses in one plane and defocuses in the orthogonal plane. Thus if a quadrupole focuses horizontally, it also defocuses vertically.

In the following quadrupoles will be considered individually, in pairs as doublets, and in threes as triplets. The characteristics of each grouping will be discussed.

4.1 Quadrupole singlets

As is indicated in Appendix Q, the 2×2 matrix that represents the action of a quadrupole in its focusing plane is

$$\begin{bmatrix} \cos\theta & \dfrac{\sin\theta}{k} \\ -k\sin\theta & \cos\theta \end{bmatrix}$$

Using the principal plane theory that we have developed, we may replace this matrix with a thin lens and two drift spaces. Using the superscript '+' to indicate that we are dealing with the focusing plane of the quadrupole, these matrices are related such that

$$\begin{bmatrix} \cos\theta & \dfrac{\sin\theta}{k} \\ -k\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & z_2^+ \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(1/f^+) & 1 \end{bmatrix} \begin{bmatrix} 1 & z_1^+ \\ 0 & 1 \end{bmatrix} .$$

From the above and equation (13) it follows that

$$f^+ = \frac{1}{k\sin\theta} = \frac{1}{k\sin kL} . \tag{15}$$

Similarly we find the location of the principal planes P_1^+ and P_2^+ relative to the ends of the quadrupole from the values of z_1^+ and z_2^+. We find

$$z_1^+ = z_2^+ = \frac{\cos kL - 1}{-k\sin kL} \tag{16}$$

From Appendix Q, the 2×2 matrix that represents the action of a quadrupole in its defocusing plane is

$$\begin{bmatrix} \cosh\theta & \dfrac{\sinh\theta}{k} \\ k\sinh\theta & \cosh\theta \end{bmatrix}$$

from which, using the same technique and using the superscript '$-$' to indicate we are dealing with the defocusing plane of the quadrupole, we find

$$f^- = -\frac{1}{k\sinh kL} \tag{17}$$

and

$$z_1^- = z_2^- = \frac{\cosh kL - 1}{k\sinh kL} \ . \tag{18}$$

The above expressions for $f^\pm$, $z_1^\pm$, and $z_2^\pm$ are completely general. However, it is of interest to consider the case when $\theta = kL \ll 1$. In this case we may expand the trigonometric and hyperbolic functions in a power series of their arguments. For the trigonometric functions we have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\operatorname{cosec} x = \frac{1}{\sin x} = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \frac{31x^5}{15120} + \cdots$$

and for the hyperbolic functions we have

$$\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \qquad \cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

$$\operatorname{cosech} x = \frac{2}{e^x - e^{-x}} = \frac{1}{x} - \frac{x}{6} + \frac{7x^3}{360} - \frac{31x^5}{15120} + \cdots$$

Using these expansions we then have in the focusing plane of the quadrupole

$$\begin{aligned} f^+ &= \frac{\operatorname{cosec}\theta}{k} \\ &= \frac{1}{k}\left[\frac{1}{\theta} + \frac{\theta}{6} + \frac{7\theta^3}{360} + \frac{31\theta^5}{15120} + \cdots\right] \\ &\approx \frac{1}{k}\left[\frac{1}{\theta} + \frac{\theta}{6}\right] \\ &= \frac{1}{k^2L} + \frac{L}{6} \end{aligned} \tag{19}$$

where we have kept only the first two terms of the expansion. Expressions are found for the positions of the principal planes in a similar manner. We obtain

$$\begin{aligned} z_1^+ = z_2^+ &= \frac{\cos\theta - 1}{-k\sin\theta} \\ &= -\frac{1}{k}\left[-\frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right]\left[\frac{1}{\theta} + \frac{\theta}{6} + \frac{7\theta^3}{360} + \cdots\right] \\ &\approx -\frac{1}{k}\left[-\frac{\theta^2}{2!} + \left[\frac{1}{4!} - \frac{1}{6(2!)}\right]\theta^3\right] \\ &= \frac{L}{2}\left[1 + \frac{k^2L^2}{12}\right] \ . \end{aligned} \tag{20}$$

Similarly, in the defocusing plane we find

$$\begin{aligned} f^+ &= \frac{\operatorname{cosech}\theta}{k} \\ &= -\frac{1}{k}\left[\frac{1}{\theta} - \frac{\theta}{6} + \frac{7\theta^3}{360} - \frac{31\theta^5}{15120} + \cdots\right] \\ &\approx -\frac{1}{k}\left[\frac{1}{\theta} - \frac{\theta}{6}\right] \\ &= -\frac{1}{k^2L} + \frac{L}{6} \end{aligned} \tag{21}$$

and

$$\begin{aligned} z_1^- = z_2^- &= \frac{\cosh\theta - 1}{-k\sinh\theta} \\ &= \frac{1}{k}\left[\frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \cdots\right]\left[\frac{1}{\theta} - \frac{\theta}{6} + \frac{7\theta^3}{360} + \cdots\right] \\ &\approx \frac{1}{k}\left[\frac{\theta}{2!} + \left[\frac{1}{4!} - \frac{1}{6(2!)}\right]\theta^3\right] \\ &= \frac{L}{2}\left[1 - \frac{k^2L^2}{12}\right]. \end{aligned} \tag{22}$$

4.1.1 First-order approximation

If only the first term is kept in each of the above expansions we have what is known as the *first-order* approximation for the quadrupole. In this case we have

$$f^+ = -f^- = f^0 = \frac{1}{k^2L} = \frac{(B\rho)_0}{(B_0/a)L} = \frac{(B\rho)_0}{gL} \tag{23}$$

and

$$z_1^+ = z_2^+ = z_1^- = z_2^- = z_0 = \frac{L}{2} \tag{24}$$

where $g = B_0/a$ is the gradient of the quadrupole field. Thus, to first order, a quadrupole may be treated as a thin lens at its geometric center with drift lengths on either side of length equal to one-half of the *effective* length of the quadrupole. Notice that in this approximation the focal lengths in the focusing and the defocusing planes have the same absolute value f^0; in the focusing plane f^0 is taken as positive whereas it is taken as negative in the defocusing plane. Figure 5 is a sketch for a quadrupole of length L.

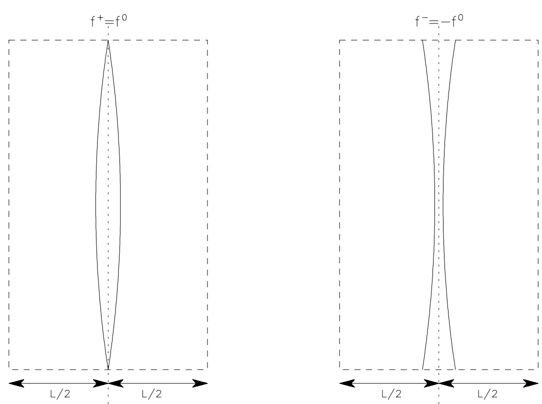

Fig. 5. First-order approximation of a quadrupole in focusing plane (left) and in defocusing plane (right).

At TRIUMF, for example, the parameters characterizing the standard 4-inch quadrupole 4Q14/8 are

$$L = 0.4090 \text{ m}, \qquad a = 0.0508 \text{ m}, \qquad B_0(\text{max}) = 8 \text{ kG} .$$

Thus

$$g = \frac{B_0}{a} = \frac{8}{0.0508} \text{ kG/m}$$

and at 500 MeV

$$(B\rho)_0 = 36.36 \text{ kG-m},$$

so that

$$f^0 \approx \frac{36.36}{(8/0.0508)} \frac{1}{0.4090} = 0.57 \text{ m}$$

and

$$z_0 = \frac{L}{2} = 0.2045 \text{ m} .$$

For most cases of beamline work at TRIUMF this first-order treatment of quadrupole is adequate. The next section considers the second-order approximation to quadrupoles and may be omitted without loss of continuity.

4.1.2 Second-order approximation

The *second-order* approximation to a quadrupole is obtained when the first two terms of the expansions of the focal length and principal plane locations are kept. This yields

$$f^{\pm} = \pm \frac{1}{k^2 L} + \frac{L}{6} = \pm f^0 + \frac{L}{6} . \tag{25}$$

Equation (25) shows that in its *focusing* plane the focal length of a quadrupole is *longer* than that of a thin lens of focal length f^0. Consequently, the focusing power of a quadrupole is less than a thin lens of a thin lens of focal length f^0. Conversely, a quadrupole has a stronger defocusing action in its defocusing plane than does a defocusing lens of focal length f^0.

Similarly, we find the positions of the principal planes from

$$z_1^{\pm} = z_2^{\pm} = \frac{L}{2}\left[1 \pm \frac{k^2 L^2}{12}\right] = \frac{L}{2}\left[1 \pm \frac{L}{12 f^0}\right] \tag{26}$$

To second order a quadrupole may be represented as shown in figure 6.

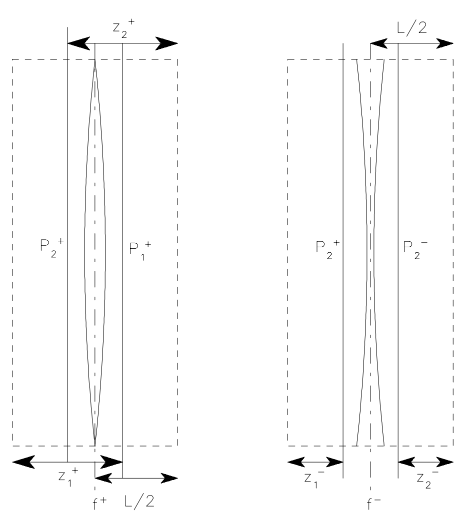

Fig. 6. Second-order approximation of a quadrupole: focusing plane (left) and defocusing plane (right).

4.2 Quadrupole doublets

We now consider the combination of two quadrupoles, one that focuses in the horizontal plane and one that focuses in the vertical plane. Assume that the effective length of each quadrupole is $2L$ and that the separation between their effective edges is d. We shall assume that the first quadrupole focuses horizontally and the second focuses vertically.

We shall examine this configuration using the thin-lens approximation for the quadrupoles. In §4.1.1 we have seen that the focal lengths of a given quadrupole in its focusing and defocusing planes are of equal magnitude but of opposite sign. Thus for the first quadrupole

$$f_1^+ \;=\; -\,f_1^- \;=\; f_1\;,$$

and for the second quadrupole

$$f_2^- \;=\; -\,f_2^+ \;=\; f_2\;.$$

The convention adapted here is that the numeric subscript designates the quadrupole and the superscript refers to the focusing (+) or defocusing (−) plane of that particular quadrupole. Then, if distances upstream of the doublet are measured to the center of the first quadrupole and those downstream are measures from the center of the second quadrupole, the doublet may be represented as shown in figure 7.

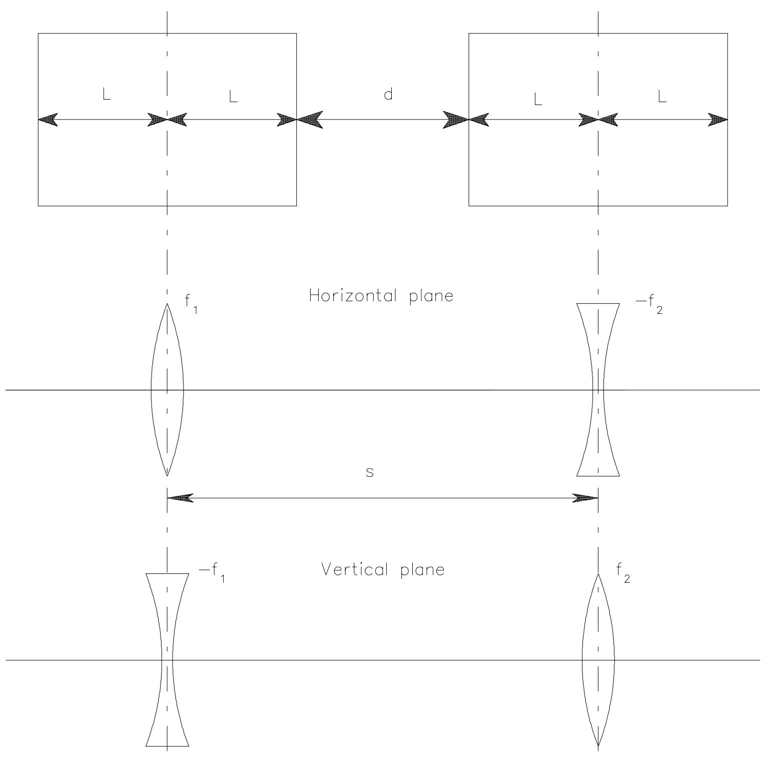

Fig. 7. First-order approximation of a horizontal-vertical quadrupole doublet.

where s is the center-to-center separation of the quadrupoles, $2L$ is the (effective) length of each quadrupole, and f_1 and f_2 are the focal lengths in the focusing planes of the first quadrupole Q_1 and the second quadrupole Q_2 respectively.

In the *horizontal* plane the transfer matrix from the entrance of lens f_1 to the exit of lens f_2 is, writing $F_i = 1/f_i$,

$$\begin{bmatrix} 1 & 0 \\ F_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -F_1 & 1 \end{bmatrix} = \begin{bmatrix} 1 - sF_1 & s \\ F_2 - F_1 - sF_1F_2 & 1 + sF_2 \end{bmatrix} \tag{27}$$

and that in the *vertical* plane is

$$\begin{bmatrix} 1 & 0 \\ -F_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ F_1 & 1 \end{bmatrix} = \begin{bmatrix} 1 + sF_1 & s \\ F_1 - F_2 - sF_1F_2 & 1 - sF_2 \end{bmatrix} . \tag{28}$$

Thus in the *horizontal* plane we have

$$-\frac{1}{f^{+-}} = -\frac{1}{f_1} + \frac{1}{f_2} - \frac{s}{f_1 f_2} \tag{29}$$

and in the *vertical* plane we find

$$-\frac{1}{f^{-+}} = -\frac{1}{f_2} + \frac{1}{f_1} - \frac{s}{f_1 f_2} \tag{30}$$

where the superscript '$+-$' indicates that the first quadrupole focuses $(+)$ and the second defocuses $(-)$ in the plane being considered, with a similar meaning for '$-+$'. Thus f^{+-} denotes the focal length in horizontal plane of an HV doublet and f^{-+} is that in its vertical plane.

Equations (29) and (30) may be rewritten as

$$\text{horizontal plane} \qquad \frac{1}{f_2} = \frac{1}{f^{+-}} \cdot \frac{f_1}{f_2 - f_1 + s} \qquad\qquad \frac{1}{f_1} = \frac{1}{f^{+-}} \cdot \frac{f_2}{f_2 - f_1 + s} \tag{31}$$

and

$$\text{vertical plane} \qquad \frac{1}{f_2} = \frac{1}{f^{+-}} \cdot \frac{f_1}{f_1 - f_2 + s} \qquad\qquad \frac{1}{f_1} = \frac{1}{f^{+-}} \cdot \frac{f_2}{f_1 - f_2 + s} \tag{32}$$

from which it follows that for *simultaneous* focusing in both the horizontal and vertical planes it is necessary that

$$|f_2 - f_1| < s \ . \tag{33}$$

Equation (12) is used to calculate the locations of the principal planes. In the horizontal plane we find

$$z_1^{+-} = -\frac{F_2}{F^{+-}} s \qquad \text{and} \qquad z_2^{+-} = \frac{F_1}{F^{+-}} s \tag{34}$$

and in the vertical plane we have

$$z_1^{-+} = \frac{F_2}{F^{+-}} s \qquad \text{and} \qquad z_2^{-+} = -\frac{F_1}{F^{+-}} s \ . \tag{35}$$

As shown in the next section, these relationships indicate that the principal planes in the focusing plane of the first quadrupole lie *upstream* of the doublet, and those in the focusing plane of the second quadrupole lie *downstream* of the doublet.

4.2.1 Antisymmetric quadrupole doublet

In most cases the quadrupole fields of a doublet are not too different. The special case in which the pole-tip fields of the quadrupoles have the same absolute value but opposite signs are called an *antisymmetric doublet.* In this case we have

$$f_1 \ = \ -f_2 \ = \ f$$

and the transfer matrices—in the thin lens approximation—reduce to

$$\begin{bmatrix} 1 \mp \dfrac{s}{f} & s \\ -\dfrac{s}{f^2} & 1 \pm \dfrac{s}{f} \end{bmatrix} \tag{36}$$

where the upper sign is used for the horizontal plane and the lower sign for the vertical plane. From this matrix the effective focal length f' of this antisymmetric doublet is found to be

$$f^{+-} = f^{-+} = f' = \frac{f^2}{s} \ . \tag{37}$$

We note that because f' is positive, an antisymmetric doublet is *always* focusing. Similarly, the positions of the principal planes are given by

$$z_1^{+-} \ = \ z_2^{-+} \ = \ -f \tag{38}$$

and

$$z_1^{-+} \ = \ z_2^{+-} \ = \ f \ . \tag{39}$$

At TRIUMF, for example, with two standard 4-inch quadrupoles at half power, we have

$$f' \approx (1.2)^2/0.8 \ = \ 1.9 \text{ m} \qquad \text{for} \qquad s \ = \ (0.2 \text{ m} + 0.4 \text{ m} + 0.2 \text{ m}) \ = \ 0.8 \text{ m} \ ,$$

so that

$$z_1^{+-} \ = \ z_2^{-+} \ = \ -f \ = \ -1.2 \text{ m}$$

and

$$z_1^{-+} \ = \ z_2^{+-} \ = \ f \ = \ 1.2 \text{ m} \ .$$

The diagram below shows an antisymmetric doublet operating in a focus-to-focus mode.

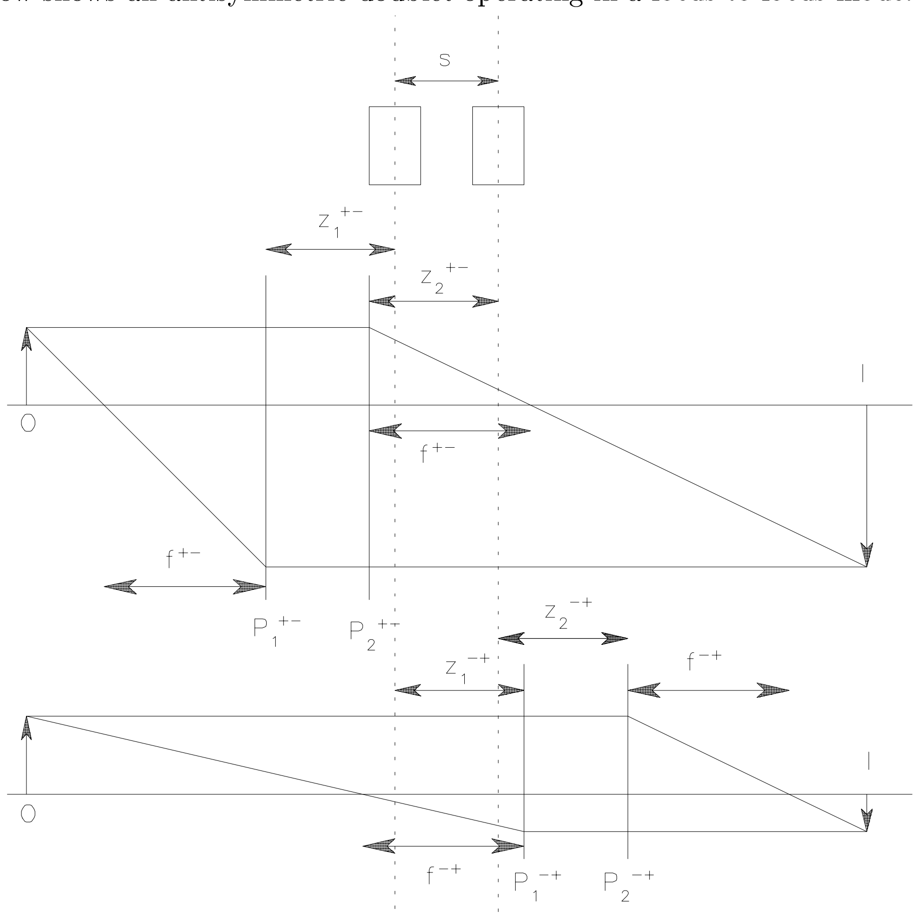

Fig. 8. An HV antisymmetric doublet operating in a focus-to-focus mode.

From the above diagram the inherent asymmetry of a quadrupole doublet should be apparent. If the doublet is operated in a focus-to-focus mode from a point L units upstream of the (center of) quadrupole

Q_1 to a point L units downstream of the (center of) quadrupole Q_2, the object and image distances in the horizontal plane are given by

$$p^{+-} = L + z_1^{+-} < L \qquad \text{and} \qquad q^{+-} = L + z_2^{+-} > L$$

because z_1^{+-} is negative and z_2^{+-} is positive. Consequently, the magnitude of the magnification in the horizontal plane is

$$|M_x| = \left|\frac{q^{+-}}{p^{+-}}\right| > 1 \ .$$

On the other hand, the object and image distances in the vertical plane are given by

$$p^{-+} = L + z_1^{-+} > L \qquad \text{and} \qquad q^{-+} = L + z_2^{-+} < L$$

because z_1^{-+} is positive and z_2^{-+} is negative. Consequently, the magnitude of the magnification in the vertical plane is

$$|M_y| = \left|\frac{q^{-+}}{p^{-+}}\right| < 1 \ .$$

Thus, in general, the magnification of a doublet operating in a focus-to-focus mode is *always* larger in the focusing plane of the first quadrupole than that in the focusing plane of the downstream quadrupole.

Exercise:

An HV antisymmetric quadrupole doublet consists of two quadrupoles, each of effective length 0.4 m and bore 10.16 cm. The distance between their effective edges is $s = 0.4$ m. It is desired to focus 500 MeV protons from an object located a distance $L = 5.0$ m upstream of the first quadrupole at a point $L = 5.0$ m downstream of the second quadrupole.

a) Using the thin lens approximation, show that the overall transfer matrix for this system is given by

$$\boldsymbol{R} = \begin{bmatrix} 1 - sF(1+LF) & 2L + s(1 - L^2F^2) & 0 & 0 \\ -sF^2 & 1 + sF(1-LF) & 0 & 0 \\ 0 & 0 & 1 + sF(1-LF) & 2L + s(1-L^2F^2) \\ 0 & 0 & -sF^2 & 1 - sF(1+LF) \end{bmatrix} ,$$

where $F = 1/f$ is the inverse focal lengths of the quadrupoles.

b) From the above show that the required focal lengths of the quadrupoles is

$$f = \frac{1}{F} = L\sqrt{\frac{s}{s+2L}}$$

and calculate the pole-tip fields required to produce this focal length, and the magnifications in each of the horizontal and vertical planes.

c) Using the principal plane technique verify the results obtained above.

Answer: $f = 1.366$ m, $B_0 = \pm 3.381$ kG, $M^{+-} = -1.752$, and $M^{-+} = -0.571$. These values are to be compared with $B_0 = \pm 3.698$ kG, $M^{+-} = -1.826$, and $M^{-+} = -0.548$ that are obtained using the full quadrupole matrix.]

4.3 Quadrupole triplets

Another combination of quadrupoles is the triplet. Consider three quadrupoles, each of effective length $2L$, that are separated by a distance d. The pole-tip fields of the quadrupoles are, respectively, B_1, B_2, and B_3, which may be converted into the focal lengths f_1, f_2, and f_3.

In the thin lens approximation the three quadrupoles are each replaced with a lens of the appropriate strength at the center of each quadrupole. The lenses are then separated by a distance $s = d + 2L$.

Consider the horizontal plane of an HVH triplet. The transfer matrix from the center of Q_1 to the center of Q_3, again with the notation $F = 1/f$, is

$$\begin{bmatrix} 1 & 0 \\ -F_3 & 1 \end{bmatrix}\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ F_2 & 1 \end{bmatrix}\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -F_1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -sF_1 + (1 - sF_1)(1 + sF_2) & s(2 + sF_2) \\ -(1 - sF_3)[F_1 - F_2(1 - sF_1)] - F_3(1 - sF_1) & -sF_3 + (1 - sF_3)(1 + sF_2) \end{bmatrix} \quad (40)$$

It is evident that even in the thin-lens approximation this expression could become unwieldly. In the following section a special case will be examined.

4.3.1 General case of equally powered outer quadrupoles

A simplification of the above expression results if the outer quadrupoles of the triplet are equally powered. We assume that all quadrupoles have equal effective lengths, that the pole-tip fields of the outer quadrupoles are B_1 and that of the inner quadrupole is B_2, and that the corresponding focal lengths are f_1 and f_2. For the purpose of illustration we assume an HVH quadrupole configuration.

In the thin lens approximation the focal lengths in the focusing and defocusing planes of a given quadrupole are equal. Consequently, the transfer matrix of the triplet in the *horizontal* plane is

$$\begin{bmatrix} 1 & 0 \\ -F_1 & 1 \end{bmatrix}\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ +F_2 & 1 \end{bmatrix}\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -F_1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - 2x_1 + x_2(1 - x_1) & s(2 + x_2) \\ \dfrac{1 - x_1}{s}\cdot[x_2(1 - x_1) - 2x_1] & 1 - 2x_1 + x_2(1 - x_1) \end{bmatrix} \quad (41)$$

where we have written $x_i = |sF_i| = |s/f_i|$. Similarly, in the vertical plane we find

$$\begin{bmatrix} 1 & 0 \\ +F_1 & 1 \end{bmatrix}\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -F_2 & 1 \end{bmatrix}\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ +F_1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 2x_1 - x_2(1 + x_1) & s(2 - x_2) \\ \dfrac{1 + x_1}{s}\cdot[2x_1 - x_2(1 + x_1)] & 1 + 2x_1 - x_2(1 + x_1) \end{bmatrix}. \quad (42)$$

From these equations we then find the locations of the principal planes to be

$$\text{Horizontal plane} \qquad z_1^{+-+} = z_2^{+-+} = \frac{s}{1 - x_1} = \frac{s}{1 - sF_1} \quad (43)$$

and

$$\text{Vertical plane} \qquad z_1^{-+-} = z_2^{-+-} = \frac{s}{1 + x_1} = \frac{s}{1 + sF_1}\,. \quad (44)$$

Equations (43) and (44) show that the field of the center quadrupole of a triplet may be varied *without affecting the positions of the principal planes.* This property has no counterpart in a quadrupole doublet configuration.

Pictorially, we have the situation indicated below. In the diagram it has been assumed that the outer quadrupoles of the triplet focus vertically and the center quadrupole focuses horizontally.

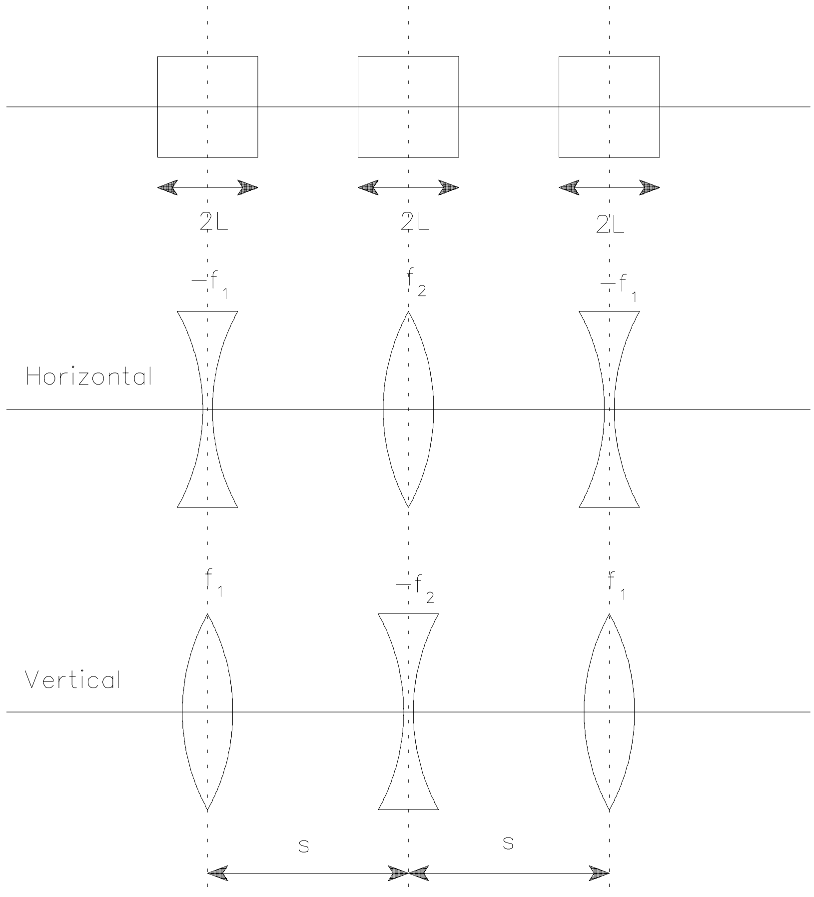

Fig. 9. Schematic of a VHV triplet configuration.

From equation (41) the effective focal length f^{+-+} in the *horizontal* plane of an HVH triplet are found to be

$$f^{+-+} = \frac{f_1 f_2}{\left(1 - \frac{s}{f_1}\right)(-f_1 + 2f_2 + s)} \ , \tag{45}$$

and we have seen that and the positions of the principal planes in the horizontal plane are given by

$$z_1^{+-+} = z_2^{+-+} = \frac{s f_1}{f_1 - s} \ . \tag{46}$$

Similarly, the effective focal length f^{+-+} in the *vertical* plane of an HVH triplet are found to be

$$f^{-+-} = \frac{f_1 f_2}{\left(1 + \frac{s}{f_1}\right)(f_1 - 2f_2 + s)} \ , \tag{47}$$

and we have seen that and the positions of the principal planes in the vertical plane are given by

$$z_1^{-+-} = z_2^{-+-} = \frac{s f_1}{f_1 + s} \ . \tag{48}$$

In the case that $s \ll f_1$ note that the positions of the principal planes in *each* of the horizontal and vertical planes can be written

$$z(+-) \ = \ \frac{s f_1}{f_1 \pm s} \to s \ ,$$

that is, for $s \ll f_1$, *all* principal planes coincide at the geometric center of the triplet. Consequently, this triplet will act more like a thin lens in *both* planes than will a doublet.

In general, provided that the focal lengths of the outer quadrupoles of a triplet are larger than twice their separation, the principal planes in both the horizontal and vertical directions lie within the triplet. As we have seen, this was not the case with the antisymmetric doublet.

Typical values of TRIUMF quadrupoles—the center quadrupole at 5 kG and the outer pair at 3 kG—are

$$f_1 \approx 1.5 \text{ m} \qquad f_2 \approx 0.9 \text{ m} \qquad s \approx 0.8 \text{ m}$$

so that

$$\begin{aligned} f_{+-+} &= \frac{(1.5)(0.9)}{(-1.5+2(0.9)+0.8)(1-(0.8/1.5))} = 2.63 \text{ m} \\ f_{-+-} &= \frac{(1.5)(0.9)}{(-2(0.9)+1.5+0.8)((1+(0.8/1.5))} = 0.59 \text{ m} \\ z_1^{+-+} = z_2^{+-+} &= \frac{(1.5)(0.8)}{1.5-0.8} = 1.71 \text{ m} \\ z_1^{-+-} = z_2^{-+-} &= \frac{(1.5)(0.8)}{1.5+0.8} = 0.52 \text{ m} \end{aligned}$$

Figure 10 shows the principal plane arrangements for a VHV triplet operating in a focus-to-focus mode with equal object and image distances.

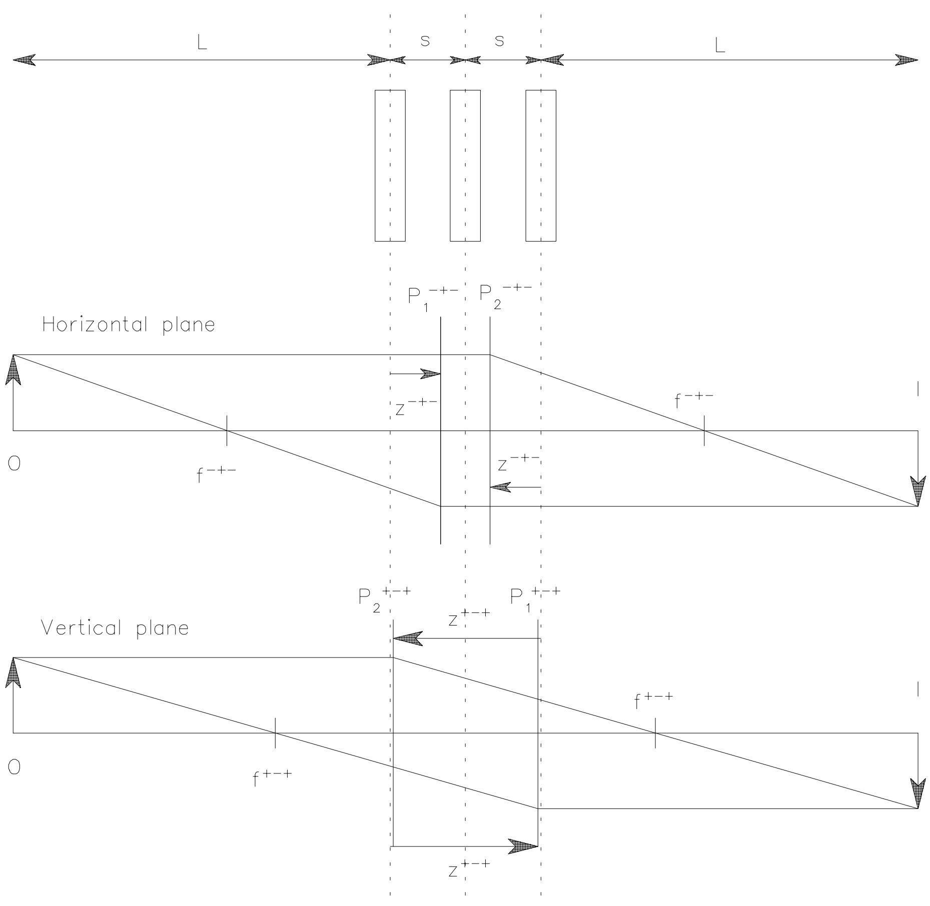

Fig. 10. Principal planes of a VHV triplet operating in a focus-to-focus mode.

We note that with the above configuration—a symmetric triplet operating in a focus-to-focus mode with equal object and image distances—the magnifications in the horizontal and vertical planes are *each* equal to -1. As we have seen, with an equivalent doublet configuration this is not possible.

Exercise:

An HVH symmetric triplet consists of three quadrupoles, each of effective length 0.4 m and bore 10.16 cm, that are separated by 0.4 m. It is desired to focus 500 MeV protons from an object situated 5 m upstream of the triplet to an image point 5 m downstream of it. The outer quadrupoles are powered equally.

a) Replace each quadrupole by a thin lens and obtain the transfer matrix from object to image. Calculate the required pole-tip fields of the quadrupoles and the magnifications in each of the horizontal and vertical planes.

b) Using the principal plane approach, verify the results obtained in part a).

Answer:

We find $f^{+-+} = 3.291$ m, $f^{-+-} = 2.881$ m, $B^{+} = 2.432$ kG, $B^{-} = -4.215$ kG, $M^{+-+} = -1.0$, and $M^{+-+} = -1.0$. These values are to be compared with $B^{+} = 2.817$kG, $B^{-} = -4.215$ kG, $M^{+-+} = -1.0$, and $M^{+-+} = -1.0$ that are obtained using the exact matrix.

c) What happens if the polarities of the quadrupoles are reversed?

4.4 Chromatic aberrations

In the foregoing it has been assumed that all particles traversing the quadrupoles have had the same momentum. This, of course, is not true in practice. Consequently, if there is a momentum spread in the beam each different momentum will be affected in a slightly different manner by the quadrupoles. In fact, different momenta will be focused at different places along a line through the axes of the quadrupoles. This is called *chromatic aberration.*

It can be shown that this effect can be neglected for most beamlines that we will meet. Other than a mention of this effect, no further consideration of quadrupole chromatic aberrations will be given here. However, for those interested ref[5)] gives a detailed investigation of this effect.

In general, however, the effect of magnets on particles of different momentum is not something that can be neglected. In fact, for some magnets it is a *necessary* consideration—as we shall see in the following section.

5. Dipole magnets

We are all familiar with the optical prism in which different wavelengths are bent through different angles with the result of a 'rainbow' effect. This is caused by the variation of the index of refraction with wavelength and is, in fact a chromatic effect. The bending power of a lens depends on the *wavelength* of the light and thus the energy of the photons (the quanta of light). [Remember that the energy of a photon is given by $E = h\nu = hc/\lambda$ where c is the speed of light, ν is its frequency, λ its wavelength, and h is a universal constant called *Planck's constant* that is equal to 6.6256×10^{-34} joule-sec. The momentum of a photon is $p = E/c$.] The beam transport analogy of this is, of course, the dipole magnet in which particles of different momentum are deflected through different angles.

In the discussion of quadrupoles it has been assumed that the particles could be treated as if they traveled parallel to a (fixed) Cartesian coordinate system. Because deviations from travel along the z-axis were assumed to be small, we were able to derive the equations of motion through a quadrupole field—at least to a first-order approximation. This procedure cannot be used with a dipole magnet because we know that such a magnet changes the direction of the incident beam.

Certainly, we could express the coordinates of particles on exit from a dipole, $\boldsymbol{x_1} = (x_1, y_1, z_1)$, in terms of those referred to an initial coordinate system, $\boldsymbol{x_0} = (x_0, y_0, z_0)$ that is set up at the entrance of the dipole. However, we are interested in a coordinate system in which the z-axis lies along the direction of the beam. In this case we must attach a coordinate system to the beam as it passes through the dipole.

The coordinate system is chosen such that the z-axis *always* points in the (instantaneous) direction in which the beam travels. This coordinate system obviously rotates with the beam and when the beam exits the dipole the z-axis will point in the beam direction. Thus at the dipole entrance we convert from a Cartesian system $\boldsymbol{x_0}$ to a rotating system. On exiting the dipole we convert from the rotating system to a new Cartesian system $\boldsymbol{x_1}$ in which the z-axis again points in the beam direction.

The procedure for developing the equations of motion through a dipole involves concepts from differential geometry. Here, however, only the results will be quoted.

5.1 General concepts

For the purposes of this report a dipole magnet will be considered to be a magnet in which the magnetic field $\boldsymbol{B}$ is given (in a Cartesian frame of reference) by

$$\boldsymbol{B} = (0, B_y, 0) \ , \tag{49}$$

that is, the magnetic field lies parallel to the positive (vertical) y-axis. Furthermore, it will be assumed that $\boldsymbol{B}$ is constant.

Qualitatively, we may describe particle motion in a dipole as follows. With the convention that the (local) z-axis lies in the direction of motion—that is, the velocity vector of the particle lies along the instantaneous z-axis—the particle feels a deflecting force given by

$$\boldsymbol{F} \ = \ q\boldsymbol{v}\times\boldsymbol{B}$$

where $\boldsymbol{v}$ is the velocity of the particle and q is its charge. The unit of q is the Coulomb, those of $\boldsymbol{v}$ are m/s, while that of $\boldsymbol{B}$ is T.

The direction of the magnetic force is *neither* in the direction of $\boldsymbol{v}$ *nor* in the direction of $\boldsymbol{B}$ but in the direction of the vector $\boldsymbol{v}\times\boldsymbol{B}$. This vector is called the *vector cross product* of $\boldsymbol{v}$ and $\boldsymbol{B}$ and is *defined* to have the magnitude $vB\sin\theta$, θ being the angle between $\boldsymbol{v}$ and $\boldsymbol{B}$, and to have a direction *perpendicular* to the plane containing $\boldsymbol{v}$ and $\boldsymbol{B}$ as determined by the right-hand rule. [Place the fingers of the right hand along the vector $\boldsymbol{v}$ and curl them into the vector $\boldsymbol{B}$. The thumb then points in the direction of the force on a *positively* charged particle.]

Because we have assumed $\boldsymbol{B}$ is parallel to the y-axis and $\boldsymbol{v}$ is parallel to the z-axis, they are perpendicular and the magnitude of $q\boldsymbol{v}\times\boldsymbol{B}$ is $qB\sin 90\text{deg} = qv_zB_y = qvB$. From the right-hand rule, the direction of the force is along the negative x-axis. Thus we have

$$F_x \ = \ -\,qvB \qquad\qquad F_y \ = \ 0 \qquad\qquad F_z \ = \ 0 \tag{50}$$

where we have dropped the subscripts of the magnetic field and the velocity. Consequently, looking along the beam, a positive particle entering the field will be deflected to the right.

Now when $\boldsymbol{v}$ and $\boldsymbol{B}$ are mutually perpendicular and the magnetic force has a constant magnitude of qvB and is always perpendicular to $\boldsymbol{v}$, it is shown in physics courses that this results in circular motion. Equating the magnetic force to the product of mass times centripetal acceleration we obtain

$$qvB = m\,\frac{v^2}{\rho} \tag{51}$$

where ρ is the radius of the circle. This we rewrite as

$$B\rho \;=\; \frac{mv}{q} \;=\; \frac{p}{q}\;. \tag{52}$$

The important quantity $B\rho$ with units of T-m is called the *magnetic rigidity* of the particle. It depends only on the ratio of the momentum of the particle to its charge. Often in beam-line literature the magnetic rigidity will be written as $(B\rho)_0$ to indicate that it is the magnetic rigidity for which the beam line is designed. In the units that we are using we have for *protons*

$$B[\mathrm{T}]\rho[\mathrm{m}] = 3.3356p[\mathrm{GeV/c}] \tag{53}$$

In particular, if a magnet is designed to deflect a beam of momentum p_0 through an angle θ_0 then we write

$$(B\rho)_0 = 3.3356p_0\;. \tag{54}$$

At this point the concept of dispersion must be introduced. In practice, a beam of particles will have a finite momentum spread. If a magnet is designed to deflect a beam of momentum p_0—called the *central* momentum—through an angle θ_0, then particles of higher momentum $p + \Delta p$ will be bent through a somewhat smaller angle. Conversely, particles of lower momentum $p - \Delta p$ will be bent through a somewhat larger angle. [Again, this effect is optically equivalent to the variation of the index of refraction with wavelength.] The result is that the beam is spread out in space—that is, the position of the beam on exiting from the dipole depends on the momentum of the particle. The beam is spatially *dispersed.* Because the direction of the beam is also changed, the beam is, in general, also dispersed in angle—that is, the angle with respect to the z-axis also depends on the momentum of the particle. The beam also has *angular* dispersion.

5.2 Transfer matrix for a wedge dipole magnet

Consider a magnet designed such that a beam in which particles with the design (central) momentum both enter and exit the magnetic field region at right angles. We then have the following situation for a bend angle of θ_0.

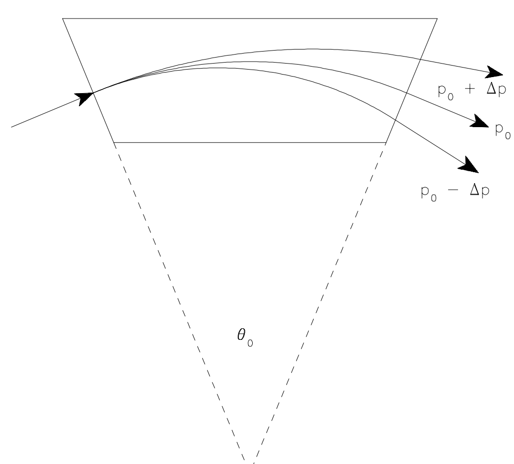

Fig. 11. Particle motion in a wedge magnet.

In the above diagram the effects of dispersion are indicated schematically. Let us define the quantity δ as

$$\delta = \frac{p - p_0}{p_0} \tag{55}$$

where p_0 is the central momentum of the beam for which the magnet will deflect a particle of that momentum through an angle θ_0 and p is the momentum of an arbitrary particle in the beam. The term δ is then a measure of the 'off-momentumness' of the particle in question. Dispersion is introduced into the matrix formalism by adding another row and column to the transfer matrix. Thus the horizontal portion of the matrix is written as

$$\begin{bmatrix} x_1 \\ \theta_1 \\ \delta_1 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \\ \delta_0 \end{bmatrix} . \tag{56}$$

In this equation $\boldsymbol{x_1}$ is the coordinate vector at the exit of the dipole and $\boldsymbol{x_0}$ is that at its entrance. The third row indicates that the momentum of a particle is not changed. This is because static magnetic fields do no work and, consequently, do not change the momenta of particles in the beam. The matrix elements R_{13} and R_{23} indicate that there is dispersion in both space and angle.

It can be shown that the transfer matrix in the bend plane of a uniform-field wedge magnet that bends a particle of momentum p_0 through and angle θ may be written as

$$\boldsymbol{R}(\text{wedge, bend plane}) = \begin{bmatrix} \cos\theta & \rho_0\sin\theta & \rho_0(1-\cos\theta) \\ -\frac{1}{\rho_0}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{bmatrix} \tag{57}$$

where ρ_0 is the radius of curvature in the magnet for particles of (design) momentum p_0.

Motion in the non-bend (vertical) plane is not affected by the magnetic field. Consequently, the momentum dependence is left out and the magnet appears as a drift space. The transfer matrix in the non-bend plane is then

$$\boldsymbol{R}(\text{wedge, non-bend plane}) = \begin{bmatrix} 1 & \rho_0\theta \\ 0 & 1 \end{bmatrix} \tag{58}$$

where $\rho_0\theta$ is the length of the trajectory within the magnet. As was done in the case of quadrupoles, these two matrices are combined into one. By convention, the momentum row and column are written last. Thus we have for the complete transfer matrix through a wedge dipole

$$\begin{bmatrix} x_1 \\ \theta_1 \\ y_1 \\ \phi_1 \\ \delta_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \rho_0\sin\theta & 0 & 0 & \rho_0(1-\cos\theta) \\ -\frac{1}{\rho_0}\sin\theta & \cos\theta & 0 & 0 & \sin\theta \\ 0 & 0 & 1 & \rho_0\theta & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \\ y_0 \\ \phi_0 \\ \delta_0 \end{bmatrix} . \tag{59}$$

5.3 Principal planes of a wedge dipole magnet

Unless indicated otherwise we shall always assume that the bend plane of a dipole is horizontal. Because the optics of the non-bend plane of a wedge magnet are those of a field-free drift region, we need only consider the its bend plane. In that plane we have

$$\begin{bmatrix} x_1 \\ \theta_1 \\ \delta_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \rho_0\sin\theta & \rho_0(1-\cos\theta) \\ -\frac{1}{\rho_0}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \\ \delta_0 \end{bmatrix}$$

from which we find

$$-\frac{1}{f} = R_{21} = -\frac{1}{\rho_0}\sin\theta \tag{60}$$

and

$$z_1 = z_2 = \frac{R_{11} - 1}{R_{21}} = -\rho_0\tan\frac{\theta}{2}\ . \tag{61}$$

Thus the principal planes are located at the cross-over point—the point of intersection of the incoming and outgoing rays—of the magnet. In the bend plane the wedge magnet ca be represented as a lens of focal length $f = \rho_0/\sin\theta$ with drift spaces of length $z_1 = z_2 = \rho_0\tan(\theta/2)$ on either side. This is illustrated in the figure below.

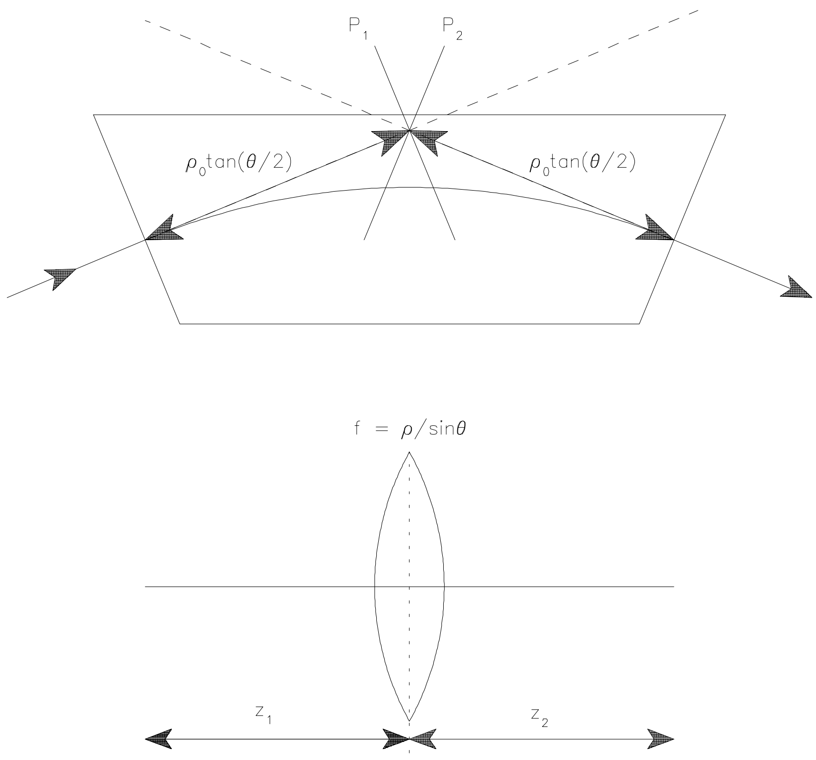

Fig. 12. Principal planes (upper) and simplified description (lower) of a wedge magnet.

Thus we may use the following simplified matrix in the bend plane of a wedge magnet *provided* distances are measured to/from its principal planes.

$$\boldsymbol{R}(\text{wedge magnet, bend plane}) = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{\rho_0}\sin\theta & 1 & \sin\theta \\ 0 & 0 & 1 \end{bmatrix}\ . \tag{62}$$

5.4 Magnets with arbitrary entry and exit angles

If particles do not enter and exit a magnet at right angles to the magnet's edge we cannot use the above prescriptions. We can, however, make such a magnet from a wedge magnet by superimposing the field of a *magnetic wedge* on that of the wedge magnet. As illustrated in the following diagram, a magnetic wedge produces a field that is positive on one side of the axis and negative on the other. This field adds to or subtracts from the field of the wedge magnet and, with judicious placement, can be used to generate the appropriate angles of entry and exit.

It can be shown (see ref[5], for example) that the transfer matrix for a magnetic wedge of angle α is

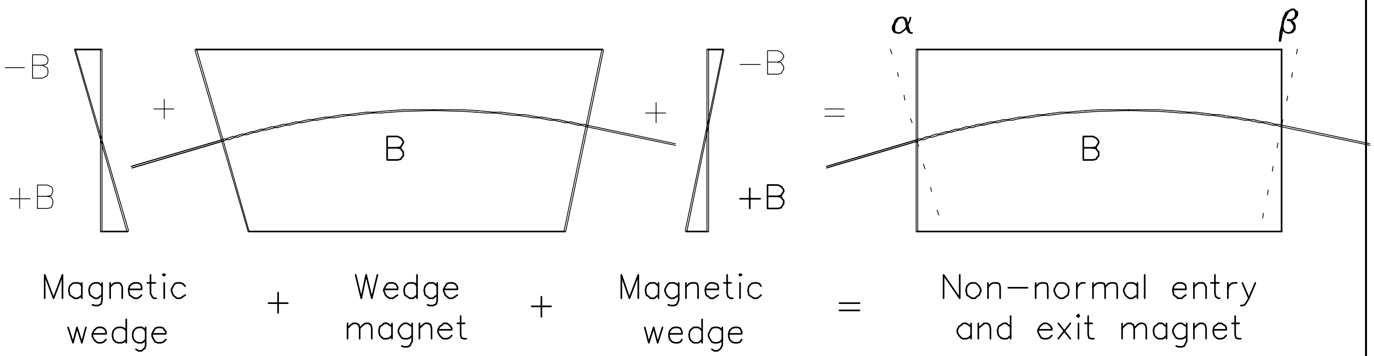

Fig. 13. Construction of a magnet with arbitrary entrance and exit angles using magnetic wedges.

$$\boldsymbol{R}(\text{pole face rotation}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \dfrac{\tan\alpha}{\rho_0} & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{\tan\alpha}{\rho_0} & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} . \tag{63}$$

[*Note:* Because we are considering the 'hard-edge' model—that is, the magnetic field drops to zero at the effective edge of the magnet—of a magnet, a correction term to the R_{43} term has been omitted.]

The angle α is *defined* as positive if the normal to the pole-face lies *outside* the entering trajectory with respect to the center of curvature. A similar matrix and sign convention applies to the exit edge.

The important result obtained from equation (63) is that edge focusing of dipoles is directly proportional to the radius of curvature of the central trajectory and inversely proportional to the tangent of the edge angle. The former is usually large (> 2 m) and the latter is usually less than 0.5, resulting in a (minimum) focal length of 4 m or more. Consequently, the focusing effect of edge angles is weak relative to that obtained with quadrupoles.

A further, important result is that a *positive* edge angle provides *vertical focusing* and *horizontal defocusing.* The converse, of course, applies for a negative edge angle.

The transfer matrix for a magnet into which a particle enters at an angle α and exits at an angle β is obtained by premultiplying the wedge-magnet transfer matrix by the pole-face rotation matrix for an angle α and postmultiplying it by the pole-face rotation matrix for an angle β. Thus

$$\boldsymbol{R}(\text{overall}) = \boldsymbol{R}(\text{pole face rotation } \beta)\boldsymbol{R}(\text{wedge magnet})\boldsymbol{R}(\text{pole face rotation } \alpha) . \tag{64}$$

5.5 The rectangular magnet

The most common magnet at TRIUMF is one that is oriented such that the entry and exit angles are each one-half of the total bend angle. The magnet is the rectangular in shape. We leave it as an exercise to obtain the transfer matrix for such a magnet.

Exercise:

A magnet bends particles through an angle θ with a radius of curvature ρ_0. The magnet is rectangular so that the entry and exit angles are $\alpha = \beta = \theta/2$. Develop the transfer matrix for such a magnet and show that it is given by

$$\boldsymbol{R}\begin{bmatrix}\text{rect.}\\\text{magnet}\end{bmatrix}=\begin{bmatrix}1 & \rho_0\sin\theta & 0 & 0 & \rho_0[1-\cos\theta]\\ 0 & 1 & 0 & 0 & 2\tan(\theta/2)\\ 0 & 0 & 1-\theta\tan(\theta/2) & \rho_0\theta & 0\\ 0 & 0 & \dfrac{\tan(\theta/2)}{\rho_0}[\theta\tan(\theta/2)-2] & 1-\theta\tan(\theta/2) & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}. \quad (65)$$

6. Achromatic systems

As has been mentioned previously, we always have some spread in momentum among the particles of the beam extracted from an accelerator. From the matrix representation above we have seen that after a beam passes through a dipole, both the position and angle in the bend plane of the dipole will be momentum dependent. If this beam falls on a target this implies that the momenta of the particles hitting one side of the target will differ from that of those striking the opposite side of the target. In most cases this is not desirable from an experimenters point of view. Consequently, we design a system in which the position of the beam on a target is independent of momentum. Such a system is termed an *achromatic* system. If both position *and* angle at the target are momentum independent, the system is termed *doubly achromatic.*

Rather than go into great details of the analysis of achromatic systems in general, their principle of operation will be illustrated by an example.

It turns out to be possible to design a transport system that includes a dipole such that a beam that is achromatic before entering a dipole is also achromatic after exiting it. One scheme is to split the dipole into two halves and place a (horizontally) focusing lens centered between the dipoles as is shown below.

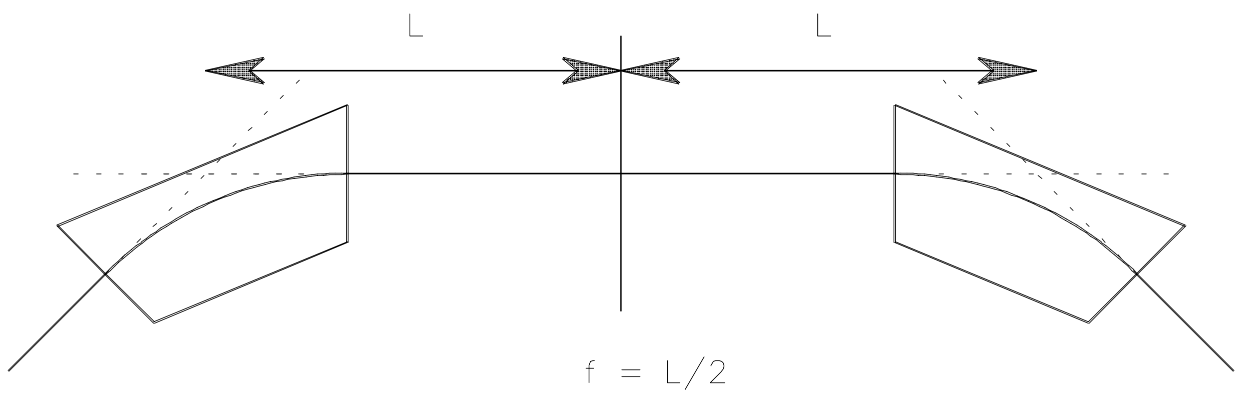

Fig. 14. Production of an achromatic beam with two wedge magnets and one quadrupole.

In this figure a (horizontal)ly focusing quadrupole is placed between two wedge magnets. Each dipole bends a beam of momentum p_0 with a radius of curvature ρ through an angle θ. The quadrupole has a focal length f and is located midway between the dipoles at a distance L from the point of intersection of the principal planes of each magnet. We study this configuration using the thin lens approximation.

Let $\boldsymbol{x_0}$ the coordinate vector of a particle at the first principal plane of the first dipole and $\boldsymbol{x_1}$ be that at its second principal plane. The vectors $\boldsymbol{x_4}$ and $\boldsymbol{x_5}$ have similar meanings for the second dipole. The vectors $\boldsymbol{x_2}$ and $\boldsymbol{x_3}$ are, respectively, the coordinate vectors immediately before and after quadrupole action. Then the transfer matrix for the system is found from the matrix equation

$$\begin{bmatrix}x_5\\\theta_5\\\delta_5\end{bmatrix}=\begin{bmatrix}\text{Magnet}\\\text{matrix}\\\#2\end{bmatrix}\begin{bmatrix}\text{Drift}\\\text{length}\\L\end{bmatrix}\begin{bmatrix}\text{Lens}\\\text{action}\\f\end{bmatrix}\begin{bmatrix}\text{Drift}\\\text{length}\\L\end{bmatrix}\begin{bmatrix}\text{Magnet}\\\text{matrix}\\\#1\end{bmatrix}\begin{bmatrix}x_0\\\theta_0\\\delta_0\end{bmatrix}.$$

Putting in the appropriate matrices and writing $s = \sin\theta$, $c = \cos\theta$, and $F = 1/f$ we have

$$\begin{bmatrix} x_5 \\ \theta_5 \\ \delta_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -s/\rho & 1 & s \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -F & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -s/\rho & 1 & s \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \\ \delta_0 \end{bmatrix}$$

$$= \begin{bmatrix} [1-LF]-(sL/\rho)[2-LF] & L[2-LF] & sL[2-LF] \\ \{[(2s/\rho)+F[1-(sL/\rho)]\}[(sL/\rho)-1] & [1-LF]-(sL/\rho)[2-LF] & s[1-(sL/\rho)][2-LF] \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x_0 \\ \theta_0 \\ \delta_0 \end{bmatrix} .$$

For the system to be doubly achromatic we require that $R_{13} = 0$ and $R_{23} = 0$. Thus we have

$$sL[2-LF] = 0 \qquad \text{and} \qquad s[1-(sL/\rho)][2-LF] = 0 ,$$

that is,

$$f = \frac{L}{2} .$$

Thus, in order to produce the doubly-achromatic condition, it is necessary to adjust the quadrupole so as to focus from the center of one dipole to the center of the other.

Insertion of the expression for f into the above matrices shows that the transfer matrix for the dipole-quadrupole-dipole doubly-achromatic system is

$$\boldsymbol{R}(\text{doubly achromatic}) = \begin{bmatrix} -1 & 0 & 0 \\ -(2/L)[1-(L/\rho)\sin\theta] & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Another technique, also involving splitting the dipole in half, is illustrated below. The diagram is taken from ref[9].

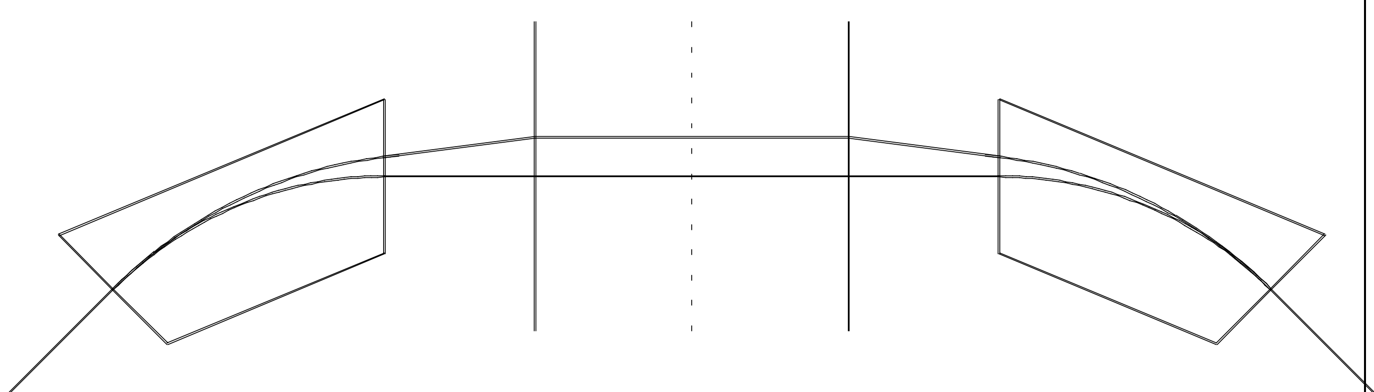

Fig. 15. An alternate technique of production of an achromatic beam.

In this case two identical focusing lenses are placed symmetrically in the space between the dipoles. The lenses are adjusted to make the beam parallel to the central trajectory in the space between them. From symmetry it should be obvious that the resulting beam will be achromatic.

The problem with each of these systems is that there is no focusing in the vertical plane—in fact there is complete defocusing. The solution is to add a vertically focusing lens at the midpoint of the system to provide the required focusing in the vertical plane. This is indicated by the dotted line in the above diagram. Of course, the optics of the system will be changed somewhat but the principle remains valid.

Often the single lens will be replaced with two equal lenses, again symmetrically located about the midpoint, to allow for a slit or diagnostic device to be inserted there. This has the advantage of allowing the insertion of a slit at the symmetry point to stop particles of unwanted momentum.

This latter technique (of adding two, equal, vertically focusing lenses) can also be used in the previous system to provide the required vertical focusing. Again, the optics will be modified somewhat but the principle remains the same.

Achromatic systems, including those using dipoles only, are treated in more detail in ref[5)].

7. Phase space and beam size determination

So far we have learned how to relate particle coordinates at the exit of a system to those at its entrance. Given a particle with coordinates $\boldsymbol{x_0}$ at the entrance of a system whose transfer matrix is $\boldsymbol{R}$, the coordinates of the particle upon exiting from the system, $\boldsymbol{x_1}$, are obtained from

$$\boldsymbol{x_1} = \boldsymbol{R} \cdot \boldsymbol{x_0} \ . \tag{66}$$

This equation relates the coordinates of *individual* particles after the action of the system to those prior to its action. If, however, we are dealing with a beam of particles, it would be impractical to apply this technique to each particle in the beam. Furthermore, we are usually interested in parameters of the beam as a whole rather than in those of any individual particle. For these reasons, the concept of phase space has been developed and the previous matrix algebra has been extended to allow us to determine its properties.

7.1 The phase space ellipse and its usefulness

Consider the collection of particles that make up the 'beam'. Each individual particle can be assigned coordinates of position, angle and momentum relative to some central trajectory. In the context of what has been discussed, we would like to write the coordinates of each particle as

$$\boldsymbol{x_1} = (x, \theta, y, \phi, \delta) \ . \tag{67}$$

For simplicity in what follows, consider a beam of particles which has no vertical size or divergence and has no momentum spread. Relative to that of some *central* particle, the position of any individual particle is completely specified by its horizontal position and divergence. We could then get the size of the beam at any point by simply plotting the x-θ distribution there. Again, however, it is clear that this would be a very tedious operation to perform.

The left diagram of figure 16, below, is meant to be such a plot, the dots representing the coordinates of individual particles. On the right, an ellipse has been drawn to enclose most of these particle coordinates. An ellipse has been chosen because of its mathematical transformation properties. Such diagrams are called a *phase-space* diagrams. In particular, the ellipse is called a *phase-space ellipse*. It displays the relationship between the horizontal size and horizontal divergence for each and every particle of the beam. It is clear that, depending upon how large an ellipse is drawn, the number of particles included within the ellipse will vary. However, once we are satisfied that the ellipse that has been drawn is a reasonable approximation to the actual beam parameters, the horizontal size of the beam can be determined by projecting the ellipse on the x-axis. Similarly, projection of the ellipse on the θ-axis will give a measure of the maximum of the divergence of the beam.

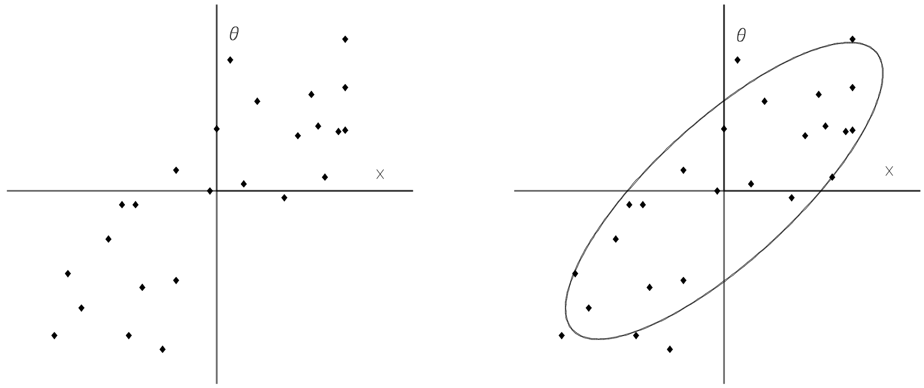

Fig. 16. A two-dimensional phase-space diagram.

It is also clear that the orientation of this ellipse will change as the beam proceeds through the transport system. Thus, for example, the case where a beam ellipse is initially upright. If a picture were taken at a second point a distance L further along the drift space, the 'snapshot' of the two beam ellipses would appear as in the following diagram.

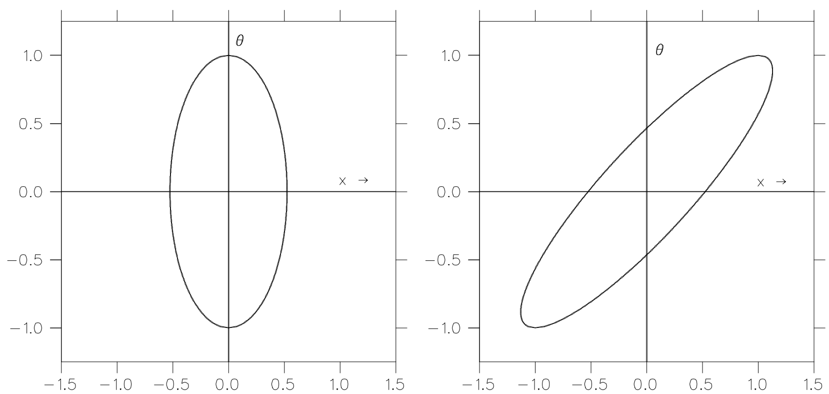

Fig. 17. The effect of a drift space on phase-space ellipse.

The reason for this change is clear. The coordinates of any particle at the second point, $\boldsymbol{x_1}$, are related to those at an earlier point, $\boldsymbol{x_0}$, by the matrix for a drift space,

$$\begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \end{bmatrix} . \tag{68}$$

The x-coordinate of a particle is altered by an amount proportional to the product of the distance between the points and the divergence of the particle at the first point. The divergence of the particle is, however, unchanged. Thus in a drift space of length L the points $(0, \pm\theta_{max})$ transform to the points $(\pm L\theta_{max}, \theta_{max})$ whereas the points $(\pm x, 0)$ are unaltered. The ellipse shears with only points lying on the x-axis unaltered and the initial upright ellipse becomes tilted to the right at the second location.

Thus the beam spreads out in space but not in angle. Estimates of maximum horizontal extent and divergence may be made by projecting the ellipse on the appropriate axis.

If, on the other hand, the beam were to encounter a (thin) lens of focal length f, coordinates after lens action would be related to those before lens action by the matrix for a thin lens

$$\begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \end{bmatrix} . \tag{69}$$

The phase-space ellipse would be transformed as shown in the following diagram.

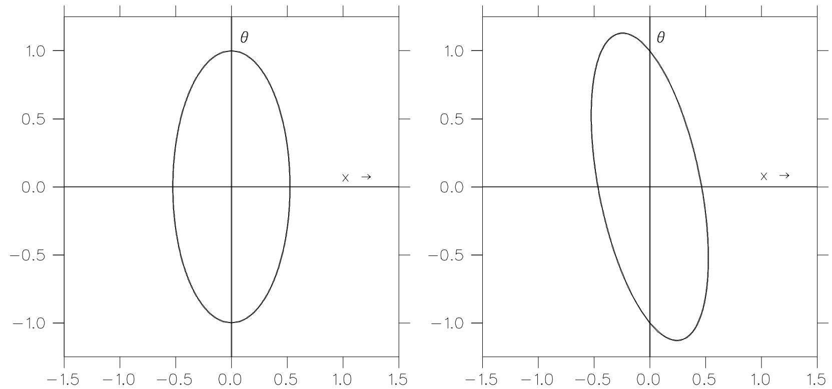

Fig. 18. The effect of a thin lens on phase-space ellipse.

In this case, the x-coordinate is unchanged but the divergence of each particle is affected. In particular, note that the points $(\pm x_{max}, 0)$ transform to the points $(\pm x_{max}, \mp x_{max}/f)$ and that the points $(0, \pm\theta_{max})$ are unaltered. Consequently, the ellipse that enclosed the beam before lens action becomes elongated along the θ-axis after lens action. The horizontal size of the beam has not changed but its divergence has.

In general, particles traverse both drift spaces and focusing and deflecting devices. Their cumulative effect on the phase-space ellipse is not obvious. However, there is one particle whose coordinates are unchanged regardless of the transport system. That particle is the *central* particle; its coordinates are (and were) $(x, \theta) = (0, 0)$.

It has been shown that any transport system can, with the principal plane approach, be reduced to a system composed of two drift lengths and one lens. Consequently, there is no need to study a more complex system. Further, if a method can be devised to carry the ellipse through the transport system, we could immediately obtain the important beam parameters (horizontal beam size and divergence) at any point in the system by simply projecting the ellipse on the coordinate axes.

The above discussion has been limited to a two-dimensional beam—that is, a beam which only had a horizontal size and divergence. Clearly, if we wanted to include momentum spread, for example, each particle would be described by the three coordinates $\boldsymbol{x} = (x, \theta, \delta)$. Rather than drawing an ellipse around the resulting three-dimensional plot, it would be necessary to draw a solid figure—an ellipsoid. We could still determine maximum beam size and divergence by projecting the ellipsoid onto the x-θ plane. Similarly, inclusion of vertical size and divergence as particle coordinates requires that we now go to a five-dimensional ellipsoid. By projecting this ellipsoid onto the appropriate planes, all important beam properties may be attained. Thus it should be clear that the methods indicated above are not restricted to the two-dimensional case.

7.2 Transformation properties of the ellipse

In the section above, an ellipse was (somewhat arbitrarily) introduced as a representation of the actual beam profile. In this section the reason for its introduction will be given.

Let the coordinate vector and its transpose be $\boldsymbol{x_0}$ and $\boldsymbol{x_0^T}$ respectively, and let $\boldsymbol{\sigma_0^{-1}}$ be the inverse of $\boldsymbol{\sigma_0}$, a real, positive-definite symmetric matrix. In Appendix A, section 6, it is shown that an equation of the form

$$\boldsymbol{x_0^T}\,\boldsymbol{\sigma_0}\,\boldsymbol{x_0} = 1 \tag{70}$$

leads to the equation of an ellipse. Now let the particle with the coordinate vector $\boldsymbol{x_0}$ be transported through a system that has a transfer matrix $\boldsymbol{R}$. Equation (70) can be rewritten as follows (recalling that

$\boldsymbol{RR^{-1}} = 1)$

$$\boldsymbol{x_0^T}\,[\boldsymbol{R^T}(\boldsymbol{R^T})^{-1}]\,\boldsymbol{\sigma_0^{-1}}[\boldsymbol{R^{-1}}\,\boldsymbol{R}]\,\boldsymbol{x_0} \quad = \quad 1\ ,$$

or,

$$[\boldsymbol{R}\,\boldsymbol{x_0}]^{\boldsymbol{T}}\,[\boldsymbol{R}\,\boldsymbol{\sigma_0}\,\boldsymbol{R^T}]^{-1}\,[\boldsymbol{R}\,\boldsymbol{x_0}] \quad = \quad 1\ .$$

But the coordinate vector at the end of the system is $\boldsymbol{x_1} = \boldsymbol{R}\,\boldsymbol{x_0}$ so that we have

$$\boldsymbol{x_1}^{\boldsymbol{T}}\,\boldsymbol{\sigma_1^{-1}}\,\boldsymbol{x_1} = 1 \tag{71}$$

where

$$\boldsymbol{\sigma_1} = \boldsymbol{R}\,\boldsymbol{\sigma_0}\,\boldsymbol{R^T}\ . \tag{72}$$

Thus, if equation (70) represents the beam at the start of the system, equation (72) will represent the beam at the end of a system that has a transfer matrix $\boldsymbol{R}$. As was shown above, all of the important parameters can be obtained from knowledge of the phase space ellipse.

7.3 Ellipse transformation in two dimensions

To simplify things we consider only the two-dimensional (x, θ) phase space. We then have

$$\boldsymbol{x^T} \;=\; [x\theta] \qquad\qquad \text{and} \qquad\qquad \boldsymbol{x} \;=\; \begin{bmatrix} x \\ \theta \end{bmatrix}$$

Suppose we draw an ellipse about this phase space such that the ellipse has its semi-major axis along the x-axis. The lengths of the semi-major and semi-minor axes are x_0 and θ_0 respectively. Now *define* the matrix $\boldsymbol{\sigma_0}$ by

$$\boldsymbol{\sigma_0} = \begin{bmatrix} \sigma_{11}(0) & \sigma_{21}(0) \\ \sigma_{21}(0) & \sigma_{22}(0) \end{bmatrix} = \begin{bmatrix} x_0^2 & 0 \\ 0 & \theta_0^2 \end{bmatrix}\ . \tag{73}$$

Notice that $\boldsymbol{\sigma_0}$ has been defined as a *symmetric* matrix—that is, $\sigma_{12}(0) = \sigma_{21}(0)$. Then the inverse of $\boldsymbol{\sigma_0}$, $\boldsymbol{\sigma_0^{-1}}$, is

$$\boldsymbol{\sigma_1^{-1}} = \frac{1}{\epsilon^2}\begin{bmatrix} \sigma_{22}(0) & -\sigma_{21}(0) \\ -\sigma_{21}(0) & \sigma_{11}(0) \end{bmatrix} = \frac{1}{\epsilon^2}\begin{bmatrix} \theta_0^2 & 0 \\ 0 & x_0^2 \end{bmatrix} \tag{74}$$

where $\epsilon^2 = \det \boldsymbol{\sigma_0} = \sigma_{11}(0)\sigma_{22}(0) - \sigma_{12}(0)\sigma_{21}(0) = x_0^2\theta_0^2$ is the determinant of the matrix $\boldsymbol{\sigma_0}$. Then the equation of the ellipse

$$\boldsymbol{x_0^T}\,\boldsymbol{\sigma_0}\,\boldsymbol{x_0} = 1$$

becomes

$$\begin{aligned} \epsilon^2 \;&=\; [x \quad \theta]\begin{bmatrix} \sigma_{22}(0) & -\sigma_{21}(0) \\ -\sigma_{21}(0) & \sigma_{11}(0) \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} \\ &=\; \sigma_{22}(0)x^2 + \sigma_{11}(0)\theta^2 \\ &=\; \theta_0^2 x^2 + x_0^2\theta^2 \\ &=\; x_0^2\theta_0^2\ . \end{aligned} \tag{75}$$

This equation shows the relationship between the elements of the matrix $\boldsymbol{\sigma}$ and the physical parameters x_0 and θ_0. The maximum values of these parameters are the *square roots* of the diagonal elements of the matrix *provided* that the ellipse has its semi-major axis along one of the coordinate axes.

The diagrams of §6.1 indicate that the phase-space ellipse will rotate as a beam traverses a transport system. In this general case—that is, if the initial ellipse is not erect—we have

$$\epsilon^2 = [\,x \quad \theta\,] \begin{bmatrix} \sigma_{22}(0) & -\sigma_{21}(0) \\ -\sigma_{21}(0) & \sigma_{11}(0) \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = \sigma_{22}x^2 - 2\sigma_{21}x\theta + \sigma_{11}\theta^2 \tag{76}$$

where we have dropped the notation '(0)'. If this ellipse is plotted we find a picture that will look similar to the following.

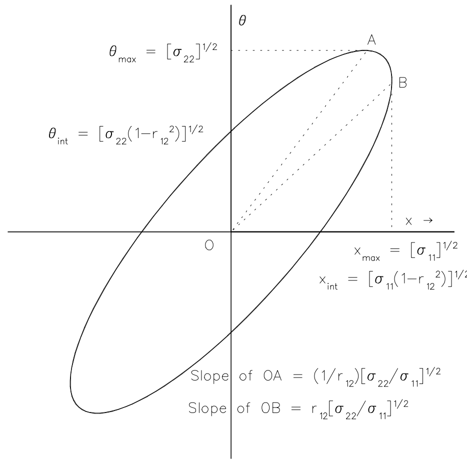

Fig. 19. Parameters of a general two-dimensional ellipse.

The area of the ellipse, A, is given by

$$A = \pi\sqrt{\det \boldsymbol{\sigma}} = \pi x_{max}\theta_{int} = \pi x_{int}\theta_{max} \tag{77}$$

where the different terms are defined in the figure. The correlation between x and θ—the orientation of the ellipse—depends on the off-diagonal term σ_{21}. This correlation, *defined* as

$$r_{21} = \frac{\sigma_{21}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}}\ , \tag{78}$$

measures the tilt of the ellipse and the intersection of the ellipse with the coordinate axes. Note that

$$-1 \le r \le\ +1\ . \tag{79}$$

Consider the special case of $r_{21} = 0$. If $r_{21} = 0$, then $\sigma_{21} = \sigma_{12} = 0$ and the ellipse is *erect*. In beam transport this situation is called a *waist*. Physically, if a beam is at a waist, we have the smallest beam size attainable for a given divergence.

Suppose that a beam described initially by equation (73) is transported through a system that has a transfer matrix $\boldsymbol{R}$. Equation (72) allows us to determine the matrix $\boldsymbol{\sigma_1}$ at the exit of the system. Thus

$$\boldsymbol{\sigma_1} = \begin{bmatrix} \sigma_{11}(1) & \sigma_{21}(1) \\ \sigma_{21}(1) & \sigma_{22}(1) \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} \sigma_{11}(0) & \sigma_{21}(0) \\ \sigma_{21}(0) & \sigma_{22}(0) \end{bmatrix} \begin{bmatrix} R_{11} & R_{21} \\ R_{12} & R_{22} \end{bmatrix}\ . \tag{80}$$

This leads to the following values for the matrix elements of $\boldsymbol{\sigma_1}$.

$$\begin{aligned} \sigma_{11}(1) &= R_{11}^2\sigma_{11}(0) + 2R_{11}R_{21}\sigma_{21}(0) + R_{12}^2\sigma_{22}(0)\ , \\ \sigma_{12}(1) = \sigma_{21}(1) &= R_{11}R_{21}\sigma_{11}(0) + [R_{11}R_{22} + R_{12}R_{21}]\sigma_{21}(0) + R_{12}R_{22}\sigma_{22}(0)\ , \\ \sigma_{22}(1) &= R_{21}^2\sigma_{11}(0) + 2R_{21}R_{22}\sigma_{21}(0) + R_{22}^2\sigma_{22}(0)\ . \end{aligned} \tag{81}$$

In the special case of an initially upright ellipse $\boldsymbol{\sigma_1}$ reduces to

$$\boldsymbol{\sigma_1} = \begin{bmatrix} R_{11}^2\sigma_{11}(0) + R_{12}^2\sigma_{22}(0) & R_{11}R_{21}\sigma_{11}(0) + R_{12}R_{22}\sigma_{22}(0) \\ R_{11}R_{21}\sigma_{11}(0) + R_{12}R_{22}\sigma_{22}(0) & R_{21}^2\sigma_{11}(0) + R_{22}^2\sigma_{22}(0) \end{bmatrix} . \tag{82}$$

Consider now some specific cases.

Suppose we start with an initially upright ellipse—that is, we start from a waist. Now let the beam drift a distance L. Insertion of the matrix elements for a drift region into equation (82) produces

$$\boldsymbol{\sigma_1} = \begin{bmatrix} \sigma_{11}(1) & \sigma_{21}(1) \\ \sigma_{21}(1) & \sigma_{22}(1) \end{bmatrix} = \begin{bmatrix} \sigma_{11}(0) + L^2\sigma_{22}(0) & L\sigma_{22}(0) \\ L\sigma_{22}(0) & \sigma_{22}(0) \end{bmatrix} . \tag{83}$$

Equation (83) then gives the following meanings for the matrix elements of $\boldsymbol{\sigma_1}$.

$$\sigma_{11}(1) = \sigma_{11}(0) + L^2\sigma_{22}(0)$$

or

$$(x_1^2)_{max} = (x_0^2)_{max} + L^2(\theta_0^2)_{max} , \tag{84}$$

and

$$\sigma_{22}(1) = \sigma_{22}(0)$$

or

$$(\theta_1^2)_{max} = (\theta_0^2)_{max} . \tag{85}$$

Remember that the initial phase space was assumed to be erect for this transformation. But x_0^2 is the square of the initial beam extent and $L^2\theta_0^2$ is that of a particle that started from $x = 0$ with maximum divergence.Thus equation (84) indicates that $\sqrt{\sigma_{11}(1)}$ can be regarded as the root mean square of the maximum displacement of the particle. Equation (85) indicates that the divergence of the particle is unchanged.

If the matrix elements for a thin lens is inserted into equation (82) we obtain

$$\boldsymbol{\sigma_2} = \begin{bmatrix} \sigma_{11}(2) & \sigma_{21}(2) \\ \sigma_{21}(2) & \sigma_{22}(2) \end{bmatrix} = \begin{bmatrix} \sigma_{11}(0) & -\dfrac{\sigma_{11}(0)}{f} \\ -\dfrac{\sigma_{11}(0)}{f} & \dfrac{\sigma_{11}(0)}{f^2} + \sigma_{22}(0) \end{bmatrix} . \tag{86}$$

from which it follows that

$$\sigma_{11}(2) = \sigma_{11}(0)$$

or

$$(x_2^2)_{max} = (x_0^2)_{max} , \tag{87}$$

and

$$\sigma_{22}(2) = \frac{\sigma_{11}(0)}{f^2} + \sigma_{22}(0)$$

or

$$(\theta_2^2)_{max} = \frac{1}{f^2}(x_0^2)_{max} + (\theta_0^2)_{max} . \tag{88}$$

Equation (87) shows that the maximum displacement of a particle does not change in a thin lens. From equation (88) we see that $\sqrt{\sigma_{22}(2)}$ may be interpreted as the root mean square maximum divergence obtained from the maximum initial divergence and the maximum change in divergence caused by the lens.

7.4 A discussion of the waist—the upright ellipse

As was noted earlier, The special case of $\sigma_{12} = \sigma_{21} = 0$ is termed a waist. We should correctly understand its meaning. For an existing beam a waist is the location of a minimum of beam size in a given region of the system. Although the waist is the minimum beam size in any given beam line, the minimum beam size attainable at a *fixed* target position (by varying the focal length of the upstream lens system) is *not* the same as the waist defined above. The figure below, taken from [1)], illustrates this point.

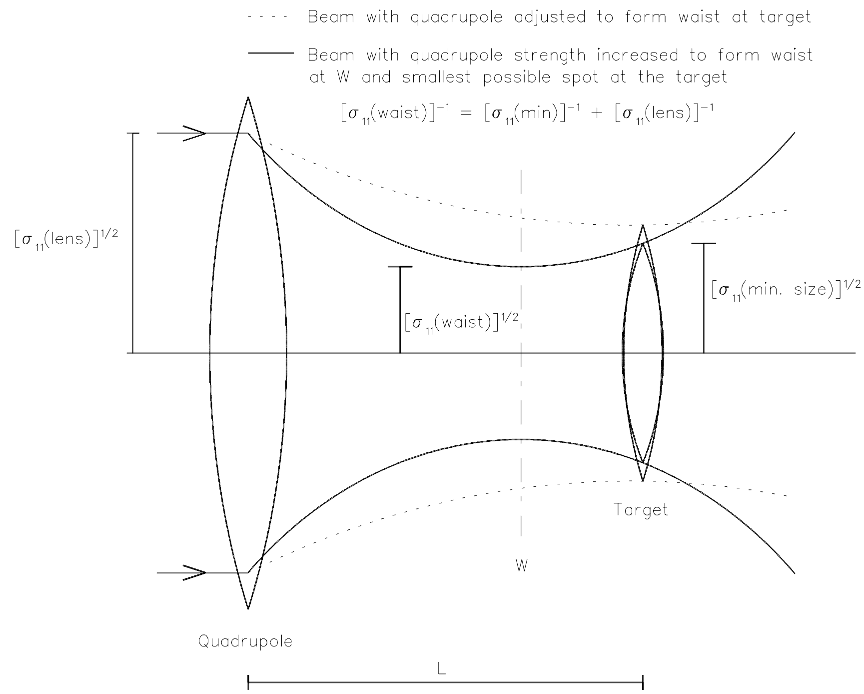

Fig. 20. The relationship between a waist and the smallest spot size at a target.

In a field-free region (a drift) the distance to a waist may be calculated if the sigma matrix is known at the location. Thus, if $\boldsymbol{\sigma_0}$ is the sigma matrix at the lens' exit and $\boldsymbol{\sigma_1}$ is that at the position of the waist, equation (81) gives (in the (x, θ) plane)

$$\sigma_{21}(1) = \sigma_{21}(0) + L\sigma_{22}(0) \tag{89}$$

and $\sigma_{21}(1)$ must be zero for there to be a waist at this position. Solving the above equation for L yields

$$L = -\frac{\sigma_{21}(0)}{\sigma_{22}(0)} = -r_{21}\,\frac{\sqrt{\sigma_{11}(0)}}{\sqrt{\sigma_{22}(0)}}\ . \tag{90}$$

Similarly, we obtain in the (y, ϕ) plane

$$L = -\frac{\sigma_{43}(0)}{\sigma_{44}(0)} = -r_{43}\,\frac{\sqrt{\sigma_{33}(0)}}{\sqrt{\sigma_{44}(0)}}\ . \tag{91}$$

It should also be noted that, in general, a waist and a point-to-point image are not one in the same. In the (x, θ) plane the transfer matrix for point-to-point imaging has been shown to be

$$\boldsymbol{R}(\text{point-to-point}) = \begin{bmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} M & 0 \\ \frac{1}{f} & \frac{1}{M} \end{bmatrix} \tag{92}$$

where $|\boldsymbol{R}| = R_{11}R_{21} = 1$ and M is the magnification. Again assuming an initially upright ellipse, the sigma matrix at the focal point is, from equation (81),

$$\boldsymbol{\sigma_1} = \begin{bmatrix} R_{11}^2\sigma_{11}(0) & R_{11}R_{21}\sigma_{11}(0) \\ R_{11}R_{21}\sigma_{11}(0) & R_{21}^2\sigma_{11}(0) + R_{22}^2\sigma_{22}(0) \end{bmatrix} . \tag{93}$$

Clearly, except for a very small source size, an image and a waist will occur only if $R_{21} = R_{21} = 0$. In order to have *two* matrix elements zero it is necessary to have (at least) two elements to vary. Given that we are at a focus, equation (90) gives the distance to a waist as

$$L = -\frac{\sigma_{21}(0)}{\sigma_{22}(0)} = -\frac{R_{11}R_{21}\sigma_{11}(0)}{R_{21}^2\sigma_{11}(0) + R_{22}^2\sigma_{22}(0)} . \tag{94}$$

If $R_{11}R_{21} = 0$, a waist and a point-to-point image coincide. If $R_{11}R_{21} < 0$, a waist follows the image.

8. Some 'building blocks' of transport systems

In the preceding sections various combinations of beam-transport elements have been discussed. It is true that they can be considered as 'building blocks' of a beam-transport line. We have, for example, considered quadrupole doublets and triplets operating in a focus-to-focus mode and a simple arrangement for the production of an achromatic beam. In this section a more general treatment of some of the standard building blocks used in developing a transport beamline will be considered. Specifically, some properties of symmetric systems will be investigated. Usually the discussions will deal with quadrupole systems only; however, in some cases quadrupole-dipole systems will be treated.

8.1 Translationally symmetric systems

For purposes of illustration, consider a quadrupole doublet positioned a distance L_1 downstream of a point A and a distance L_2 upstream of a point B. The transfer matrix for this particular configuration may be found using techniques that have been discussed earlier; let that transfer matrix be written as $\boldsymbol{R}$ where

$$\boldsymbol{R} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} .$$

We now ask the question "What happens if another, identical doublet configuration is installed between the points B and C?" as is indicated in the figure below.

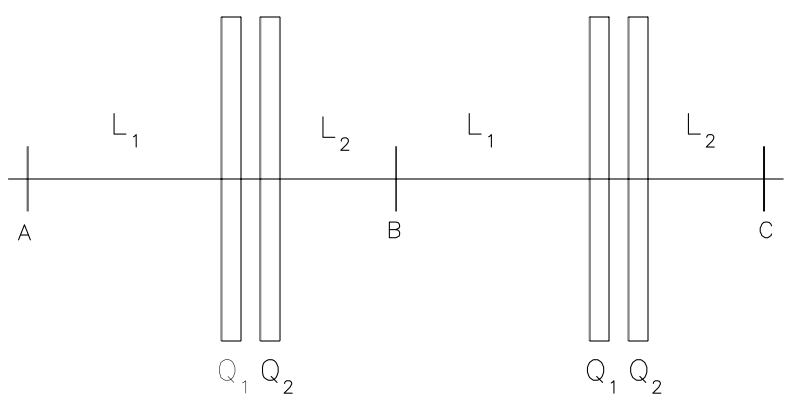

Fig. 21. A translationally-symmetric quadrupole system.

The transfer matrix for the complete system from A to C is found from

$$\boldsymbol{R}(A\rightarrow C) = \boldsymbol{R}^2 = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$$
$$= \begin{bmatrix} R_{11}(R_{11}+R_{22})-1 & R_{12}(R_{11}+R_{22}) \\ R_{21}(R_{11}+R_{22}) & R_{22}(R_{11}+R_{22})-1 \end{bmatrix} . \quad (95)$$

Two special situations become immediately apparent when we look at this overall transfer matrix.

If the original (doublet system)98) between A and B is designed such that $R_{11} = -R_{22}$, then the overall transfer matrix becomes

$$\boldsymbol{R}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad [R_{11} = -R_{22}] . \quad (96)$$

In this case the transfer matrix between A and C is $-\boldsymbol{I}$— the negative unity matrix. Beam conditions at C are exact inversions of those at A. Such a system is called a *unit section.* Notice that although a quadrupole doublet structure was used *any* system designed such that $R_{11} = -R_{22}$ could have been used.

The second special case occurs if the original system is designed such that $R_{12} = R_{21} = 0$ at the location B. In this case the overall transfer matrix becomes

$$\boldsymbol{R}^2 = \begin{bmatrix} R_{11}^2 & 0 \\ 0 & \dfrac{1}{R_{11}^2} \end{bmatrix} \qquad [R_{12} = R_{21} = 0] . \quad (97)$$

Thus if $|R_{11}| > 1$ at the midpoint of the section we obtain a magnified image at the point C but with the divergence there reduced, relative to that at A, by a factor equal to the magnification. This type of system is called *telescopic* (the first case discussed is also telescopic). Conversely, if R_{11} at B is less than unity, we obtain an image reduced in size but having a greater divergence.

A point in terminology: In the previous sections we often have talked about objects and images with the meaning that $R_{12} = 0$. In the jargon of the trade, this is called *point-to-point* imaging. The terminology is obvious: under that condition one point on the object is reproduced at one point on the image. In both cases discussed here we had $R_{21} = 0$. This type of focusing is called *parallel-to-parallel* imaging. In this case the divergences of particles at the image depend only on those at the object, regardless of the point on the object where they the particles originated. We also have *parallel-to-point* optics. In this case we have $R_{11} = 0$. Rays originating from the object with a given divergence are focused at one point on the object. That position is independent of the initial position at the object. The final imaging condition is called *point-to-parallel.* Here we have $R_{22} = 0$. Consequently, the divergence at the image depends only on the position at the object and is independent of the initial divergence of the particle.

Exercise:

A quadrupole pair, the first focusing horizontally and the second vertically, have a center-to-center separation s. The doublet is located midway between points A and B. Another identical doublet is placed between B and C. Thus the distances $\overline{AB}$ and $\overline{BC}$ are equal.

a) Using the thin-lens approximation determine the focal lengths of the quadrupoles that are required to produce a unit section between A and C.

b) Two targets are separated by 17.7 m. Use the results of part a) to determine what pole-tip fields are required to transport a 500 MeV proton beam from one target to the other. Assume each quadrupole has an effective length of 0.5 m and the doublets are separated by 0.4 m.

c) The distance between targets 1AT1 and 1AT2 on beamline 1A at TRIUMF is 17.7 m. Five quadrupoles, 1AQ9–13, are located in this region. If quadrupole 1AQ9 is turned off, quadrupoles 1AQ10–11 and 1AQ12–13 can be treated as doublets. Do the results of b) bear any resemblance to the actual settings used for 1AQ10–13?

d) repeat the procedure of part a) for the requirement that $R_{12} = R_{21} = 0$ at the point B.

8.2 Mirror-symmetric systems

Rather than put two identical systems together in the same order, we could make the second system the mirror image of the first. In this case we would have a combined system shown in the next figure.

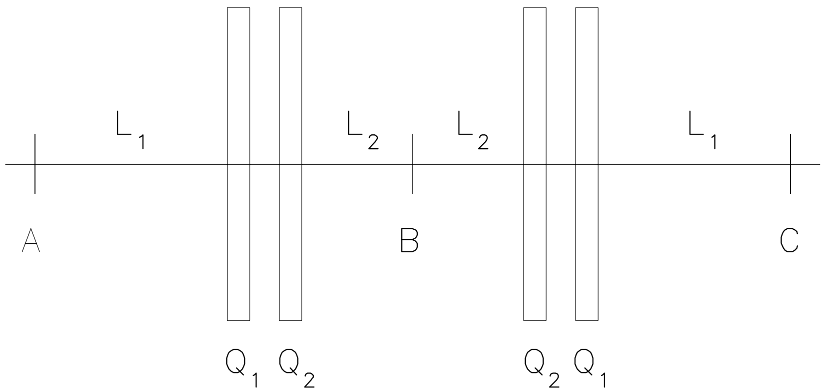

Fig. 22. A mirror-symmetric quadrupole configuration.

The transfer matrix from A to B for this system is

$$\begin{aligned}\boldsymbol{R}(A\to B) &= \begin{bmatrix} 1 & L_2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ +F_2 & 1 \end{bmatrix}\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -F_1 & 1 \end{bmatrix}\begin{bmatrix} 1 & L_1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 - x_1 + L_2 F^{+-} & s + L_1(1-x_1) + L_2(1 + x_2 + L_1 F^{+-}) \\ F^{+-} & 1 + x_2 + L_1 F^{+-} \end{bmatrix} \end{aligned} \tag{98}$$

where $F_i = \dfrac{1}{f}$, $x_i = s|F_i|$, and $F^{+-} = \dfrac{x_2(1-x_1) - x_1}{s}$.

In the above it has been assumed that the first quadrupole focuses horizontally and that the second focuses vertically. Now consider the transfer matrix that would result if the system were traversed from B to A. We have

$$\begin{aligned}\boldsymbol{R}(B\to A) &= \begin{bmatrix} 1 & L_1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -F_1 & 1 \end{bmatrix}\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ +F_2 & 1 \end{bmatrix}\begin{bmatrix} 1 & L_2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 + x_2 + L_1 F^{-+} & s + L_2(1-x_2) + L_1(1 - x_1 + L_2 F^{-+}) \\ F^{-+} & 1 - x_1 + L_2 F^{-+} \end{bmatrix} . \end{aligned} \tag{99}$$

But $F^{-+} = F^{+-}$ [*Exercise:* Prove this.] and

$$s + L_2(1-x_2) + L_1(1 - x_1 + L_2 F^{-+}) = s + L_1(1-x_1) + L_2(1 + x_2 + L_1 F^{+-}) ,$$

so that if we rewrite equation (98) as

$$\boldsymbol{R}(A\to B) = * \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} , \tag{100}$$

then equation (99) becomes

$$\boldsymbol{R}(B{\rightarrow}A) \quad = * \begin{bmatrix} R_{22} & R_{12} \\ R_{21} & R_{11} \end{bmatrix} . \tag{101}$$

Equations (100) and (101) recall to memory the relationship between a 2×2 matrix and its inverse:

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}^{-1} = \begin{bmatrix} R_{22} & -R_{12} \\ -R_{21} & R_{11} \end{bmatrix} ,$$

where the fact that $|\boldsymbol{R}| = 1$ has been used. Although we might expect $\boldsymbol{R}(B{\rightarrow}A)$ to be the inverse of $\boldsymbol{R}(A{\rightarrow}B)$, it is clear that this is not the case. The matrix obtained from traversing a system in reverse differs from the inverse of the matrix obtained from traversing the system in the forward direction in that the signs of the off-diagonal elements are reversed.

With a little thought, the reason for this becomes clear. Consider, for example, a system operating in a focus-to-focus mode. The beam diverges from the object and converges to the image. However, traveling in the reverse direction the beam diverges from the image and converges to the object. In other words, when traveling in the reverse direction *angles* are reversed with respect to those found when traveling in the forward direction. It can be shown that the transfer matrices for travel ind the forward and reverse directions are related by

$$\boldsymbol{R}(B{\rightarrow}A) \quad = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} [\boldsymbol{R}(A{\rightarrow}B)] \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} . \tag{102}$$

Remember, however, that this result [equation (102)] is valid *only* for a mirror-symmetric system!

We now realize that the trajectory for B to C will be identical to that from B to A. Consequently,

$$\boldsymbol{R}(B{\rightarrow}C) \quad = \quad \boldsymbol{R}(B{\rightarrow}A) \tag{103}$$

The overall transfer matrix for this system then becomes

$$\begin{aligned} \boldsymbol{R}(A{\rightarrow}C) &= \boldsymbol{R}(B{\rightarrow}C)\boldsymbol{R}(A{\rightarrow}B) \\ &= \boldsymbol{R}(B{\rightarrow}A)\boldsymbol{R}(A{\rightarrow}B) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} [\boldsymbol{R}(A{\rightarrow}B)]^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \boldsymbol{R}(A{\rightarrow}B) , \end{aligned} \tag{104}$$

for the case of a 2×2 matrix and

$$\boldsymbol{R}(A{\rightarrow}C) \quad = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [\boldsymbol{R}(A{\rightarrow}B)]^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{R}(A{\rightarrow}B) \tag{105}$$

for the case of a 3×3 matrix.

Explicitly, for the case of a 2×2 matrix the overall transfer matrix $\boldsymbol{M}$ is given by

$$\boldsymbol{M} \quad = \begin{bmatrix} 2R_{11}R_{22} - 1 & 2R_{12}R_{22} \\ 2R_{11}R_{21} & 2R_{11}R_{22} - 1 \end{bmatrix} , \tag{106}$$

and for the case of a 3×3 matrix by

$$M = \begin{bmatrix} 2R_{11}R_{22}-1 & 2R_{12}R_{22} & 2R_{12}R_{23} \\ 2R_{11}R_{21} & 2R_{11}R_{22}-1 & 2R_{11}R_{23} \\ 0 & 0 & 1 \end{bmatrix} . \tag{107}$$

There are two important things to notice about these two expressions. First, notice that $M_{11} = M_{22}$. This is a property of *all* mirror-symmetric systems. Second, notice that if $R_{23} = 0$ then $M_{11} = M_{22} = 0$. The latter shows that in order to make a mirror-symmetric system *doubly achromatic* it is necessary that the *angular* dispersion at the mid-plane (symmetry plane) be zero. If R_{23} cannot be made zero at the mid-plane it may be concluded that double achromaticity is impossible.

Two other conditions of interest may be obtained from equations (106) and (107). Suppose we design the system such that $R_{11} = R_{22} = 0$ at the mid-plane. In this case equation (106) becomes

$$M = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad [R_{11} = R_{22} = 0] . \tag{108}$$

If the design is such that $R_{12} = R_{21} = 0$ at the mid-plane, then the overall transfer matrix is

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad [R_{12} = R_{21} = 0] . \tag{109}$$

Thus in both cases the total system is telescopic. Whether inversion occurs depends on which matrix elements are made zero at the mid-plane.

Exercise:

In §6 the conditions necessary for double-achromaticity in a simple system were obtained. Use the technique developed above to verify the results obtained previously.

Exercise:

Between target locations 4BT1 and 4BT2 on beamline 4B at TRIUMF the configuration sketched below will be found.

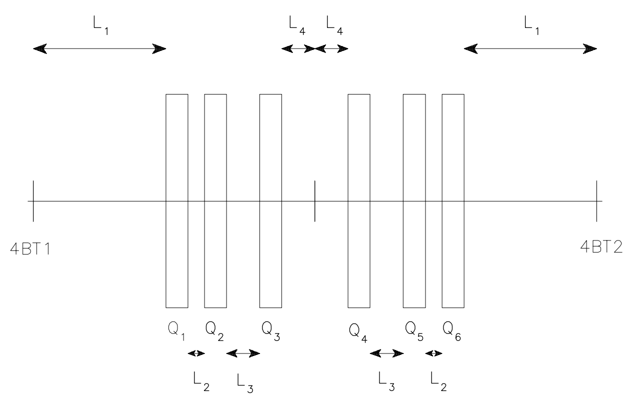

Fig. 23. Configuration between 4BT1 and 4BT2 on beamline 4B.

The purpose of this array will be discussed later. Quadrupoles Q_1 and Q_2 focus horizontally and are equally powered. A similar condition holds for quadrupoles Q_3 and Q_4 (but their fields are not equal to those of Q_1 and Q_2). Quadrupoles Q_2 and Q_5 focus in the vertical plane and, again, their fields are identical.

Clearly, this system is a mirror-symmetric beam-transport section. Its design is such that the transfer matrix in the horizontal plane is the unit matrix $\boldsymbol{I}$ while that in the vertical plane is the negative unit matrix $-\boldsymbol{I}$. Given that the effective length of all quadrupoles is 0.4090 m, $L_1 = 2.4096$ m, $L_2 = 0.3048$ m, $L_3 = 0.6348$ m, and $L_4 = 0.5750$ m, use the technique discussed in this section together with the thin-lens approximation to determine the quadrupole fields necessary to produce this condition. Assume a beam energy of 500 MeV.

Design values for the quadrupole fields are $B(Q_1) = B(Q_6) = 5.127$ kG, $B(Q_2) = B(Q_5) = -7.005$ kG, and $B(Q_1) = B(Q_6) = 8.000$ kG.

9. Waist-to-waist transport

In §7.3 the transformation properties of the phase-space ellipse were discussed. In particular, it was shown in equation (82) that if an initial phase-space ellipse was erect and described by the matrix $\boldsymbol{\sigma_0}$, then $\boldsymbol{\sigma_0}$ is transformed by a system with the transfer matrix $\boldsymbol{R}$ into the ellipse $\boldsymbol{\sigma_1}$ given by

$$\boldsymbol{\sigma_1} \quad = \quad \begin{bmatrix} R_{11}^2\sigma_{11}(0) + R_{12}^2\sigma_{22}(0) & R_{11}R_{21}\sigma_{11}(0) + R_{12}R_{22}\sigma_{22}(0) \\ R_{11}R_{21}\sigma_{11}(0) + R_{12}R_{22}\sigma_{22}(0) & R_{21}^2\sigma_{11}(0) + R_{22}^2\sigma_{22}(0) \end{bmatrix} . \tag{110}$$

In order that there be a waist at position 1 it is necessary that $\sigma_{21}(1) = 0$. This implies that

$$R_{11}R_{21}\sigma_{11}(0) + R_{12}R_{22}\sigma_{22}(0) \quad = \quad 0 . \tag{111}$$

From this equation three important special cases can arise.

First, if the transport system is a unit section or any system for which the matrix $\boldsymbol{R}$ has the form

$$\boldsymbol{R} \quad = \quad \begin{bmatrix} M & 0 \\ 0 & \dfrac{1}{M} \end{bmatrix} , \tag{112}$$

we will have a waist at position 1 because $\boldsymbol{\sigma_1}$ will have the form

$$\boldsymbol{\sigma_1} \quad = \quad \begin{bmatrix} M^2 & 0 \\ 0 & \dfrac{1}{M^2} \end{bmatrix} \boldsymbol{\sigma_0} . \tag{113}$$

The beam size at position 1 is then

$$\sqrt{\sigma_{11}(1)} \quad = \quad M\sqrt{\sigma_{11}(0)} . \tag{114}$$

The second case arises if neither R_{12} nor R_{21} is zero but the system is designed such that

$$R_{11}R_{21}\sigma_{11}(0) \quad = \quad -R_{12}R_{22}\sigma_{22}(0)$$

that is,

$$\frac{\sigma_{22}(0)}{\sigma_{11}(0)} \quad = \quad -\frac{R_{11}}{R_{22}}\cdot\frac{R_{21}}{R_{12}} . \tag{115}$$

In this case the beam size at location 1 is

$$\sqrt{\sigma_{11}(1)} \quad = \quad \sqrt{\frac{R_{11}}{R_{22}}}\cdot\sqrt{\sigma_{11}(0)} . \tag{116}$$

The third special case is that in which $R_{11} = R_{22}$. Then equation (111) reduces to

$$\sigma_{22}(0) \quad = \quad -\frac{R_{21}}{R_{12}}\cdot\sigma_{11}(0) . \tag{117}$$

But

$$\sigma_{11}(0)\sigma_{22}(0) \;=\; \det \boldsymbol{\sigma_0} \;=\; |\,\boldsymbol{\sigma_0}\,|$$

so that

$$\frac{|\,\boldsymbol{\sigma_0}\,|}{\sigma_{11}(0)} \;=\; -\frac{R_{21}}{R_{12}}\cdot\sigma_{11}(0) \;,$$

or

$$\sigma_{11}(0) \;=\; \sqrt{-\frac{R_{12}}{R_{21}}\cdot|\,\boldsymbol{\sigma_0}\,|}\;. \tag{118}$$

From equation (118) it follows immediately that

$$\sigma_{11}(1) \;=\; \sigma_{11}(0) \;, \tag{119}$$

that is, the waist at location 1 is identical to that at position 0.

9.1 Waist-to-waist transport in one plane

Consider the simple transport system shown below.

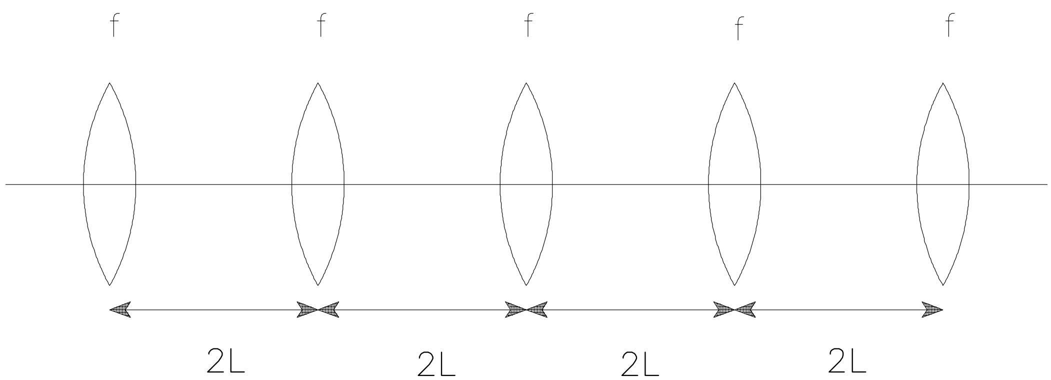

Fig. 24. A one-dimensional line of focusing lenses.

This system consists of a series of (focusing) lenses of focal length f that are spaces a distance $2L$ units apart. We wish to calculate the required value of f such that a waist is produced at the principal plane of each lens and all waists are of the same minimum size.

Because of the repetitive nature of this system it is convenient to think of it as composed of a series of subsystems. Each subsystem comprises two lenses, each of focal length $2f$, that are separated by a distance of $2L$. One such subsystem is pictured below.

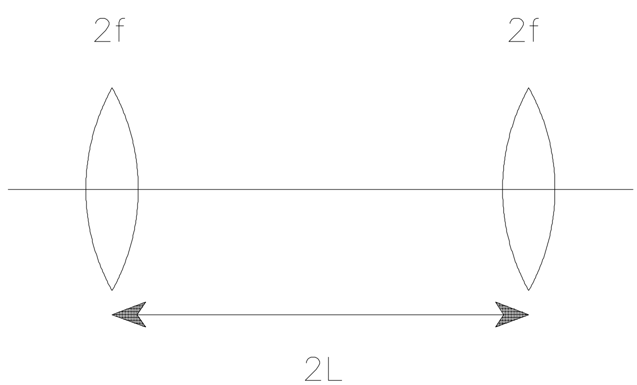

Fig. 25. A subsystem of the one-dimensional line of focusing lenses.

The transfer matrix of this subsystem is

$$\boldsymbol{R} = \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{2f} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{2f} & 1 \end{bmatrix} = \begin{bmatrix} 1 - \dfrac{L}{f} & 2L \\ -\dfrac{1}{f}\left(1 - \dfrac{L}{2f}\right) & 1 - \dfrac{L}{f} \end{bmatrix} . \tag{120}$$

From equations (118) and (119) it follows that

$$\sigma_{11}(1) = \sigma_{11}(0) = \sqrt{-\frac{R_{12}}{R_{21}} \cdot |\, \boldsymbol{\sigma_0} \,|} = \sqrt{\frac{4Lf^2}{2f - L} \cdot |\, \boldsymbol{\sigma_0} \,|} . \tag{121}$$

Squaring this equation and substituting $|\, \boldsymbol{\sigma_0} \,| = \sigma_{11}(0)\sigma_{22}(0)$ yields

$$\Psi = \frac{\sigma_{11}(0)}{\sigma_{22}(0)} = \frac{4Lf^2}{2f - L} .$$

To obtain the minimum value for $\sigma_{11}(0)$ we differentiate this expression with respect to f and set it equal to zero. Thus we have

$$\frac{\mathrm{d}\Psi}{\mathrm{d}f} = \frac{8Lf}{(2f - L)^2}\,(f - L) = 0$$

which requires $f = L$. Substitution of this result into the expression for Ψ leads to

$$f = L = \frac{1}{2}\sqrt{\frac{\sigma_{11}(0)}{\sigma_{22}(0)}} . \tag{122}$$

Equation (122) expresses the values of f and L in terms of the parameters at the first waist.

This result is quoted in ref[1] as an example of waist-to-waist transport in one plane. In the following section we will examine a more useful configuration.

9.2 Waist-to-waist transport in two planes — the F0D0 array

The example of §9.1 is interesting from an academic point of view, but it is not practical for beamline transport problems. We know that a quadrupole focuses in one direction and defocuses in the authorial plane. A logical extension of the one-dimensional array is sketched below.

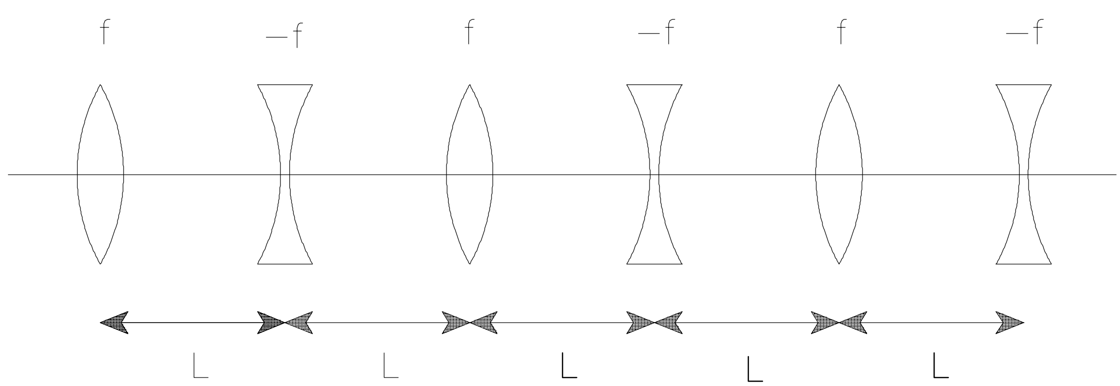

Fig. 26. A two-dimensional F0D0 array.

In this arrangement the focal lengths of all quadrupoles are equal in magnitude but they are arranged so that their focusing planes alternate in sign. The center-to-center separation of the quadrupoles is L. This structure is termed a *F0D0* array because, for example, if the first quadrupole is focusing (F) it is followed by a drift space (0)—that is, no focusing. The next quadrupole is defocusing (D) and it too is followed by a drift space (0). The configuration repeats again.

a F0D0 array is used to transport a beam over long distances while keeping the beam within the available apertures. We design the system such that there is a of the same size at the principal planes of the coursing quadrupoles. This size is adjusted so as to be within the allowable apertures of the quadrupoles and the beam tube.

As with the one-dimensional array it is convenient to consider the F0D0 array to be formed from subsystems. One such subsystem is illustrated below.

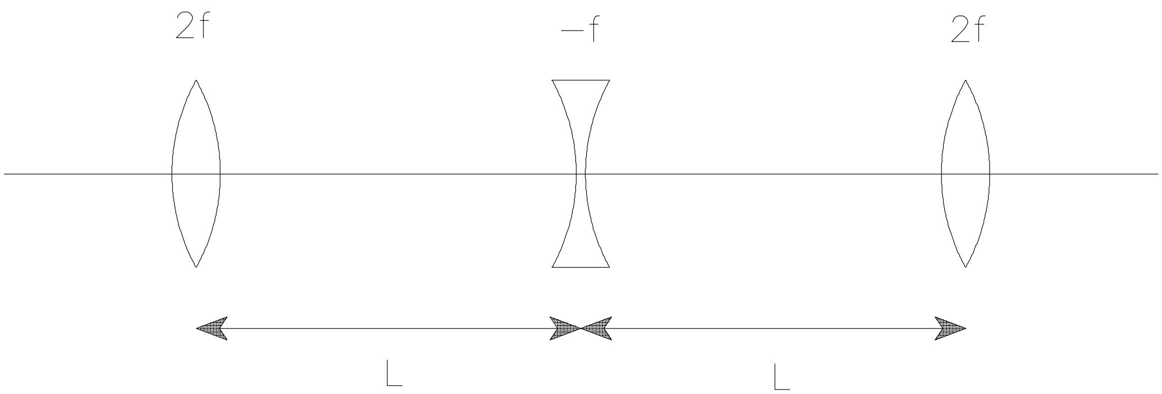

Fig. 27. A subsystem of a two-dimensional F0D0 array.

The transfer matrix of this subsystem is give by

$$\begin{aligned}\boldsymbol{R}_{\mathrm{HVH}}(\mathrm{cell}) &= \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{2f} & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dfrac{1}{2f} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dfrac{1}{2f} & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{2f} & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1-\dfrac{L^2}{2f^2} & L\left(2+\dfrac{L}{f}\right) \\ -\dfrac{L}{4f^2}\left(2-\dfrac{L}{f}\right) & 1-\dfrac{L^2}{2f^2} \end{bmatrix} = \begin{bmatrix} \dfrac{2f^2-L^2}{2f^2} & \dfrac{L}{f}(L+2f) \\ -\dfrac{L}{4f^3}(2f-L) & \dfrac{2f^2-L^2}{2f^2} \end{bmatrix}, \end{aligned} \tag{123}$$

where the center lens (2) has been replaced by two half-lenses. For reference, the transfer matrix at the midpoint of the center lens is

$$\boldsymbol{R}_{\mathrm{VH}}(\mathrm{mid}) = \begin{bmatrix} 1 & 0 \\ \dfrac{1}{2f} & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{2f} & 1 \end{bmatrix} = \begin{bmatrix} \dfrac{2f-L}{2f} & L \\ -\dfrac{L}{4f^2} & \dfrac{2f+L}{2f} \end{bmatrix}. \tag{124}$$

We have seen above that given an initial waist at the center of the first lens, a waist will exist at the center of the third lens provided that

$$\Psi_{\mathrm{HVH}} = \left.\frac{\sigma_{11}(1)}{\sigma_{22}(1)}\right|_{\mathrm{HVH}} = -\left.\frac{R_{12}(\mathrm{cell})}{R_{21}(\mathrm{cell})}\right|_{\mathrm{HVH}} = 4f^2\left[\frac{2f+L}{2f-L}\right]. \tag{125}$$

To minimize the waist size we again differentiate Ψ_{HVH} with respect to f, set the result equal to zero, and solve for f in terms of the element separations L. Thus we find

$$\frac{\mathrm{d}\Psi_{\mathrm{HVH}}}{\mathrm{d}f} = -8f^2\frac{L+2f}{(L-2f)^2} - \frac{8f(L+3f)}{L-2f} = -\frac{8f}{(L-2f)^2}[L^2+2f(L-2f)] = 0 ,$$

from which we obtain

$$f_{\mathrm{HVH}} = \begin{cases} \frac{L}{4}(1-\sqrt{5}) \\ 0 \\ \frac{L}{4}(1+\sqrt{5}) \end{cases} . \tag{126}$$

Because we have taken both f and L to be positive, and discarding the trivial solution of $f_{\mathrm{HVH}} = 0$, we finally have

$$\frac{L}{f_{\mathrm{HVH}}} = \frac{4}{1+\sqrt{5}} \tag{127}$$

It is readily shown that there is a waist at the center of the second lens for we have

$$\begin{aligned} \sigma_{12}(2) &= R_{11}(\mathrm{mid})R_{21}(\mathrm{mid})\sigma_{11}(1) + R_{12}(\mathrm{mid})R_{22}(\mathrm{mid})\sigma_{22}(1) \\ &= \left[R_{11}(\mathrm{mid})R_{21}(\mathrm{mid}) - R_{12}(\mathrm{mid})R_{22}(\mathrm{mid})\frac{R_{21}(\mathrm{mid})}{R_{12}(\mathrm{mid})}\right]\sigma_{11}(1) \\ &= \left[-\frac{L}{4f^2}\frac{2f-1}{2f} - L\frac{2f+1}{2f}\frac{L(L-2f)}{4f^3}\frac{f}{L(L+2f)}\right]\sigma_{11}(1) \\ &= [0]\sigma_{11}(1) . \end{aligned} \tag{128}$$

Thus $\sigma_{12}(2) = 0$, meaning that there is also a waist at the center of the (defocusing) lens 2.

It is instructive to compare the sizez of the waists at the centers of the focusing and defocusing lenses. We have

$$\begin{aligned} \sigma_{12}(2) &= R_{11}^2(\mathrm{mid})\sigma_{11}(1) + R_{12}^2(\mathrm{mid})\sigma_{22}(1) \\ &= \left[R_{11}^2(\mathrm{mid}) - R_{12}^2(\mathrm{mid})\frac{R_{21}(\mathrm{mid})}{R_{12}(\mathrm{mid})}\right]\sigma_{11}(1) \\ &= \left[\frac{(L-2f)^2}{4f^2} - L^2\frac{L(L-2f)}{4f^3}\frac{f}{L(L+2f)}\right]\sigma_{11}(1) \\ &= -\frac{L-2f}{L+2f}\sigma_{11}(1) , \end{aligned} \tag{129}$$

or

$$\frac{\sigma_{11}(1)}{\sigma_{11}(2)} = -\frac{L+2f}{L-2f} . \tag{130}$$

Substituting the result from equation (127) above we have

$$\frac{\sigma_{11}(1)}{\sigma_{11}(2)} = -\frac{3+\sqrt{5}}{1-\sqrt{5}} = 4.236 , \tag{131}$$

and the ratio of the beam sizes at the two locations is

$$\frac{x(1)}{x(2)} = \sqrt{\frac{\sigma_{11}(1)}{\sigma_{11}(2)}} = 2.058 . \tag{132}$$

Exercise: Consider now the *vertical* plane of a F0D0 array. Show that the transfer matrix for a cell is

$$\boldsymbol{R}_{\mathrm{VHV}}(\mathrm{cell}) \quad = \quad \begin{bmatrix} \dfrac{2f^2 - L^2}{2f^2} & \dfrac{L}{f}(2f - L) \\ -\dfrac{L}{4f^3}(2f + L) & \dfrac{2f^2 - L^2}{2f^2} \end{bmatrix} , \tag{133}$$

where the center lens (2) has been replaced by two half-lenses, and that the the transfer matrix at the midpoint of the center lens is

$$\boldsymbol{R}_{\mathrm{HV}}(\mathrm{mid}) \quad = \quad \begin{bmatrix} \dfrac{2f + L}{2f} & L \\ -\dfrac{L}{4f^2} & \dfrac{2f - L}{2f} \end{bmatrix} . \tag{134}$$

Hence show that

$$\Psi_{\mathrm{VHV}} \;=\; \frac{\sigma_{11}(1)}{\sigma_{22}(1)} \;=\; -\frac{R_{12}(\mathrm{cell})}{R_{21}(\mathrm{cell})} \;=\; -\,4f^2\left[\frac{L - 2f}{L + 2f}\right] ,$$

from which we see that the conditions on the initial phase-space ellipse for waist-to-waist transport in the vertical plane differ from those in the horizontal plane. Proceed as before to show that

$$\frac{\mathrm{d}\Psi_{\mathrm{VHV}}}{\mathrm{d}f} \;=\; 8f^2\frac{L - 2f}{(L + 2f)^2} - \frac{8f(L - 3f)}{L + 2f} \;=\; -\,\frac{8f}{(L + 2f)^2}[L^2 - 2f(L + 2f)] \;.$$

Equate this to zero and show that the relationship between f_{VHV} and L are

$$f_{\mathrm{VHV}} \quad = \quad \begin{cases} \dfrac{L}{4}(\sqrt{5} - 1) \\ 0 \\ -\dfrac{L}{4}(1 + \sqrt{5}) \end{cases} . \tag{135}$$

Comparison of equations (125) and (134) shows that if the system is designed to mimimize the beam size in the *horizontal* plane, that in the vertical plane is not simultaneously minimized. However, the ratio of the extrema of the vertical and horizontal ellipses is given by

$$\frac{\Psi_{\mathrm{VHV}}}{\Psi_{\mathrm{HVH}}} \;=\; \left.\frac{\sigma_{11}(1)}{\sigma_{22}(1)}\right|_{\mathrm{VHV}} \Bigg/ \left.\frac{\sigma_{11}(1)}{\sigma_{22}(1)}\right|_{\mathrm{HVH}} \;=\; \left[\frac{-4f^2(L - 2f)}{L + 2f}\right]\left[\frac{L - 2f}{-4f^2(L + 2f)}\right] \;=\; \left[\frac{L - 2f}{L + 2f}\right]^2 . \tag{136}$$

Using the value given in equation (126) for f_{HVH} we find

$$\frac{\Psi_{\mathrm{VHV}}}{\Psi_{\mathrm{HVH}}} \;=\; \left[\frac{1 - \sqrt{5}}{3 + \sqrt{5}}\right]^2 \;=\; 0.05573 \;. \tag{137}$$