Solenoid, with embedded dipole

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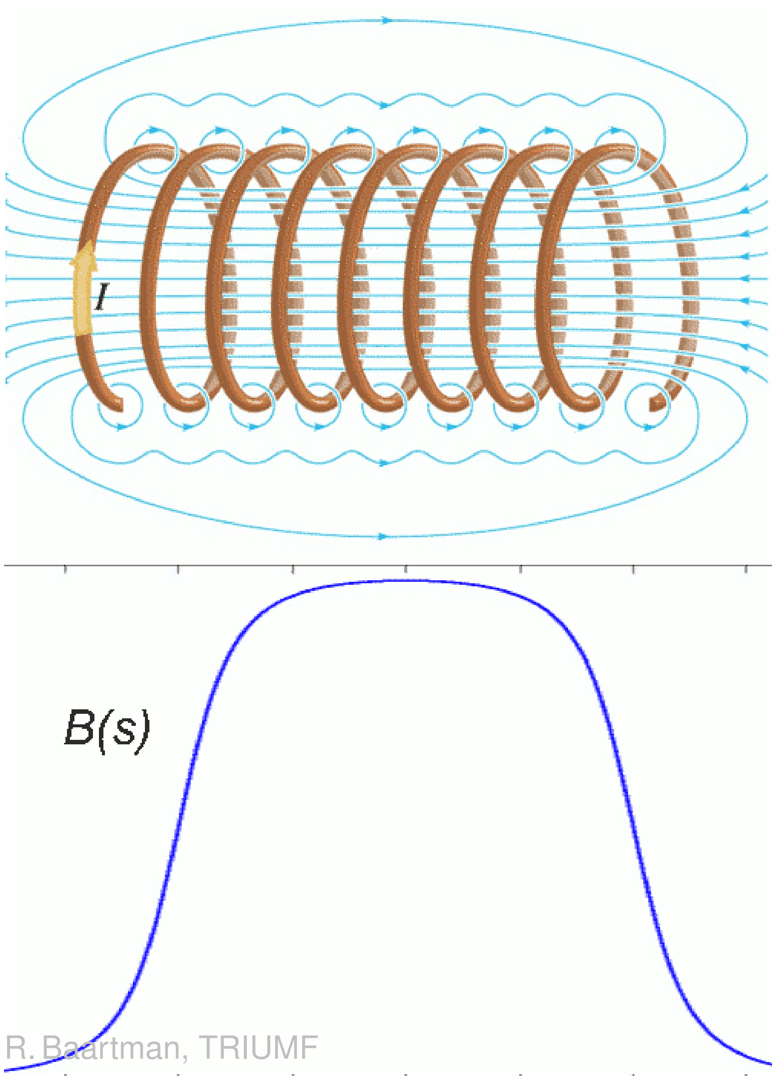
General Hamiltonian

$$H_s = -qA_s - \left(1 + \frac{x}{\rho}\right)\sqrt{-m_0^2c^2 - (P_x - qA_x)^2 - (P_y - qA_y)^2 + \left(\frac{E - q\Phi}{c}\right)^2} \quad (1)$$

No electric field, straight trajectory: $\Phi = 0$, $1/\rho = 0$:

$$H_s = -qA_s - \sqrt{-m_0^2c^2 - (P_x - qA_x)^2 - (P_y - qA_y)^2 + \left(\frac{E}{c}\right)^2} \quad (2)$$

Plain Solenoid



By symmetry, we know the field on axis is purely in the axis' direction. By Lorentz equation, it's obvious that particles travelling on the axis will stay on the axis, so the axis is the reference trajectory $\hat{s}$.

Denote the on-axis field as $B(s)$. The vector potential that generates such a field is

$$\vec{A} = \frac{B(s)}{2}(-y, x, 0) \tag{3}$$

The off-axis field has a radial component that "kicks" the particles into a spiralling motion.

The Hamiltonian can be found from eqn. 2 with $A_s = 0$. Noting that E is a constant, we see the longitudinal motion is trivial and just replace $\sqrt{\frac{E^2}{c^2} - m_0^2 c^2}$ with the total momentum P_o:

$$H_s = -\sqrt{P_\mathrm{o}^2 - \left(P_x + \frac{qB}{2}y\right)^2 - \left(P_y - \frac{qB}{2}x\right)^2} \tag{4}$$

We can normalize all momenta by dividing by P_o:

$$H = -\sqrt{1 - (P_x + \kappa y)^2 - (P_y - \kappa x)^2} \tag{5}$$

where $\kappa = \kappa(s) = \frac{qB}{2P_\mathrm{o}} = \frac{B(s)}{2(B\rho)}$ ($(B\rho)$ is the "rigidity" so you see that κ has units of inverse length). Expanding but throwing out the constant,

$$H = \frac{1}{2}(P_x + \kappa y)^2 + \frac{1}{2}(P_y - \kappa x)^2 \tag{6}$$

In terms of the "infinitesimal transfer matrix":

$$\mathbf{F} = \begin{pmatrix} 0 & 1 & \kappa & 0 \\ -\kappa^2 & 0 & 0 & \kappa \\ -\kappa & 0 & 0 & 1 \\ 0 & -\kappa & -\kappa^2 & 0 \end{pmatrix} \tag{7}$$

Compare with an imaginary case where there is continuous focusing $K(s)$ equally in both directions:

$$\mathbf{F} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -K & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -K & 0 \end{pmatrix} \tag{8}$$

You see that the solenoid matrix has coupling between x and y motions coming from the cross terms $\kappa(P_x y - P_y x)$ in the Hamiltonian. These can be eliminated by going to a rotating frame $(x_R, P_{xR}, y_R, P_{yR})$.

$$\begin{aligned} x &= x_R \cos(\textstyle\int \kappa \mathrm{d}s) - y_R \sin(\textstyle\int \kappa \mathrm{d}s) \\ P_x &= P_{xR} \cos(\textstyle\int \kappa \mathrm{d}s) - P_{yR} \sin(\textstyle\int \kappa \mathrm{d}s) \\ y &= x_R \sin(\textstyle\int \kappa \mathrm{d}s) + y_R \cos(\textstyle\int \kappa \mathrm{d}s) \\ P_y &= P_{xR} \sin(\textstyle\int \kappa \mathrm{d}s) + P_{yR} \cos(\textstyle\int \kappa \mathrm{d}s) \end{aligned} \tag{9}$$

The generating function on the pattern $P_x x + P_y y$, with old momenta, new positions, is:

$$\begin{aligned} F(x_R, P_x, y_R, P_y) &= P_x\left[x_R\cos(\textstyle\int\kappa\mathrm{d}s) - y_R\sin(\textstyle\int\kappa\mathrm{d}s)\right] + \\ &+ P_y\left[x_R\sin(\textstyle\int\kappa\mathrm{d}s) + y_R\cos(\textstyle\int\kappa\mathrm{d}s)\right] \end{aligned}$$

Plug the transformation into the Hamiltonian and all the cosines and sines drop out, and the added term $\partial F/\partial s$ is precisely the cross terms of opposite sign $-\kappa(P_x y - P_y x)$, so we are left with

$$H = \frac{P_{xR}^2}{2} + \kappa^2\frac{x_R^2}{2} + \frac{P_{yR}^2}{2} + \kappa^2\frac{y_R^2}{2} \tag{10}$$

Remember: $\kappa = \kappa(s)$.
Rotation angle: $\theta = \int\kappa\mathrm{d}s$
Focal strength is κ^2; for short solenoid, $\frac{1}{f} = \int\kappa^2\mathrm{d}s$

Add a distributed dipole field

Field in y-direction, has no x-dependence, but a (e.g. bell-shaped) s-dependence, call it $B_y(s)$. The vector potential adds to the solenoid. It is

$$\vec{A} = (0, 0, -xB_y(s)) \tag{11}$$

Go back to eqn. 2, and divide by P_o. This adds a new term to the Hamiltonian (6):

$$H = Cx + \frac{1}{2}(P_x + \kappa y)^2 + \frac{1}{2}(P_y - \kappa x)^2 \tag{12}$$

where $C = C(s) = \frac{B_y(s)}{B\rho}$.

Equations of motion:

$$
\begin{aligned}
x' = \frac{\partial H}{\partial P_x} &= \kappa y + P_x \\
P_x' = -\frac{\partial H}{\partial x} &= -\kappa^2 x + \kappa P_y - C \\
y' = \frac{\partial H}{\partial P_y} &= -\kappa x + P_y \\
P_y' = -\frac{\partial H}{\partial y} &= -\kappa^2 y - \kappa P_x
\end{aligned}
\tag{13}
$$

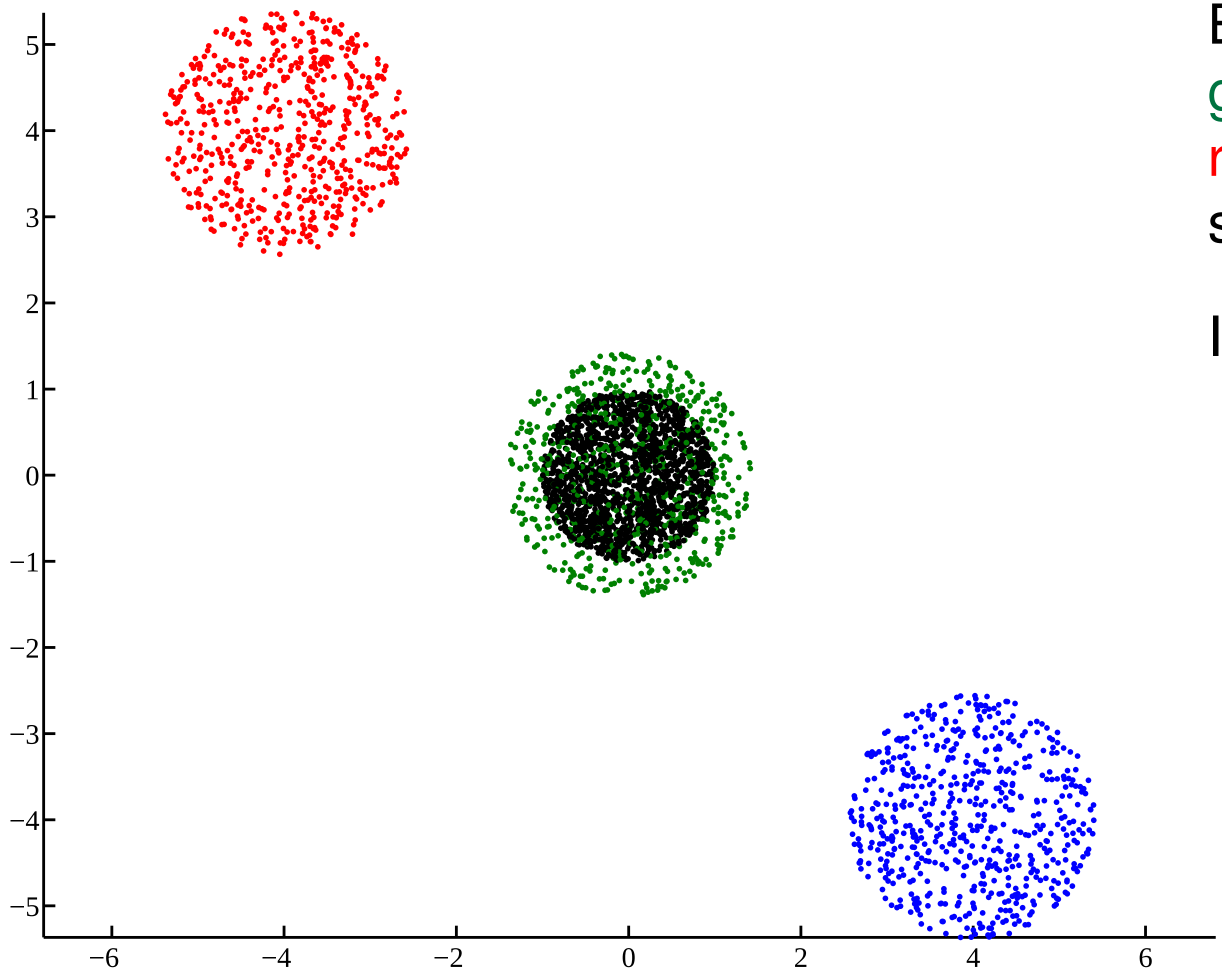
I have numerically solved these equations of motion. There is no emittance growth: an input emittance simply comes out off axis, just as if the solenoid and dipole were separate.

In the example, the 4D phase space is populated as a hypersphere shell (a Kapchinsky-Vladimirsky distribution). The steerer $C(s)$ is a narrow gaussian ($\sigma = 0.22$) and solenoid $\kappa(s)$ is also gaussian with $\sigma = 1$.

MATLAB code; $\mathrm{y} = \vec{y}$ has 4 elements (x, P_x, y, P_y):

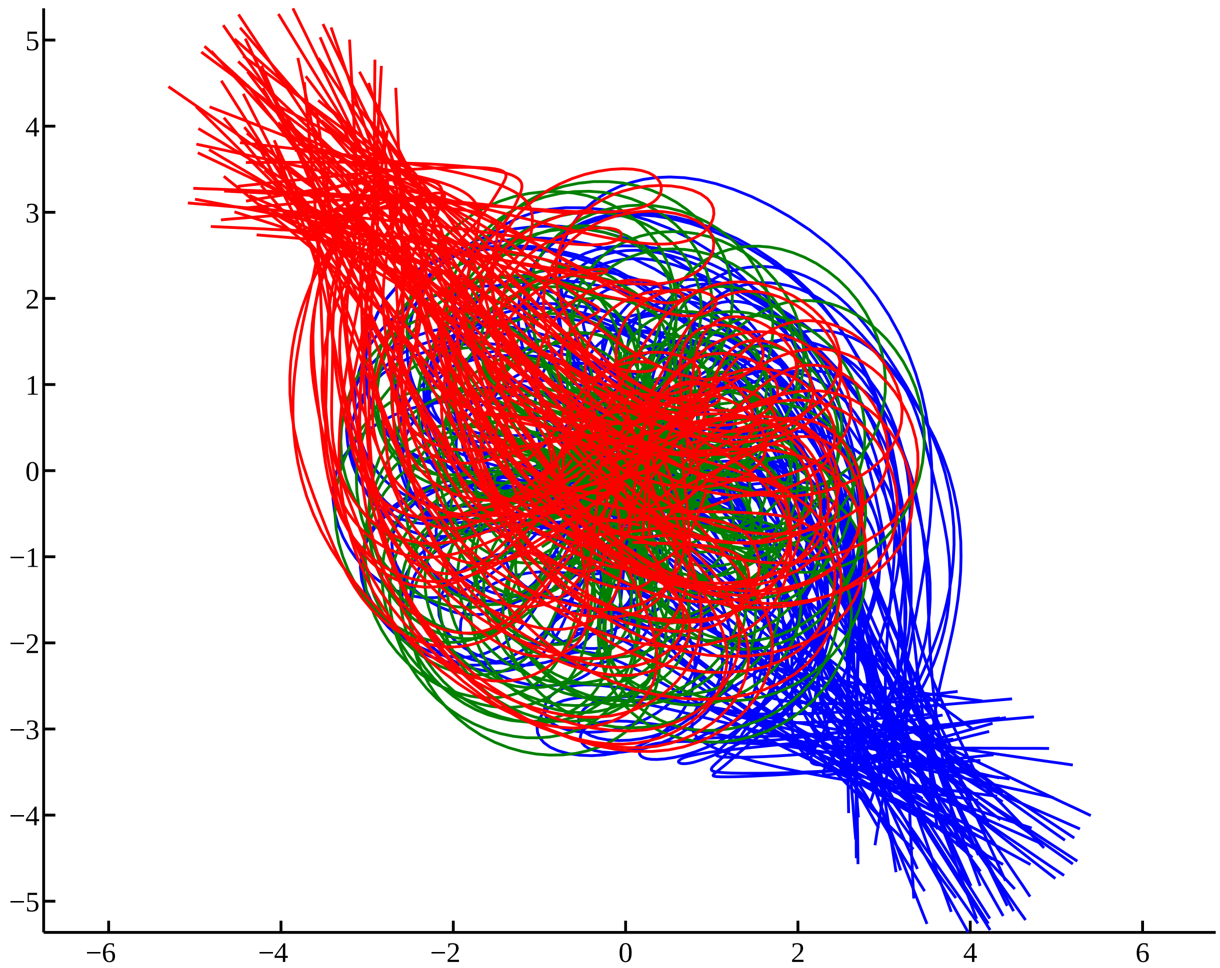
```
global steer, offset;
kappa=0.62666*exp(-s^2/2);C=steer*exp(-(s-offset)^2*10);
dy = [    y(2)+kappa*y(3);
     -C+(y(4)-kappa*y(1))*kappa;
         y(4)-kappa*y(1);
       -(y(2)+kappa*y(3))*kappa];
```

The solenoid strength 0.62666 makes $\theta = \int \kappa \mathrm{d}s$ exactly $\pi/2$, which is handy.

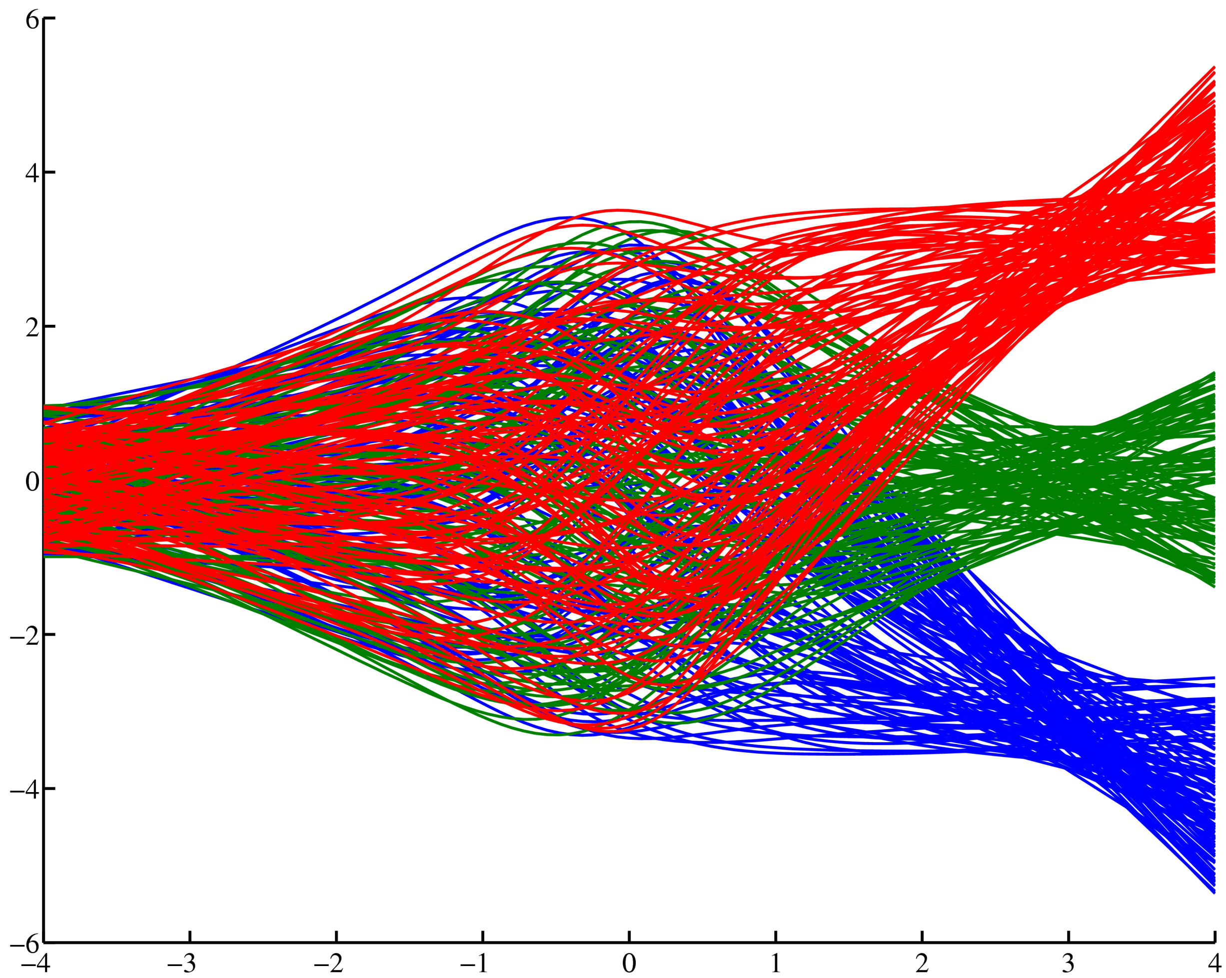


Black is initial,
green is unsteered,
red and blue are
steered $+3$ and -3.

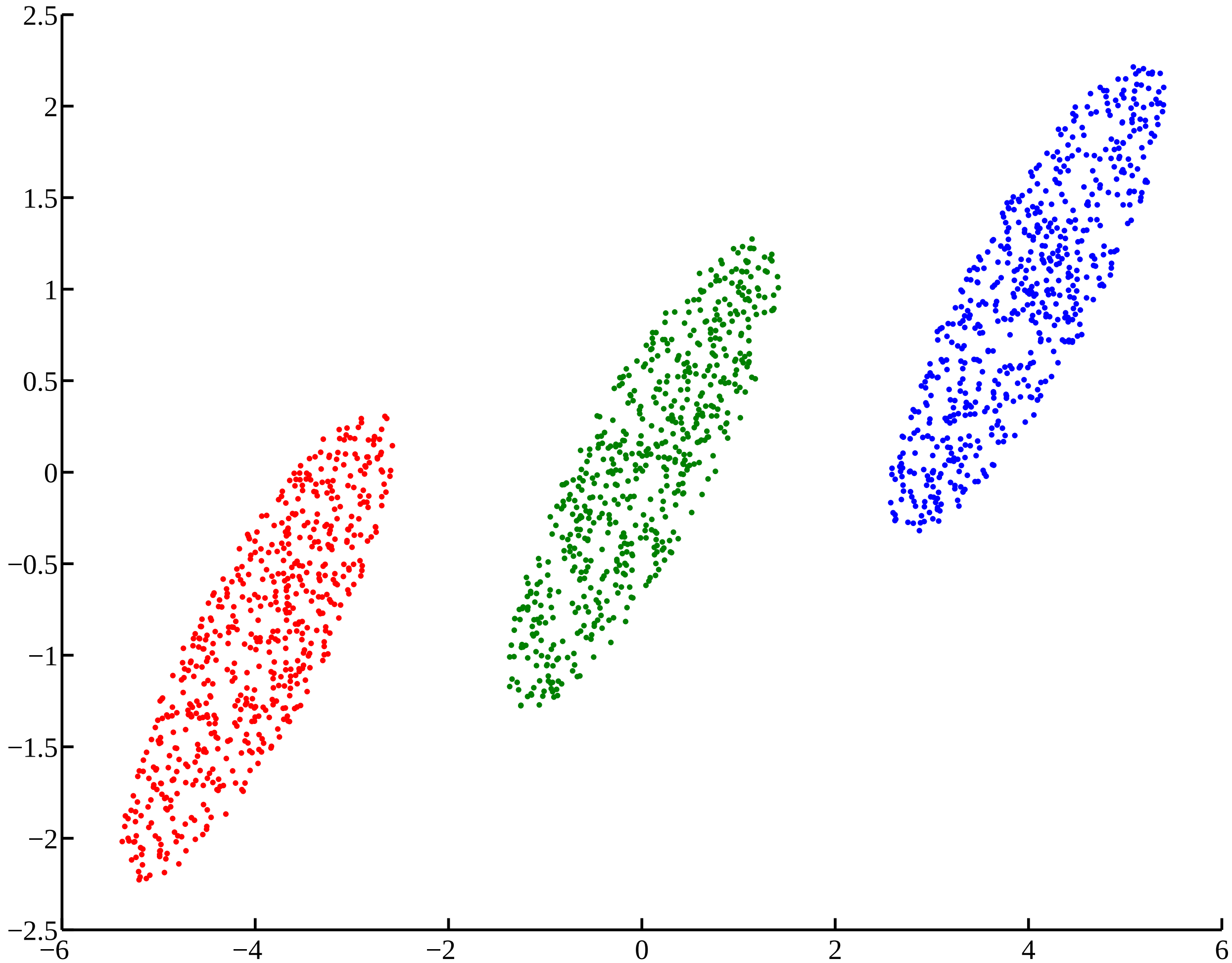
In real (x, y) space.



Trajectories in xy space.



Trajectories
from the
side (ys space).



Final (x, P_x) or (x, x') phase space. (y, P_y) is exactly the same.

with dipole, in rotating frame

Now let's apply the same transformation as before, i.e., to the rotating frame (dropping the R subscript to avoid clutter, $\theta = \int \kappa \mathrm{d}s$):

$$H = C\left(x\cos\theta - y\sin\theta\right) + \frac{P_x^2}{2} + \frac{\kappa^2}{2}x^2 + \frac{P_y^2}{2} + \frac{\kappa^2}{2}y^2 \tag{14}$$

There's nothing really pathological about this Hamiltonian. It does not couple x and y. It is just a uniform focusing device $\kappa^2(s)$ plus an embedded dipole of envelope $C(s)$, twisting at a rate of $\frac{\mathrm{d}\theta}{\mathrm{d}s} = \kappa(s)$.

So it should not be a surprise that there is no emittance growth. The beam particles cannot gain angular momentum when passing through the solenoid.