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DESIGN NOTE	NAME W. Joho	DATE July 1982.	FILE NO. TRI-DN-82-12	PAGE 1 of 19

SUBJECT

Simulating Cyclotron Sector Magnets with the Computer Program TRANSPORT

Aims

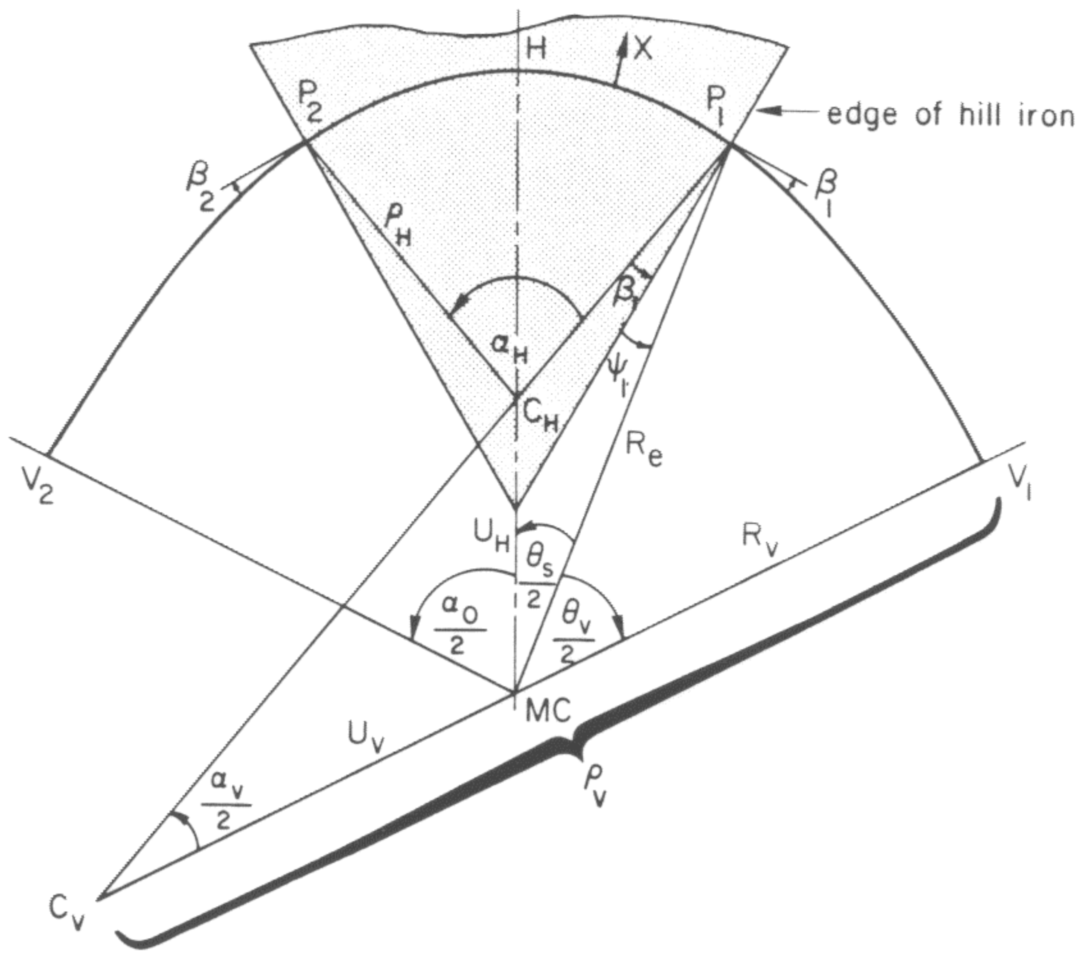
1. From the magnet data $B_{Hill}(R)$, $B_{valley}(R)$, gap (R) and spiral angle $\delta(R)$ to calculate the hard-edge geometry and get a rough estimate of ν_x, ν_y.

2. By varying magnet parameters in TRANSPORT to get a fit to known or specified values of ν_x, ν_y.

3. Calculate the eigenellipse and the dispersion trajectory with TRANSPORT.

4. Obtain tolerance criteria for sector magnets from the sensitivity of ν_x, ν_y and of the equilibrium orbit to changes in magnet parameters.

Example of sector magnet without spiral



MC = machine center

C_H , C_V = center of curvature in hill and valley respectively

H = hill midpoint

V_1 , V_2 = valley midpoints

$\overline{P_1HP_2} \equiv \ell$ = orbit length in hill field B_H

$\overline{V_1P_1} + \overline{P_2V_2} \equiv \ell_v$ = orbit length in valley field B_v

$\ell_p \equiv \ell + \ell_v$ = length of one period

all angles are shown with positive values.

α_H = deflection in Hill region
α_V = deflection in valley region

1. $\alpha_o = \alpha_H + \alpha_V = \frac{2\pi}{N}$ = total deflection angle

N = number of sectors
$L \equiv N\ell_p$ = total circumference of orbit

2. $R \equiv \frac{L}{2\pi} = \frac{\ell_p}{\alpha_o}$ = average radius

θ_s = sector angle
δ = spiral angle (= 0 on drawing)

define "sector flare angle" ψ_s

3. $\psi_s = \psi_1 + \psi_2 = R_e \frac{d\theta_s}{dR_e} \approx R \frac{d\theta_s}{dR}$

the edge focusing angles are then given by:

4. $\beta_1 = \frac{1}{2}(\alpha_H - \theta_s - \psi_s) + \delta$

$\beta_2 = \frac{1}{2}(\alpha_H - \theta_s - \psi_s) - \delta$

Further relationships are:

5. $(B\rho) = B_H\,\rho_H = B_V\,\rho_V \equiv B_o(R)\,R$ (for $B_V<0$: $\rho_V<o$)

$B_o(R) = \gamma\, B_{cu}$ = average field
$\omega_o = \frac{e}{m_o\gamma} B_o(R)$ = revolution frequency

$\omega_{rf} = 2\pi\nu_{rf}$ where ($\nu_{rf} \equiv$ frequency of RF-system)

$h = \frac{\omega_{rf}}{\omega_o}$ = harmonic

6.
$$R_\infty = \frac{c}{\omega_o} = \frac{ch}{2\pi\nu_{rf}}$$

$$R = \beta R_\infty$$

$$B_{cu} R_\infty = \frac{m_o c}{e} = \frac{E_o}{ec} \quad (= 3.1297 \text{ Tm for protons})$$

The connection with the kinetic values is

7.
$$\gamma = (1-\beta^2)^{-1/2}$$
$$E = (\gamma-1)E_o = \text{Kinetic energy}$$

$$E_o = m_o c^2 = \text{rest energy}$$

$$pc = \sqrt{E(E+2E_o)} \qquad p = \text{momentum}$$

$$pc = \sqrt{(\gamma-1)E_o\ ((\gamma-1)E_o+2E_o)} = \beta\gamma E_o$$

$$(B\rho) = \beta\gamma\ (B_{cu} R_\infty) \quad (= pc\ 3.3356 \frac{\text{Tm}}{\text{GeV}} \text{ for charge } = 1)$$

The problem is usually that $B_H(R)$ and $B_V(R)$ are specified and that we want to deduce from this the corresponding angles which define the pole tip geometry $\alpha_H(R)$, $\alpha_V(R)$ and $\theta_s(R)$. The spiral angle $\delta(R)$ being an independent parameter.

Define the following ratios:

8.
$$\eta_H \equiv \frac{B_H(R)}{B_o(R)} = \frac{R}{\rho_H}$$

$$\eta_V \equiv \frac{B_V(R)}{B_o(R)} = \frac{R}{\rho_V}$$

Now $\alpha_H = B_H\ \ell_H/(B\rho) = B_H \ell_H/(B_o R)$ using (5)

$$\frac{\alpha_H B_o}{B_H} = \frac{\alpha_H}{\eta_H} = \frac{\ell_H}{R} \text{ ; similarly } \frac{\alpha_V}{\eta_V} = \frac{\ell_V}{R}$$

9. Hence $\alpha_o = \frac{\alpha_H}{\eta_H} + \frac{\alpha_V}{\eta_V}$

We can combine this with (1.) to obtain:

10.
$$\alpha_V = \frac{\eta_V(\eta_H - 1)}{\eta_H - \eta_V}\,\alpha_o$$

$$\alpha_H = \frac{\eta_H(1-\eta_V)}{\eta_H - \eta_V}\,\alpha_o = \alpha_o - \alpha_V$$

11.
$$\ell = \rho_H \alpha_H = \frac{R}{\eta_H}\,\alpha_H = \text{magnet length}$$

$$\ell_V = \rho_V \alpha_V = \frac{R}{\rho_V}\alpha_V = R\alpha_o - \ell = \text{valley length}$$

to get θ_s we use the relations

12.
$$\rho_H \sin\frac{\alpha_H}{2} = R_e \sin\frac{\theta_s}{2}$$

$$\rho_V \sin\frac{\alpha_V}{2} = R_e \sin\left(\frac{\alpha_o}{2} - \frac{\theta_s}{2}\right)$$

eliminating R_e we obtain after some manipulation:

13.
$$\sin\frac{\theta_s}{2} = \left\{\left(M + \cos\frac{\alpha_o}{2}\right)^2 + \sin^2\frac{\alpha_o}{2}\right\}^{-1/2} \sin\frac{\alpha_o}{2}$$

where $M \equiv \dfrac{\eta_H \alpha_o - \alpha_H}{2\sin\frac{\alpha_H}{2}}$

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Evaluation of θ_s at different radii gives $\psi_s = \frac{R\, d\theta_s}{dR}$ which, through (4.), allows the evaluation of the edge angles β_1 and β_2. The flutter of the sector geometry is thus taken into account through θ_s.

For TRANSPORT calculation one needs in addition the field indices n_H and n_V.

14. $$n_H \equiv -\frac{\rho_H}{B_H}\frac{dB_H}{dx} \approx -\frac{\rho_H}{B_H}\frac{dB_H}{dR}$$

$$n_V \equiv -\frac{\rho_V}{B_V}\frac{dB_V}{dx} \approx -\frac{\rho_V}{B_V}\frac{dB_V}{dR}$$

The radii R_V, R_H and R_e are obtained as follows:

from (8.): $\rho_H = \frac{R}{\eta_H}$, $\rho_V = \frac{R}{\eta_V}$ (for $B_V \neq 0$)

from (12.)

15. $$R_e = \rho_H \frac{\sin \frac{\theta_s}{2}}{\sin \frac{\alpha_H}{2}}$$

Defining U_H as the distance between C_H and MC in Fig. 1 and similarly U_V one uses:

$U_H \sin \frac{\theta_s}{2} = \rho_H \sin\left(\frac{\alpha_H}{2} - \frac{\theta_s}{2}\right)$ and $R_H = \rho_H + \eta_H$ to get

16. $$R_H = \rho_H \; 1\left[+ \frac{\sin 0.5\,(\alpha_H - \theta_S)}{\sin 0.5\ \theta_s}\right]$$

from $U_v \sin \frac{\alpha_o - \theta_s}{2} = \rho_v \sin \frac{\alpha_H - \theta_s}{2}$ one gets

17. $$R_v = \rho_v \left[1 - \frac{\sin 0.5\,(\alpha_H - \theta_s)}{\sin 0.5\,(\alpha_o - \theta_s)}\right] \quad \text{for } B_v \neq 0$$

$$R_v = R_e \cos\left(\frac{\alpha_o - \theta_s}{2}\right) \quad \text{for } B_v = 0$$

In Summary to obtain an isochronous sector geometry

Given: 1) N = number of sectors $\rightarrow \alpha_o = \frac{2\pi}{N}$

2) $R_\infty = \frac{c\ h}{2\pi\nu_{rf}} \rightarrow B_{cu} = \frac{(B_{cu}\ R_\infty)}{R_\infty}$

3) Hill field $B_H(R)$ which yields $\rightarrow n_H(R)$

4) Valley field $B_V(R)$ which yields $\rightarrow n_V(R)$

5) Spiral angle $\delta(R)$

Wanted: $E(R)$, $\alpha_H(R)$, $\alpha_V(R)$, $\theta_s(R)$, $\ell(R)$, $\ell_V(R)$
$\beta_1(R)$, $\beta_2(R)$ for TRANSPORT runs

Procedure: Specify R which yields $\rightarrow \beta = \frac{R}{R_\infty}$, $\gamma = (1-\beta^2)^{-1/2}$

$$E = (\gamma-1)E_o, \quad pc = \sqrt{E(E+2E_o)}$$

$$B_o(R) = \gamma B_{cu}$$

$$\eta_H = \frac{B_H}{B_o}, \qquad \eta_V = \frac{B_V}{B_o}$$

$$\alpha_V = \frac{\eta_V(\eta_H-1)}{\eta_H-\eta_V}\alpha_o, \quad \alpha_H = \alpha_o-\alpha_V$$

$$\ell = \frac{R}{\eta_H}\alpha_H = \text{magnet length}$$

$$\ell_p = R\alpha_o = \text{period length}$$

$$\ell_V = \ell_p-\ell = \text{valley length}$$

$\sin\frac{\theta s}{2}$ from (13)

$$\psi_s = R\frac{d\theta s}{dR}$$

$$\beta_{1/2} = 1/2(\alpha_H-\theta_s-\psi_s)\pm\delta$$

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The use of the program TRANSPORT to obtain the focusing frequencies as well as the eigenellipses for a given periodic beam line.

Given:
- periodic beamline from $Z = 0$ to $Z = \ell_p$
- TRANSPORT-RUN with R-matrix from $R(0) \equiv 1$
- beam emittances $\pi\varepsilon_x$, $\pi\varepsilon_y$

Wanted:
- phase advance angles μ_x, μ_y over one period.

  For cyclotrons with N sectors: $\alpha_o = \frac{2\pi}{N}$

  $\nu_x = \mu_x/\alpha_o$, $\nu_y = \mu_y/\alpha_o$
- periodic dispersion trajectory

  $D(Z) \equiv R_{16}(Z)$, $D'(Z) = R_{26}(Z) = R'_{16}(Z)$

To illustrate the procedure we assume that x- and y- motions are decoupled and take only the x-ellipse as an example.

Units: x in mm , x' in mrad, $\frac{\Delta P}{P}$ in ‰

Z in m, $' \equiv \frac{d}{dZ}$

The transfer matrix R is usually written in K. Brown's notation as

(18) $$R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \equiv \begin{pmatrix} C(Z) & S(Z) \\ C'(Z) & S'(Z) \end{pmatrix} \qquad R(0) = \mathbb{1}\ , \quad \det R = 1$$

C(Z) and S(Z) are the cosine - and sine like trajectories.

For a periodic section we use the Twiss-formalism.

(19) $$R = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

which can be written formally as

$$R = \cos\mu\,\mathbb{1} + \sin\mu . J \qquad J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$ and $\mathbb{1}$ is the identity matrix

$$(\det J = 1 \rightarrow 1 + \alpha^2 = \beta\gamma)$$

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The phase advance angle μ is thus given by

(20) $$\boxed{\cos\mu = 0.5(R_{11} + R_{22})}$$

Since β is always positive, we get μ (mod 2π) by observing that

(21) $$\text{sign}(\sin\mu) = \text{sign}(R_{12})$$

For $|R_{11} + R_{22}| < 2$ the motion is stable, since μ is real.

The Twiss parameters α, β, γ are obtained from R by:

(22) $$\beta = \frac{R_{12}}{\sin\mu} \qquad \alpha = \frac{R_{11} - R_{22}}{2\sin\mu} \qquad \gamma = \frac{1+\alpha^2}{\beta}$$

The eigen ellipse is then given by the formula

(23) $$X = X_m \cos\phi \qquad (0 \leq \phi \leq 2\pi)$$
$$X' = \theta_m \sin(\phi + \chi) \qquad \phi = \text{running parameter}$$

(24) $$X_m = \sqrt{\varepsilon\beta} = \sqrt{\varepsilon} \cdot \frac{\sqrt{R_{12}}}{\sin\mu} \qquad = \text{max. beam amplitude}$$
$$\tan\chi = -\alpha = \frac{R_{22} - R_{11}}{2\sin\mu}$$
$$\theta_m = \frac{\varepsilon}{X_m \cos\chi} \qquad = \text{max. beam divergence}$$

(25) $r_{12} = \sin\chi$ is the usual correlation parameter printed in a TRANSPORT-run.

For an individual particle on the periphery of the ellipse we have the connection:

$$\phi(\ell p) = \Phi(0) + \mu$$

The periodic solution for the dispersion trajectory D,D' is obtained by solving the equations

$$D_\theta = R_{11}\, D_0 + R_{12}\, D_0' + R_{16} = D_0$$

$$D_\theta' = R_{21}\, D_0 + R_{22}\, D_0' + R_{26} = D_0'$$

yielding

(26)

$$D = \frac{R_{12}\, R_{26} + R_{16}\, (1 - R_{22})}{\Delta} \qquad [\text{mm}/‰\ \frac{\Delta P}{P}]$$

$$D' = \frac{R_{21}\, R_{16} + R_{26}\, (1 - R_{11})}{\Delta} \qquad [\text{mrad}/‰\ \frac{\Delta P}{P}]$$

where $\Delta = 2 - (R_{11} + R_{22}) = 4 \sin^2 \frac{\mu}{2}$

To display the periodic dispersion trajectory $R_{16}(Z)$ start another TRANSPORT-run with $R_{16}(0) = D$ and $R_{26}(0) = D'$.

As a check one has for isochronous cyclotrons:

(27) $\langle R_{16} \rangle \approx \frac{R}{\gamma^2}$ [m or mm/‰]

(28) $R_{56} = 0$ or at least $|R_{56}| \ll \frac{\ell p}{\gamma^2}$

warning: to get correct bunch length and R_{56}, be sure to specify the mass of the particle. The default value is electron mass!

Example of TRANSPORT-run for 3 GeV ring N = 15 sectors → $\alpha_0 = \frac{2\pi}{N} = 24°$

1. Run with D = D' = 0 → COMMAND: EIGEN prints new D, D'

2. Run with D = .755, D' = -.16

cards type

```
'FIT OF RING1 AT 2.711 GEV  02.6.82'
0
15. 1. 'MM' .1 ;
15. 6. 'PM' .1 ;
16. 3. 1837. 'MASS' ;
1. 0.760 .424 1.168 0.380  .000  0.00 3.527 'BEAM' ;
12. -.670 .0 .0 .0 .0 .775 +.0 -.00 .0 .0 .0 .0 .0 .0 .0 'COR' ;
14. 1. .0 .0 .0 .0 0.755 1.0 'DISP' ;
14. 0. 1. .0 .0 .0 -.16 2. 'DPR' ;
16. 5. 65. 'GV2' ;
16. 7. .45 'K1' ;
16. 8. 4. 'K2' ;
3. 0.00 'D1' ;
4. 1.362 -4.0 -20 'BVAL' ;
2. -60.0 'B1V' ;
-16. 5. 65. 'G2' ;
2. +60.0 'BET1' ;
4. 1.466 41.00 -0.14 'SM' ;
2. -58.0 'BET2' ;
-16. 5. 65. 'GV2' ;
2. +58.0 'B2V' ;
4. 1.362 -4. -20 'BVAL' ;
3. .00 'VAL' ;
SENTINEL
SENTINEL
```

```
ENTER   COMMAND:
EIGEN   → input Beam
Z =    4.191 M ; LABEL = 'VAL '
XM=  0.0   X     =  0.760 CM                    (D=    0.4921 CM/PC) (DP=  -0.2548 MR/PC)
TM=  0.0   THETA=  0.424 MR -0.670
YM=  0.0   Y     =  1.168 CM  0.000 -0.000
PM=  0.0   PHI   =  0.380 MR  0.000  0.000  0.775
LM=  0.0   L     = -0.206 CM -0.074  0.077 -0.000 -0.000
DM=  0.0   DELTA=  0.004 PC  0.002 -0.001  0.000  0.000*******
```

Beam matrix at beginning and end are the same → equilibrium

```
ENTER   COMMAND:
B (.001)
Z =        0.001 M
XM=  0.0   X     =  0.760 CM
TM=  0.0   THETA=  0.424 MR -0.670
YM=  0.0   Y     =  1.168 CM  0.0     0.0
PM=  0.0   PHI   =  0.380 MR  0.0     0.0     0.775
LM=  0.0   L     =  0.0   CM  0.0     0.0     0.0     0.0
DM=  0.0   DELTA=  0.0   PC  0.0     0.0     0.0     0.0     0.0
ENTER   COMMAND:
B ('VAL')
Z =    4.191 M ; LABEL = 'VAL '
XM=  0.0   X     =  0.761 CM
TM=  0.0   THETA=  0.424 MR -0.671
YM=  0.0   Y     =  1.168 CM  0.0     0.0
PM=  0.0   PHI   =  0.380 MR  0.0     0.0     0.775
LM=  0.0   L     =  0.039 CM -0.876   0.945   0.0     0.0
DM=  0.0   DELTA=  0.0   PC  0.0     0.0     0.0     0.0     0.0
```

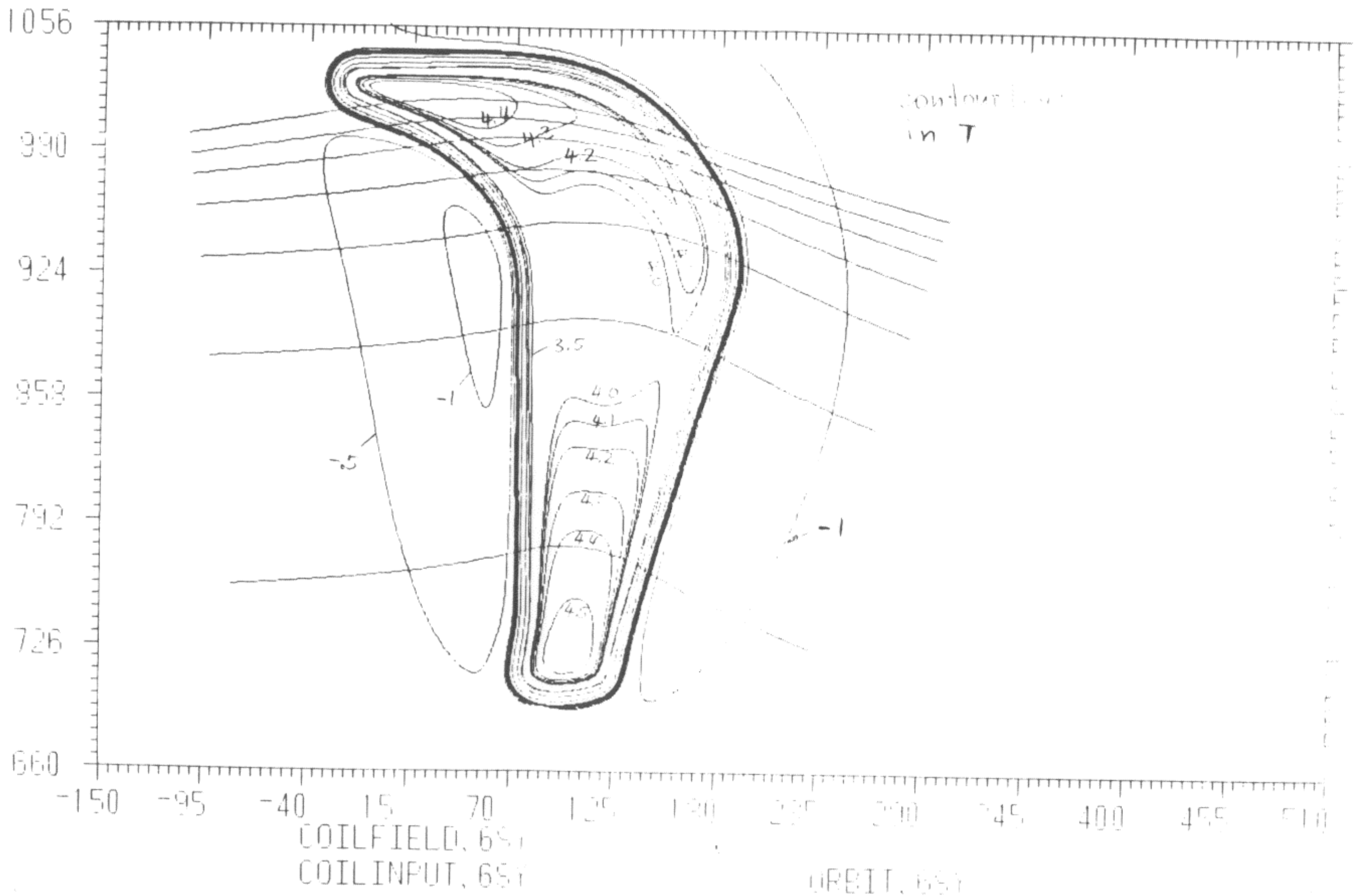
Transfer matrix at beginning and end: D, D' = periodic

```
ENTER   COMMAND:
M -1 (.001)
Z =       0.001 M
    1.0000    0.0       0.0       0.0       0.0       0.7550 :     0.0     DEG
    0.0       1.0000    0.0       0.0       0.0      -0.1600 :
    0.0       0.0       1.0000    0.0       0.0       0.0    :     0.0     DEG
    0.0       0.0       0.0       1.0000    0.0       0.0    :
    0.0       0.0       0.0       0.0       1.0000    0.0    :
    0.0       0.0       0.0       0.0       0.0       1.0000 :
ENTER   COMMAND:
M -1('VAL')                                                       μx
Z =    4.191 M ; LABEL = 'VAL '                                   ↑
    0.7095    2.3774    0.0       0.0       0.0       0.7486 :  100.3199 DEG
   -0.7393   -1.0678    0.0       0.0       0.0      -0.1630 :
    0.0       0.0      -0.3394    3.8099    0.0       0.0    :   51.5809 DEG
    0.0       0.0      -0.4034    1.5822    0.0       0.0    :    ↓
   -0.0598   -0.1167    0.0       0.0       1.0000    0.0009 :    μy
    0.0       0.0       0.0       0.0       0.0       1.0000 :
```

$\nu_x = \mu_x / \alpha_0 = 4.18$

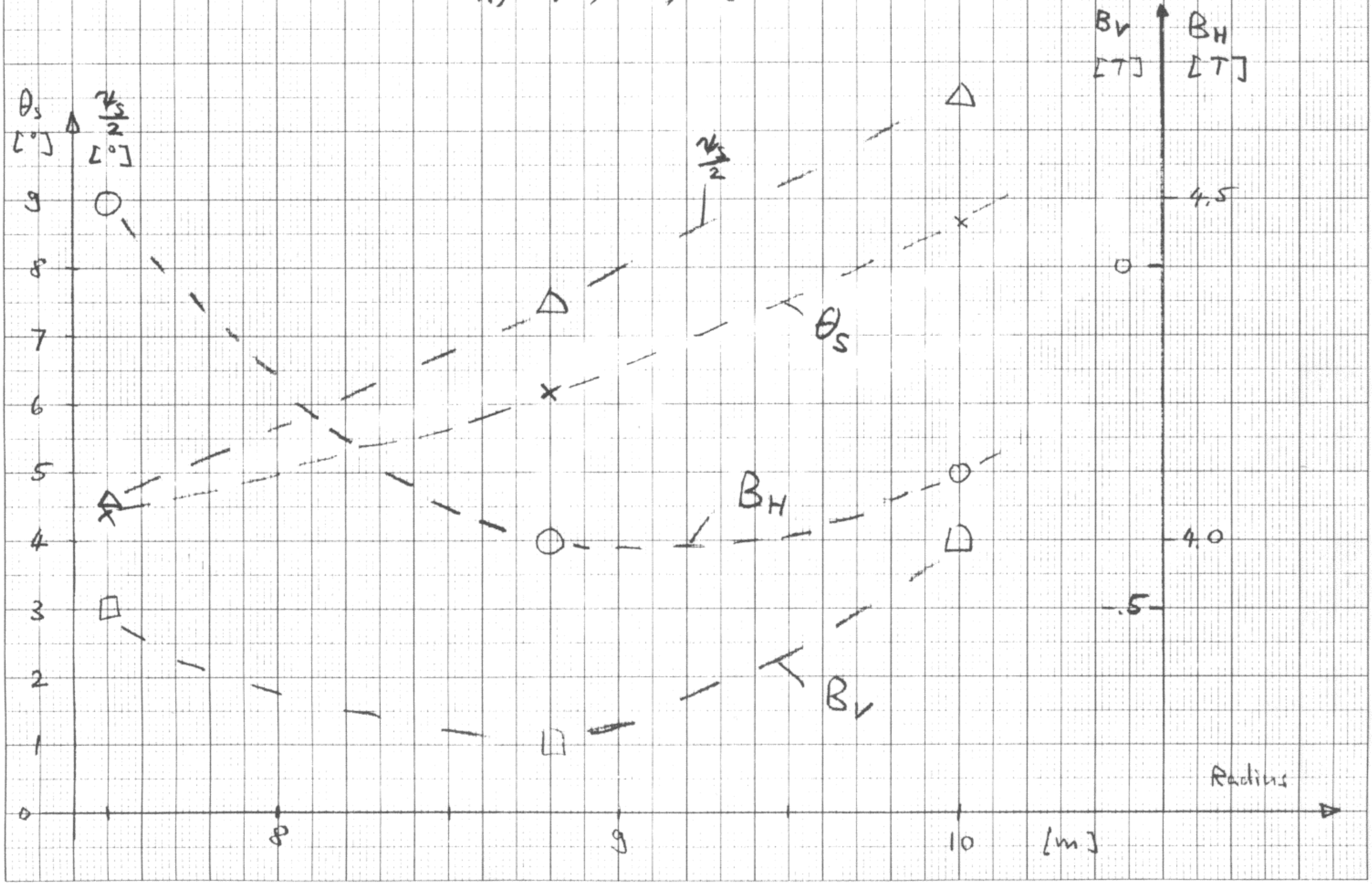
$\nu_y = \mu_y / \alpha_0 = 2.15$

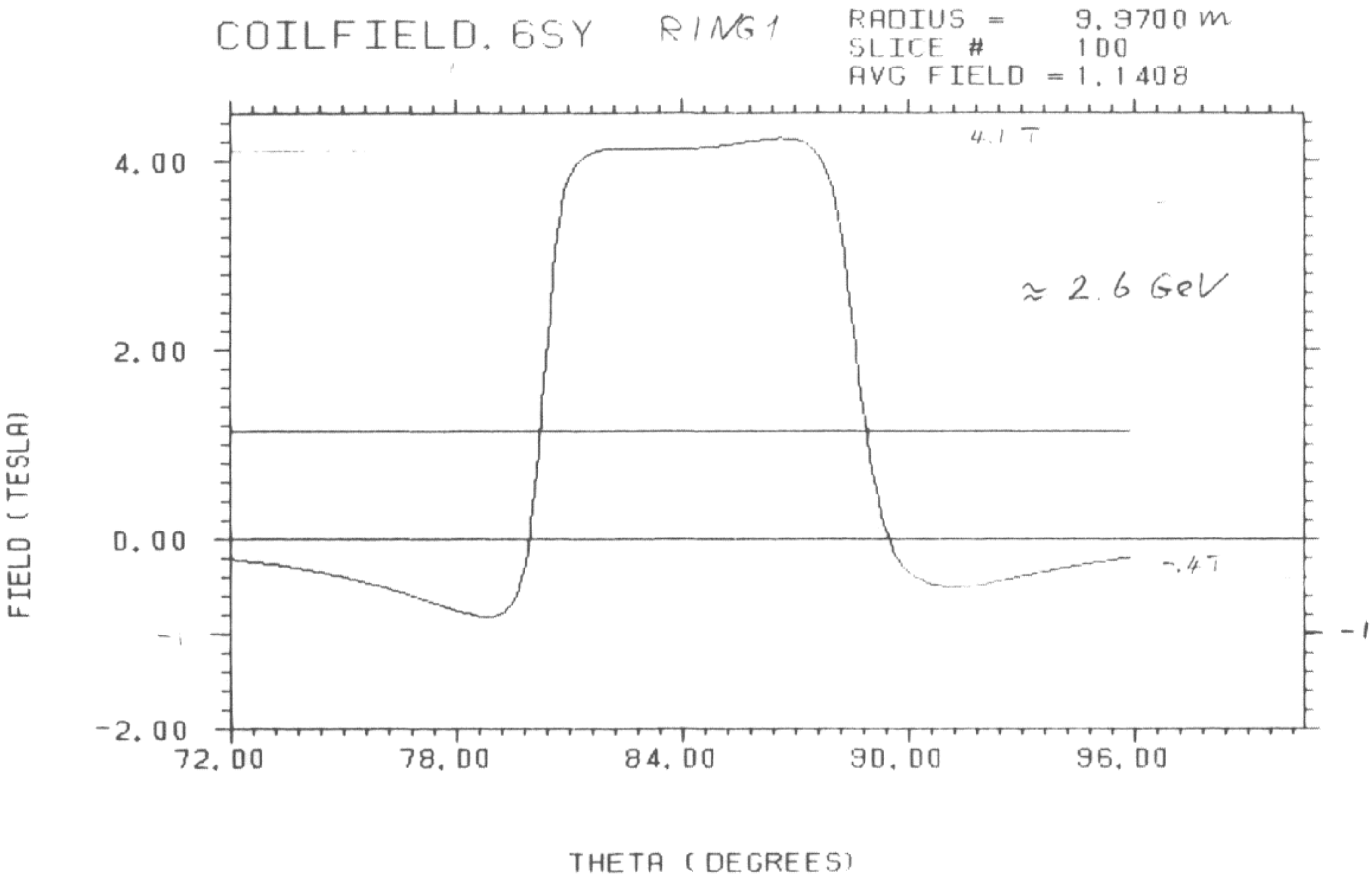
RING CYCLOTRON WITH 15 SECTORS

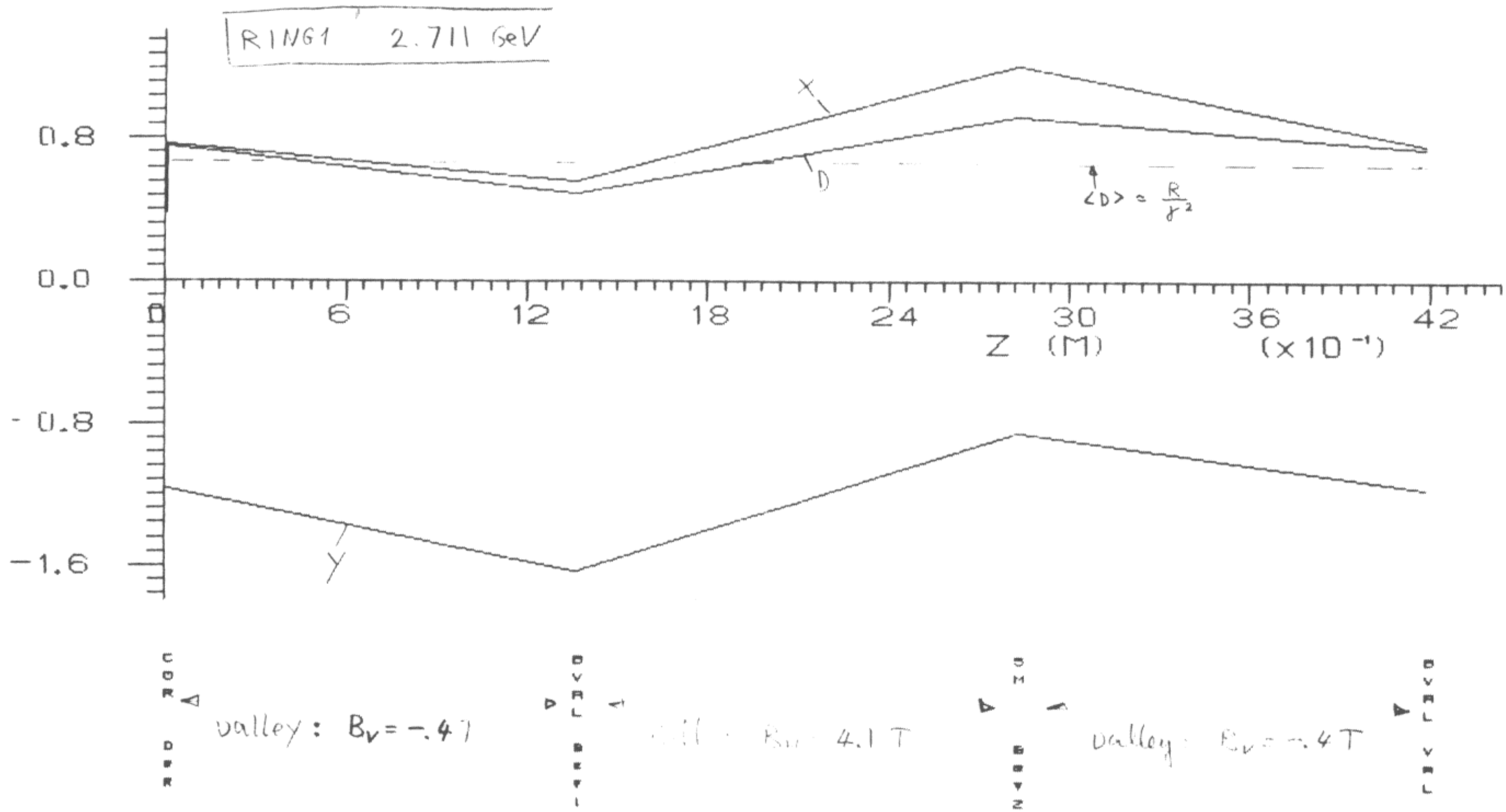


from previous [illegible] we obtain

RING 1 : B_H, B_V, θ_s, ν_s




COILFIELD.6SY RING1
RADIUS = 9.9700 m
SLICE # 100
AVG FIELD = 1.1408
4.1 T
≈ 2.6 GeV
-.4 T
-1
4.00
2.00
0.00
-2.00
72.00
78.00
84.00
90.00
96.00
FIELD (TESLA)
THETA (DEGREES)


RING1 2.711 GeV
0.8
0.0
- 0.8
-1.6
0
6
12
18
24
30
36
42
Z (M)
(x10⁻¹)
X
D
Y
<D> = R/γ²
Valley: Bv = -.4T
4.1 T
Valley
.4 T

Influence of sector geometry on focusing ν_x ν_y

RING 1   2.71 GeV   R = 10.00 m

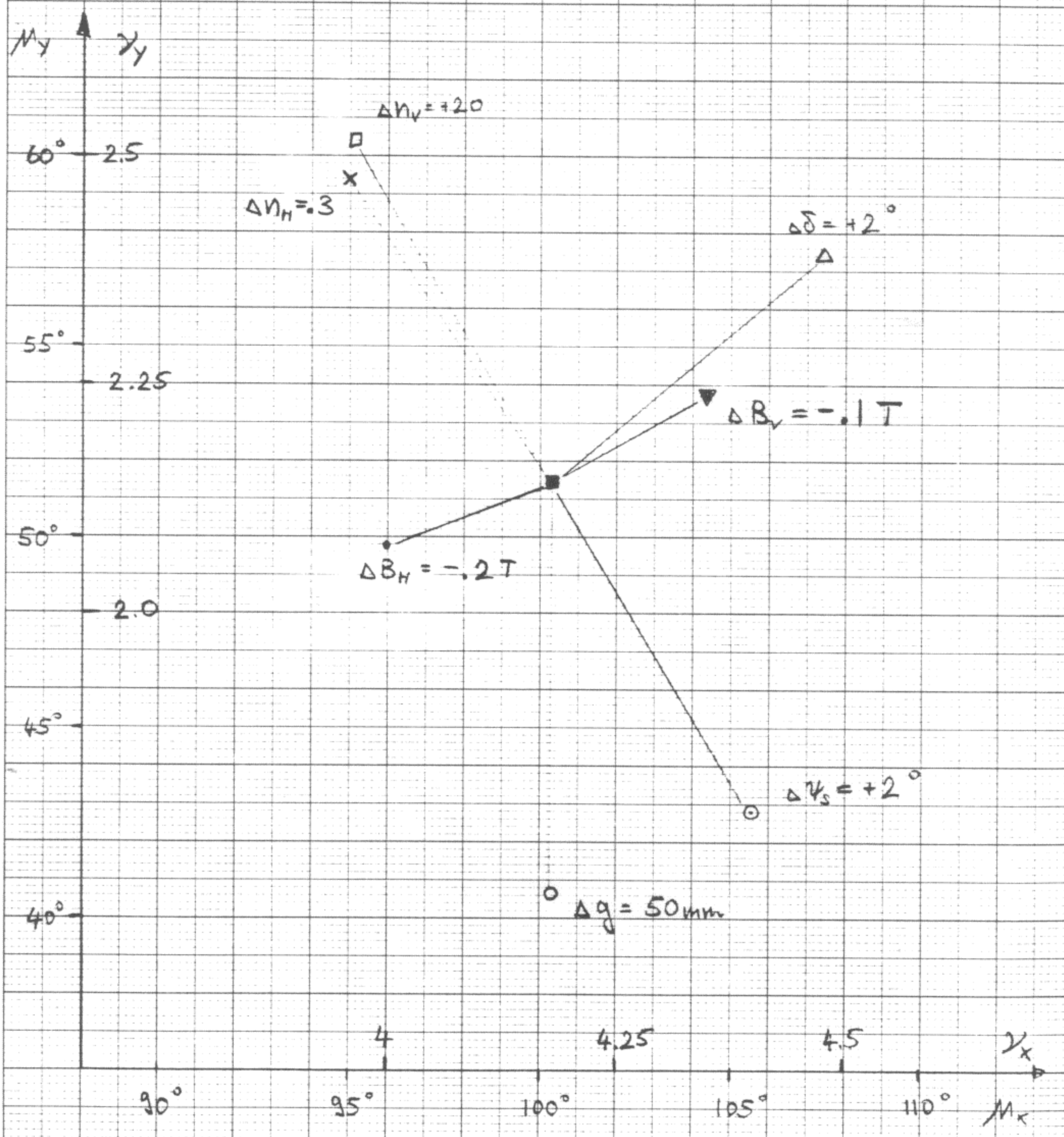
$\ell = 1.466$ m $B_H = 4.1$ T $n = -.14$

$\ell_v = 2.724$ m $B_v = -.4$ T $n = -20$

$\delta = 59°$

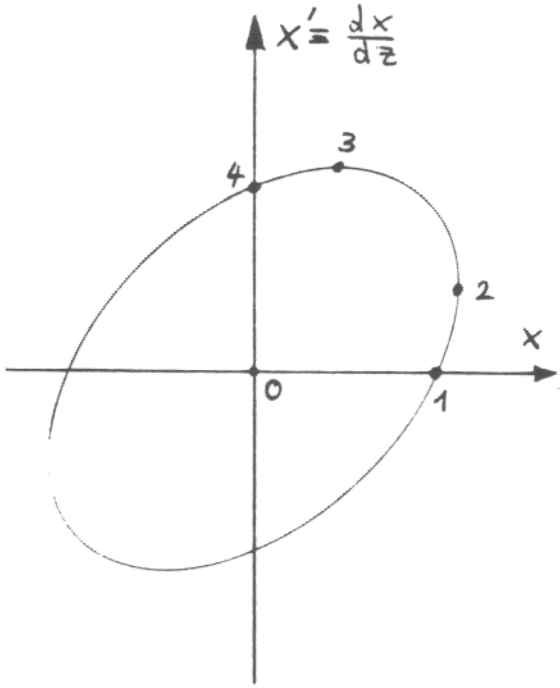
$\psi_s = 19°$

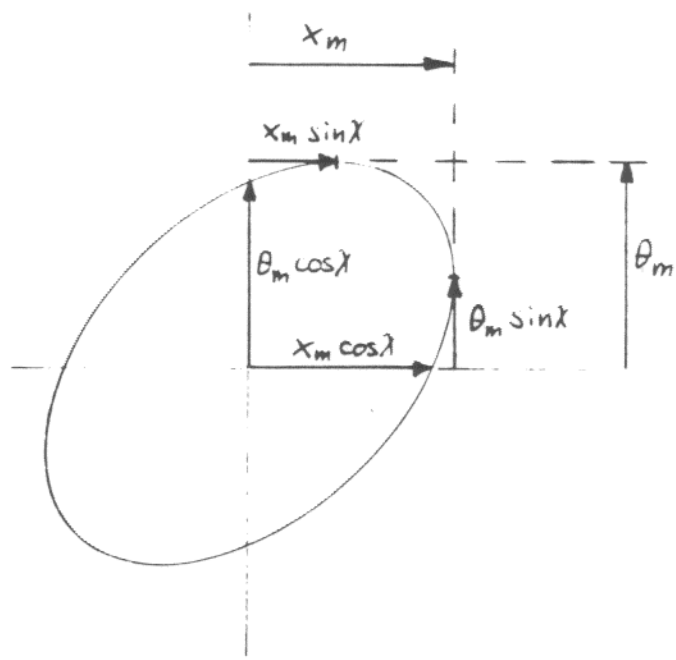
g = 130 mm   (gap)



W.J. 2.6.82

The meaning of the ellipse parameters is best illustrated with the coordinates of the four specific points 1, 2, 3, 4 on the ellipse. Points 1 and 3 as well as points 2 and 4 are conjugate pairs, i.e. the tangent at point 1 is parallel to line 0-3, the tangent at point 4 is parallel to line 0-2. Notice the nice symmetry of the coordinates in the parametric representation. The slope of the envellope $x_m(z)$ is given by x_2' .





Parametric				Courant-Snyder	
$\tan\chi \equiv \frac{x_2'}{x_4'} = \frac{x_3}{x_1}$				$\alpha \equiv -\frac{x_2'}{x_4'} = -\frac{x_3}{x_1}$	
point , δ		x	x'	x	x'
1)	$-\chi$	$x_m\cos\chi$	0	$\sqrt{\frac{\varepsilon}{\gamma}}$	0
2)	0	x_m	$\theta_m\sin\chi$	$\sqrt{\varepsilon\beta}$	$-\alpha\sqrt{\frac{\varepsilon}{\beta}}$
3)	$90^0-\chi$	$x_m\sin\chi$	θ_m	$-\alpha\sqrt{\frac{\varepsilon}{\gamma}}$	$\sqrt{\varepsilon\gamma}$
4)	90^0	0	$\theta_m\cos\chi$	0	$\sqrt{\frac{\varepsilon}{\beta}}$