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Subject Partial longitudinal polarization with an SDSD system		

1. Introduction

In 1986 a solenoid-dipole-solenoid-dipole system (SDSD) was installed in the vault section of beamline 4 to allow the production of completely longitudinal and sideways polarized beams in the proton experimental area[1−4]. Recently, proponents of the parity experiment have asked for solutions of this system in which there is partial longitudinal polarization and partial sideways or vertical polarization. The reason for this, apparently, is that the SDSD system works too well and they cannot obtain any non-longitudinal components of polarization with which to calibrate their polarimeters. This note presents solutions for longitudinal-sideways and longitudinal-vertical polarizations.

2. Basic relationships

We begin by stating the basic relationships that have been developed. For the production of a completely longitudinally polarized beam from one that is completely vertically polarized, we have from ref[1]

$$\sin\phi_1 = \frac{\cos\Theta_2}{\sin\Theta_1} \qquad \text{and} \qquad \cos\phi_2 = \frac{1}{\tan\Theta_1\cdot\tan\Theta_2}$$

In these expressions, the ϕ_i are the spin precession angles in the solenoids and the Θ_i are those in the dipoles. Explicitly, we have

$$\phi = \mu\,\frac{B_0 L}{(B\rho)_0} \qquad \text{and} \qquad \Theta = \gamma G\theta$$

in which:

	μ	=	the magnetic moment of the particle
	B_0	=	the axial field of the solenoid
	L	=	the effective length of the solenoid
	$(B\rho)_0$	=	the magnetic rigidity of the particle
	G	=	(g/2 − 1) = 1.79285
	θ	=	the bend angle of the dipole

and γ has its usual relativistic meaning.

Similarly, from ref[2], a beam of complete sideways polarization is produced from one of complete vertical polarization using the relationships

$$\sin(\phi_1) = -\frac{\sin\Theta_2}{\sin\Theta_1} \qquad \text{and} \qquad \cos\phi_2 = -\frac{\tan\Theta_2}{\tan\Theta_1}$$

Finally, ref[3] gives the general spin transfer through an SDSD system of a beam with initial polarization $\mathbf{P_0} = (P_{x0}, P_{y0}, P_{z0})$ in a coordinate system with the z-axis vertical, the y-axis along the beam direction and the x-y plane horizontal as

$$\mathbf{P_f} = \begin{bmatrix} P_{xf} \\ P_{yf} \\ P_{zf} \end{bmatrix} = \mathbf{R_{SDSD}}(\Theta_1,\phi_1,\Theta_2,\phi_2)\,\mathbf{P_0} = \mathbf{R_{SDSD}}(\Theta_1,\phi_1,\Theta_2,\phi_2)\begin{bmatrix} P_{x0} \\ P_{y0} \\ P_{z0} \end{bmatrix}$$

$\mathbf{P_f} = (P_{xf}, P_{yf}, P_{zf})$ is the resulting polarization vector and the elements of the $\mathbf{R_{SDSD}}(\Theta_1,\phi_1,\Theta_2,\phi_2)$

matrix are

$$
\begin{aligned}
R_{11} &= \cos(\phi_1)\,[\cos(\phi_2)\,\cos(\Theta_1)\,\cos(\Theta_2) - \sin(\Theta_1)\,\sin(\Theta_2)] - \sin(\phi_1)\,\sin(\phi_2)\,\cos(\Theta_2) \\
R_{21} &= \cos(\phi_1)\,[\cos(\phi_2)\,\cos(\Theta_1)\,\sin(\Theta_2) + \sin(\Theta_1)\,\cos(\Theta_2)] - \sin(\phi_1)\,\sin(\phi_2)\,\sin(\Theta_2) \\
R_{31} &= -[\sin(\phi_1)\,\cos(\phi_2) + \cos(\phi_1)\,\sin(\phi_2)\,\cos(\Theta_1)] \\
\\
R_{12} &= -[\cos(\Theta_1)\,\sin(\Theta_2) + \cos(\phi_2)\,\sin(\Theta_1)\,\cos(\Theta_2)] \\
R_{22} &= \cos(\Theta_1)\,\cos(\Theta_2) - \cos(\phi_2)\,\sin(\Theta_1)\,\sin(\Theta_2) \\
R_{32} &= \sin(\phi_2)\,\sin(\Theta_1) \\
\\
R_{13} &= \sin(\phi_1)\,[\cos(\phi_2)\,\cos(\Theta_1)\,\cos(\Theta_2) - \sin(\Theta_1)\,\sin(\Theta_2)] + \cos(\phi_1)\,\sin(\phi_2)\,\cos(\Theta_2) \\
R_{23} &= \sin(\phi_1)\,[\cos(\phi_2)\,\cos(\Theta_1)\,\sin(\Theta_2) + \sin(\Theta_1)\,\cos(\Theta_2)] + \cos(\phi_1)\,\sin(\phi_2)\,\sin(\Theta_2) \\
R_{33} &= \cos(\phi_1)\,\cos(\phi_2) - \sin(\phi_1)\,\sin(\phi_2)\,\cos(\Theta_1)
\end{aligned}
$$

In what follows, it is assumed that the initial beam is *completely* vertically polarized. Consequently, because the initial polarization vector $\mathbf{P_0}$ has the components (0, 0, 1), only the R_{i3} matrix elements will be involved in further calculations.

3. Longitudinal and Vertical Polarizations

Here we consider the production of a beam with a final polarization vector $\mathbf{P_f} = (0,\, P_{yf},\, P_{zf})$. We begin by solving for $\sin(\phi_2)$ in the equation $R_{13} = 0$. Thus

$$\sin(\phi_2) = -\frac{\sin(\phi_1)[\cos(\phi_2)\,\cos(\Theta_1)\,\cos(\Theta_2) - \sin(\Theta_1)\,\sin(\Theta_2)]}{\cos(\phi_1)\,\cos(\Theta_2)}$$

This we substitute into the equation

$$P_{yf} = R_{23}\,P_{z0} = R_{23}$$

to obtain

$$\sin(\phi_1) = \frac{\cos(\Theta_2)}{\sin(\Theta_1)}\,P_{yf} \tag{1}$$

This relationship then determines the setting of the first solenoid for any given value of P_{yf}. We note that equation (1) is identical to that for $\sin(\phi_1)$ in ref[1)] if complete longitudinal polarization is desired—that is, $P_{yf} = 1$ and $P_{zf} = 0$.

An expression for $\cos(\phi_2)$ is obtained by substituting equation (1) into the equation

$$P_{zf} = R_{33}\,P_{z0} = R_{33}$$

to obtain

$$\cos(\phi_2) = \frac{P_{yf}^2\,\cos(\Theta_1)\,\sin(\Theta_1)\,\cos(\Theta_2)\,\sin(\Theta_2) + P_{zf}\cos(\phi_1)\,\sin^2(\Theta_1)}{P_{yf}^2\,\cos^2(\Theta_1)\,\cos^2(\Theta_2) + \cos^2(\phi_1)\,\sin^2(\Theta_1)} \tag{2}$$

We note that $\cos(\phi_2)$ is now defined in terms of ϕ_1 as well as Θ_1 and Θ_2 instead of only the latter two as was the case for complete longitudinal polarization.

As a check on our calculation, we evaluate $\cos(\phi_2)$ for the case of completely longitudinal polarization by setting $P_{zf} = 0$ and $P_{yf} = 1$. This yields

$$\cos(\phi_2) = \frac{\cos(\Theta_1)\,\sin(\Theta_1)\,\cos(\Theta_2)\,\sin(\Theta_2)}{\sin(\Theta_1)\,\sin^2(\Theta_2)} = \frac{1}{\tan(\Theta_1)\,\tan(\Theta_2)}$$

This is, again, the same expression that is given in ref[1].

Table 1 presents the results of numerical calculations for a beam energy of 222 MeV on beamline 4A2 for a mixture of longitudinal and vertical polarizations.

4. Longitudinal and Sideways Polarizations

In this instance we want the final polarization vector to have the components $\mathbf{P_f} = (P_{xf}, P_{yf}, 0)$. We begin by solving for $\sin(\phi_1)$ in the equation $R_{33} = 0$. This produces

$$\sin(\phi_1) = \frac{\cos(\phi_1)\,\cos(\phi_2)}{\cos(\Theta_1)\,\sin(\phi_2)}$$

This result is substituted into the equation

$$P_{yf} = R_{23}\,P_{z0} = R_{23}$$

to give

$$\cos(\phi_1) = \frac{\cos(\Theta_1)\,\sin(\phi_2)}{\cos(\Theta_1)\,\sin(\Theta_2) + \sin(\Theta_1)\,\cos(\Theta_2)\,\cos(\phi_2)}\,P_{yf}$$

These values for $\sin(\phi_1)$ and $\cos(\phi_1)$ are now substited into the eqnation $P_{xf} = R_{13}\,P_{z0} = R_{13}$ to provide a solution for $\cos(\phi_2)$ of

$$\cos(\phi_2) = \frac{\cos(\Theta_1)[\cos(\Theta_2)\,P_{yf} - \sin(\Theta_2)\,P_{xf}]}{\sin(\Theta_1)[\sin(\Theta_2)\,P_{yf} + \cos(\Theta_2)\,P_{xf}]} = \frac{1}{\tan(\Theta_1)} \cdot \frac{\cos(\Theta_2)\,P_{yf} - \sin(\Theta_2)\,P_{xf}}{\sin(\Theta_2)\,P_{yf} + \cos(\Theta_2)\,P_{xf}} \tag{3}$$

Substitution of $P_{xf} = 0$ and $P_{yf} = 1$ into this expression yields

$$\cos(\phi_2) = \frac{1}{\tan(\Theta_1) \cdot \tan(\Theta_2)}$$

which is the condition on $\cos(\phi_2)$ for the production of complete longitudinal polarization. Substitution of $P_{xf} = 1$ and $P_{yf} = 0$ into the expression yields

$$\cos(\phi_2) = -\frac{\tan(\Theta_2)}{\tan(\Theta_1)}$$

and this is the condition on ϕ_2, given in ref[2], for the production of complete sideways polarization.

This result for $\cos(\phi_2)$ is now substituted into the expression for $\sin(\phi_1)$ to yield

$$\sin(\phi_1) = \frac{\cos(\Theta_2)\,P_{yf} - \sin(\Theta_2)\,P_{xf}}{\sin(\Theta_1)} \tag{4}$$

Evaluating this expression for $P_{xf} = 0$ and $P_{yf} = 1$ produces

$$\sin(\phi_1) = \frac{\cos(\Theta_2)}{\sin(\Theta_1)}$$

and agrees with the requirement on $\sin(\phi_1)$ for the production of complete longitudinal polarization. Similarly, evaluation of this expression for $P_{xf} = 1$ and $P_{yf} = 0$ produces

$$\sin(\phi_1) = -\frac{\sin(\Theta_2)}{\sin(\Theta_1)}$$

which is the expression for $\sin(\phi_1)$ from ref[2] for the production of complete sideways polarization.

Table 2 presents the results of numerical calculations for a beam energy of 222 MeV on beamline 4A2 for a mixture of longitudinal and sideways polarizations.

Tables 3 and 4 give similar results with finer steps of P_{zf} and P_{xf}.

References

1. M.K. Craddock, *TRIUMF Design Note* TRI-DN-84-8.
2. G.M. Stinson, *TRIUMF Design Note* TRI-DNA-84-3.
3. G.M. Stinson, *TRIUMF Design Note* TRI-DNA-86-1.
4. R. Abegg et al., *Nuclear Instruments and Methods*, **A273** (1988).

Table 1

Solutions for beamline 4A2 with longitudinal and vertical polarizations

P_{yf}	P_{zf}	ϕ_1 (°)	B_{Sol1} (kG)	ϕ_2 (°)	B_{Sol2} (kG)
1.000000	0.000000	63.425767	18.061992	87.366682	24.879735
0.950000	0.312250	58.172289	16.565939	51.600606	14.694497
0.900000	0.435890	53.602526	15.264591	40.970672	11.667371
0.850000	0.526783	49.482025	14.091180	34.310069	9.770606
0.800000	0.600000	45.682912	13.009293	29.492029	8.398555
0.750000	0.661438	42.126261	11.996452	25.729129	7.326980
0.700000	0.714143	38.759212	11.037605	22.641474	6.447697
0.650000	0.759934	35.544218	10.122059	20.017460	5.700447
0.600000	0.800000	32.453388	9.241872	17.727612	5.048358
0.550000	0.835165	29.465245	8.390928	15.687163	4.467292
0.500000	0.866025	26.562755	7.564375	13.837658	3.940602
0.450000	0.893029	23.732054	6.758265	12.137058	3.456315
0.400000	0.916515	20.961601	5.969313	10.554037	3.005512
0.350000	0.936750	18.241591	5.194726	9.064520	2.581337
0.300000	0.953939	15.563527	4.432084	7.649466	2.178367
0.250000	0.968246	12.919913	3.679252	6.293404	1.792196
0.200000	0.979796	10.304011	2.934312	4.983420	1.419147
0.150000	0.988686	7.709664	2.195510	3.708442	1.056067
0.100000	0.994987	5.131136	1.461213	2.458703	0.700174
0.050000	0.998749	2.562994	0.729873	1.225327	0.348941
0.000000	1.000000	0.000000	0.000000	0.000000	0.000000

These calculations are for a proton beam of energy 222 MeV, ($\gamma = 1.236605$, $\mu = 2.792847$ and $G = (g/2-1) = 1.792847$). Spin precession angles in the dipoles that bend 40.0° and 12.0° are 88.682° and 26.605°, respectively. The effective lengths of the solenoids are 0.49962 m.

Table 2

Solutions for beamline 4A2 with longitudinal and sideways polarizations

P_{yf}	P_{xf}	ϕ_1 (°)	B_{Sol1} (kG)	ϕ_2 (°)	B_{Sol2} (kG)
1.000000	0.000000	63.425767	18.061992	87.366682	24.879735
0.950000	0.312250	45.215872	12.876292	88.672152	25.251499
0.900000	0.435890	37.565197	10.697581	88.986294	25.340958
0.850000	0.526783	31.616472	9.003540	89.188621	25.398576
0.800000	0.600000	26.533142	7.555942	89.341887	25.442222
0.750000	0.661438	21.991970	6.262735	89.467675	25.478043
0.700000	0.714143	17.827349	5.076761	89.576111	25.508923
0.650000	0.759934	13.940834	3.969983	89.672804	25.536458
0.600000	0.800000	10.268113	2.924089	89.761215	25.561635
0.550000	0.835165	6.764280	1.926289	89.843656	25.585112
0.500000	0.866025	3.396369	0.967197	89.921771	25.607358
0.450000	0.893029	0.139190	0.039638	89.996795	25.628723
0.400000	0.916515	−3.027154	−0.862054	90.069703	25.649485
0.350000	0.936750	−6.118841	−1.742485	90.141301	25.669874
0.300000	0.953939	−9.149370	−2.605500	90.212289	25.690090
0.250000	0.968246	−12.130278	−3.454384	90.283303	25.710312
0.200000	0.979796	−15.071655	−4.292011	90.354948	25.730715
0.150000	0.988686	−17.982526	−5.120951	90.427826	25.751469
0.100000	0.994987	−20.871142	−5.943553	90.502563	25.772752
0.050000	0.998749	−23.745217	−6.762014	90.579830	25.794756
0.000000	1.000000	−26.612128	−7.578435	90.660379	25.817694

These calculations are for a proton beam of energy 222 MeV, ($\gamma = 1.236605$, $\mu = 2.792847$ and $G = (g/2 - 1) = 1.792847$). Spin precession angles in the dipoles that bend 40.0° and 12.0° are 88.682° and 26.605°, respectively. The effective lengths of the solenoids are 0.49962 m.

Table 3

Solutions for beamline 4A2 with longitudinal and vertical polarizations

P_{yf}	P_{zf}	ϕ_1 (°)	B_{Sol1} (kG)	ϕ_2 (°)	B_{Sol2} (kG)
1.000000	0.000000	63.425767	18.061992	87.366682	24.879735
0.995000	0.099875	62.858640	17.900489	74.796978	21.300214
0.990000	0.141067	62.302259	17.742046	69.840489	19.888736
0.985000	0.172554	61.755985	17.586482	66.183309	18.847267
0.980000	0.198997	61.219239	17.433631	63.210972	18.000824
0.975000	0.222205	60.691495	17.283343	60.681548	17.280510
0.970000	0.243105	60.172272	17.135482	58.469119	16.650469
0.965000	0.262250	59.661129	16.989922	56.497888	16.089114
0.960000	0.280000	59.157661	16.846547	54.717885	15.582216
0.955000	0.296606	58.661496	16.705253	53.094043	15.119788
0.950000	0.312250	58.172289	16.565939	51.600606	14.694497
0.945000	0.327070	57.689721	16.428517	50.217980	14.300761
0.940000	0.341174	57.213496	16.292900	48.930852	13.934221
0.935000	0.354648	56.743338	16.159012	47.726982	13.591390
0.930000	0.367560	56.278991	16.026778	46.596405	13.269432
0.925000	0.379967	55.820216	15.896131	45.530886	12.966000
0.920000	0.391918	55.366787	15.767006	44.523529	12.679131
0.915000	0.403454	54.918495	15.639344	43.568493	12.407162
0.910000	0.414608	54.475143	15.513089	42.660783	12.148670
0.905000	0.425412	54.036545	15.388188	41.796091	11.902429
0.900000	0.435890	53.602526	15.264591	40.970672	11.667371
0.895000	0.446066	53.172922	15.142251	40.181249	11.442564
0.890000	0.455961	52.747577	15.021124	39.424935	11.227186
0.885000	0.465591	52.326344	14.901168	38.699177	11.020509
0.880000	0.474974	51.909084	14.782343	38.001700	10.821886
0.875000	0.484123	51.495666	14.664613	37.330473	10.630738
0.870000	0.493052	51.085964	14.547940	36.683670	10.446545
0.865000	0.501772	50.679859	14.432292	36.059648	10.268840

These calculations are for a proton beam of energy 222 MeV, (γ = 1.236605, μ = 2.792847 and $G = (g/2-1)$ = 1.792847). Spin precession angles in the dipoles that bend 40.0° and 12.0° are 88.682° and 26.605°, respectively. The effective lengths of the solenoids are 0.49962 m.

Table 4

Solutions for beamline 4A2 with longitudinal and sideways polarizations

P_{yf}	P_{xf}	ϕ_1 (°)	B_{Sol1} (kG)	ϕ_2 (°)	B_{Sol2} (kG)
1.000000	0.000000	63.425767	18.061992	87.366682	24.879735
0.995000	0.099875	57.687467	16.427875	87.916876	25.036416
0.990000	0.141067	55.307753	15.750195	88.096550	25.087583
0.985000	0.172554	53.479573	15.229577	88.220570	25.122900
0.980000	0.198997	51.936487	14.790147	88.317223	25.150424
0.975000	0.222205	50.575340	14.402528	88.397126	25.173179
0.970000	0.243105	49.343253	14.051662	88.465574	25.192671
0.965000	0.262250	48.208826	13.728607	88.525636	25.209775
0.960000	0.280000	47.151608	13.427539	88.579265	25.225047
0.955000	0.296606	46.157401	13.144415	88.627788	25.238865
0.950000	0.312250	45.215872	12.876292	88.672152	25.251499
0.945000	0.327070	44.319221	12.620950	88.713058	25.263148
0.940000	0.341174	43.461393	12.376663	88.751039	25.273964
0.935000	0.354648	42.637573	12.142061	88.786515	25.284067
0.930000	0.367560	41.843860	11.916032	88.819818	25.293550
0.925000	0.379967	41.077039	11.697661	88.851218	25.302492
0.920000	0.391918	40.334424	11.486184	88.880936	25.310955
0.915000	0.403454	39.613744	11.280954	88.909158	25.318992
0.910000	0.414608	38.913060	11.081417	88.936040	25.326647
0.905000	0.425412	38.230696	10.887098	88.961714	25.333959
0.900000	0.435890	37.565197	10.697581	88.986294	25.340958
0.895000	0.446066	36.915288	10.512504	89.009879	25.347675
0.890000	0.455961	36.279845	10.331546	89.032553	25.354132
0.885000	0.465591	35.657868	10.154424	89.054392	25.360351
0.880000	0.474974	35.048468	9.980883	89.075462	25.366351
0.875000	0.484123	34.450846	9.810696	89.095822	25.372149
0.870000	0.493052	33.864283	9.643658	89.115525	25.377760
0.865000	0.501772	33.288129	9.479584	89.134616	25.383196

These calculations are for a proton beam of energy 222 MeV, (γ = 1.236605, μ = 2.792847 and $G = (g/2-1) = 1.792847$). Spin precession angles in the dipoles that bend 40.0° and 12.0° are 88.682° and 26.605°, respectively. The effective lengths of the solenoids are 0.49962 m.