Fourier expansion of Long Range Beam-Beam Hamiltonian, DRAFT Triumf Note

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1 Introduction

The two-dimensional coefficients $c_{m,k}$ (resonance basis) in the Fourier expansion of the long-range beam-beam Hamiltonian are expressed through *generalized modified Bessel functions depending on two variables*, u and v, see e.g. [1] and [2]:

$$\mathbf{I}_m(u,v) \equiv \sum_{k=-\infty}^{\infty} I_{m-2k}(u) I_k(v). \tag{1}$$

These $\mathbf{I}_m(u,v)$ have properties very similar to the ordinary Bessel functions $I_m(u)$, but are much less familiar. Below, formulas for the generating function, recursion and derivatives are used, derived by simply transforming results from [2] – a paper devoted the very similar two-dimensional Bessel J functions [1].

To our knowledge, analytic formulas utilizing $I_m(u)$ for a long-range beam-beam potential were *first* derived by the authors of [3] in one-dimension, i.e. coefficients c_m. The same basis was independently used in [4] to construct approximate 1D long-range invariant (the exhibition below follows closely the notations in this paper). Formulas for the tune shift away from resonances, i.e. c_0 only, had been previously derived in [8].

In two dimensions, for a round-beam collision point (a.k.a. IP) [2] the expression derived in this paper is:

$$c_{m,k} = \int_0^1 \frac{dt}{t}\left[\delta(m)\delta(k) - \; i^{m+k}\prod_z e^{u_2^{(z)}+u_3^{(z)}}\mathbf{I}_j(u_1^{(z)},u_2^{(z)})\right]. \tag{2}$$

Here $u_1^{(z)} = ta_zd_z$, $u_2^{(z)} = -\frac{t}{4}a_z^2$, $u_3^{(z)} = -\frac{t}{2}d_z^2$; a_z, d_z are the normalized particle amplitude and separation. Under the product sign $z = x$ or y and if $z = x$, then $j = m$; if $z = y$, then $j = k$.

The form (2) indicates that one might attempt to find $c_{m,k}$ recursively using the relation

$$u_1\left[\mathbf{I}_{m-1} - \mathbf{I}_{m+1}\right] + 2u_2\left[\mathbf{I}_{m-2} - \mathbf{I}_{m+2}\right] = 2m\mathbf{I}_m, \tag{3}$$

where arguments $(u_1^{(z)}, u_2^{(z)})$ of all $\mathbf{I}$ are assumed. Sample *Mathematica* script to test (3) for some randomly chosen u_1, u_2 is given in the Appendix.

In case of success, i.e. in case that all $c_{m,k}$ can be expressed through low order ones, there is an interesting consequence. Assume "multipole" is defined in the usual way, but in this new resonance basis. Since $\mathbf{I}_m$ for all m can be found given only the first four such functions: $\mathbf{I}_m$ for $m = 0,1,2,3$, it appears that the high-order resonance coefficients are not independent. They can be all expressed through the ones of order up to and including order m=3 (the redefined "octupole").

Imagine that as a result of some lumped correction, *local*, i.e. at this IP, compensation of all terms to order $m = 3$ has taken place. This would mean that all resonance terms are also canceled.

On the other hand, the two derivative properties

$$\frac{\partial}{\partial u_1}\mathbf{I}_m = 1/2\left[\mathbf{I}_{m-1} + \mathbf{I}_{m+1}\right]; \quad \frac{\partial}{\partial u_2}\mathbf{I}_m = 1/2\left[\mathbf{I}_{m-2} + \mathbf{I}_{m+2}\right] \tag{4}$$

allow to compute the two-dimensional tune-shifts (footprint) by knowing $\mathbf{I}_m$ for m=0,1 and 2 (see below).

[1] Remark: the $\mathbf{I}_m(u,v)$ also obey a differential equation

[2] valid also for flat-beam IP, see barred variables below

One may then conclude that, if terms up to order 2 ("sextupole") are locally minimized, then the footprint is reduced. If in addition the $m = 3$ term is minimized, then this leads to all resonance terms being small. Such observations have been reported in [5].

2 2D Hamiltonian

The Hamiltonian describing a long-range beam-beam kick (or -potential) divided by $\frac{N_b r_0}{\gamma}$ can be written as:

$$H(x,y) = \int_0^\infty \frac{dq}{\sqrt{q+2\sigma_x^2}\sqrt{q+2\sigma_y^2}} \left[1 - \exp\left(-\frac{(x+D_x)^2}{q+2\sigma_x^2} - \frac{(y+D_y)^2}{q+2\sigma_y^2}\right)\right], \tag{5}$$

which, with $q = 2\left(-1+\frac{1}{t}\right)\sigma_x{}^2$, $\frac{dq}{dt} = -\frac{2\sigma_x{}^2}{t^2}$ and $r \equiv \frac{\sigma_y}{\sigma_x}$, is transformed to

$$H(x,y) = \int_0^1 \frac{dt}{t\sqrt{1+(-1+r^2)\,t}} \left[1 - \exp\left(-\frac{t(x+D_x)^2}{2\sigma_x^2} - \frac{t(y+D_y)^2}{2\sigma_y^2\left[1+(-1+r^2)\,t\right]}\right)\right]. \tag{6}$$

Here $D_{x,y}$ is the real space offset of the collision point in x or y direction and σ_x and σ_y are the ones of the strong beam. This Hamiltonian, when differentiated over x or y can be shown to produce the Bassetti-Erskine kick [9].

Let $d_{x,y} = D_{x,y}/\sigma_{x,y}$ be the normalized offsets – relative separations between the orbits of the colliding bunches. With $r \equiv \frac{\sigma_y}{\sigma_x}$, (6) becomes

$$H(x,y) = \int_0^1 \frac{1 - e^{-\frac{1}{2}t\left[\left(\frac{x}{\sigma_x}+d_x\right)^2 + \frac{r^2\left(\frac{y}{\sigma_y}+d_y\right)^2}{1+(-1+r^2)t}\right]}}{t\sqrt{1+(-1+r^2)\,t}}\,dt. \tag{7}$$

For the LHC, in case of a round-beam IR optics, we have the relations $\sigma_x^w = \sigma_y$, $\sigma_y^w = \sigma_x$; $\sigma_{x,y}^w \equiv \sqrt{\epsilon\beta_{x,y}^w}$, with $\beta_{x,y}^w$ being betas at the IP for the weak beam and ϵ the emittance, assumed to be the same for weak and strong beam. Introduce action-angle variables $(J_{x,y}, \phi_{x,y})$, or alternatively $(I_{x,y}, \phi_{x,y})$, with

$$x = \sqrt{2\beta_x^w J_x}\sin\phi_x = \sigma_x^w\sqrt{2I_x}\sin\phi_x = \sigma_x^w\ a_x\ \sin\phi_x = \sigma_y\ a_x\ \sin\phi_x = r\sigma_x\ a_x\ \sin\phi_x \tag{8}$$

(since, from the definition of r, $\sigma_y = r\sigma_x$). In the same way:

$$y = \sqrt{2\beta_y^w J_y}\sin\phi_y = \sigma_y^w\sqrt{2I_y}\sin\phi_y = \sigma_y^w\ a_y\ \sin\phi_y = \sigma_x\ a_y\ \sin\phi_y = \frac{\sigma_y}{r}\ a_y\ \sin\phi_x \tag{9}$$

The "n-sigma" variable, particle amplitude measured in units of beam size, is $a_z = \sqrt{2J_z/\epsilon} = \sqrt{2I_z}$, (z=x,y).

The Hamiltonian (7) is formally expanded in Fourier series:

$$\begin{aligned} H(x,y) = H(a_x\sigma_x \sin\phi_x, a_y\sigma_y \sin\phi_y) &= \int_0^1 \frac{1-e^{-t(P_x+P_y)}}{t\sqrt{1+(-1+r^2)\,t}}\,dt = \\ &= \sum_{m=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} c_{m,k}e^{im\phi_x+ik\phi_y}. \end{aligned}$$

where:

$$\begin{aligned} P_x = \frac{1}{2}\left(d_x + ra_x\sin\phi_x\right)^2 = \\ = \frac{1}{2}\left(\bar{d_x} + \bar{a_x}\sin\phi_x\right)^2 \end{aligned} \tag{10}$$

$$\begin{aligned} P_y &= \frac{1}{2}\left(d_y + \frac{a_y}{r}\sin\phi_y\right)^2 \frac{r^2}{\left[1+(-1+r^2)\,t\right]} \\ &= \frac{1}{2}\left(\bar{d_y} + \bar{a_y}\sin\phi_y\right)^2 \end{aligned} \tag{11}$$

with $g(t) = \sqrt{1+(r^2-1)\,t}$, $\bar{d_x} = d_x$, $\bar{a_x} = ra_x$, $\bar{d_y} = d_y\frac{r}{g(t)}$, $\bar{a_y} = a_y\frac{1}{g(t)}$,

We need to find the coefficients:

$$c_{m,k} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} e^{-im\phi_x-ik\phi_y}\int_0^1 \frac{1-e^{-t(P_x+P_y)}}{t\,g(t)}\,dt\;d\phi_x d\phi_y \tag{12}$$

with m,k = integer.

For this, change the order of integration and re-group

$$c_{m,k} = \int_0^1 \frac{dt}{t\,g(t)}\,\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} e^{-im\phi_x-ik\phi_y}\left(1-e^{-t(P_x+P_y)}\right)\,d\phi_x d\phi_y. \tag{13}$$

The unity produces a product of delta-functions that does not contribute to coefficients other than $c_{0,0}$ (for $c_{0,0}$ it turns into 1) [3].

$$c_{m,k} = \int_0^1 \frac{dt}{t\,g(t)}\left[\delta(m)\delta(k) - \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} e^{-im\phi_x-ik\phi_y}e^{-t(P_x+P_y)}\;d\phi_x d\phi_y\right]. \tag{14}$$

with m,k = 0,1,2,...

[3]

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-im\phi}d\phi = \delta(m) \equiv \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m \neq 0. \end{cases}$$

3 Coefficients via two-dimensional Bessel

Rewrite (14) as:

$$c_{m,k} = \int_0^1 \frac{dt}{t\,g(t)} \left[\delta(m)\delta(k) - \mathbf{Q}_m^{(x)}\,\mathbf{Q}_k^{(y)}\right], \tag{15}$$

Under the integral

$$\mathbf{Q}_m^{(x)} = \frac{1}{2\pi}\int_0^{2\pi} e^{-im\phi_x}\, e^{-tP_x} d\phi_x, \quad \mathbf{Q}_k^{(y)} = \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\phi_y}\, e^{-tP_y} d\phi_x. \tag{16}$$

These can be represented as Bessel series [3],[4]. In case of $\mathbf{Q}_m^{(x)}$, rewrite the factor of the exponent as

$$-tP_x = -u_1^{(x)}\sin\phi_x + 2u_2^{(x)}\sin^2\phi_x + u_3^{(x)}, \tag{17}$$

$$-tP_y = -u_1^{(y)}\sin\phi_y + 2u_2^{(y)}\sin^2\phi_y + u_3^{(y)}, \tag{18}$$

where

$$\begin{array}{ll} u_1^{(x)} = t\bar{a}_x\bar{d}_x & u_1^{(y)} = t\bar{a}_y\bar{d}_y \\ u_2^{(x)} = -\frac{t}{4}\bar{a}_x{}^2 & u_2^{(y)} = -\frac{t}{4}\bar{a}_y{}^2 \\ u_3^{(x)} = -\frac{t}{2}\bar{d}_x^2 & u_3^{(y)} = -\frac{t}{2}\bar{d}_y^2 \end{array} \tag{19}$$

and use the integral representation of the two-dimensional modified Bessel function. In the Appendix it is shown that the result is:

$$\begin{aligned} \mathbf{Q}_m^{(x)}(a_x, d_x, t) &= i^m e^{-\frac{1}{4}t\left(2d_x^2 + a_x{}^2\right)} \sum_{q=-\infty}^{\infty} I_{m-2q}(ta_xd_x) I_q(-\frac{t}{4}a_x^2) = \\ &= i^m e^{-\frac{1}{4}t\left(2d_x^2 + a_x{}^2\right)}\, \mathbf{I}_m(ta_xd_x, -\frac{t}{4}a_x^2) = \\ &= i^m e^{u_2^{(x)} + u_3^{(x)}} \mathbf{I}_m^{(x)}(u_1^{(x)}, u_2^{(x)}) \end{aligned} \tag{20}$$

The above can be expressed via generalized Bessel $\mathbf{\Lambda}_n(u) = e^{-u}I_n(u)$ in two dimensions $\mathbf{\Lambda}_n(u_1,u_2) = e^{-u_1-u_2}I_n(u_1,u_2)$. For this, notice that

$$u_2 + u_3 = -t/2(a_x - d_x)^2 - u_1 - u_2,$$

which follows from

$$-t/2(a_x - d_x)^2 = -t/2a_x^2 + ta_xd_x - t/2d_x^2 = u_1 + 2u_2 + u_3$$

to rewrite (20) as

$$\begin{aligned}\mathbf{Q}_m^{(x)}(a_x,d_x,t) &= i^m e^{u_2^{(x)}+u_3^{(x)}}\mathbf{I}_m^{(x)}(u_1^{(x)},u_2^{(x)}) = \\ &= i^m e^{-\frac{t}{2}(a_x-d_x)^2} e^{-u_1-u_2}\mathbf{I}_m^{(x)}(u_1^{(x)},u_2^{(x)}) = \\ &= i^m e^{-\frac{t}{2}(a_x-d_x)^2}\mathbf{\Lambda}_m(u_1^{(x)},u_2^{(x)})\,. \end{aligned} \tag{21}$$

The expressions for y

$$\mathbf{Q}_k^{(y)} = \mathbf{Q}_m^{(x)}(d_x \to \bar{d}_y, a_x \to \bar{a}_y, m \to k)$$

are identical to (20) with index x replaced by y and a_x replaced by a_y in (19).

We finally get the Eqn. 2, already advertised in in the Introduction:

$$c_{m,k} = \int_0^1 \frac{dt}{t\,g(t)}\left[\delta(m)\delta(k) - \; i^{m+k} e^{u_3^{(x)}+u_3^{(y)}} e^{u_2^{(x)}+u_2^{(y)}}\mathbf{I}_m(u_1^{(x)},u_2^{(x)})\;\mathbf{I}_k(u_1^{(y)},u_2^{(y)})\right]. \tag{22}$$

We used $\mathbf{\Lambda}$ instead of $\mathbf{I}^{(x)}$ then $a_x = d_x$ means: traverses the strong B core

For round beam $r = 1$, this expression (14) is symmetric ($x \leftrightarrow y$) and only one Q need be computed. We see that in case of a round beam IP, in order to find some coefficient, the Bessel expansion (20) is multiplied by an identical one, the result is substituted in (15), i.e. it is divided by t and integrated over t from 0 to 1. It's easy to see that under the integral there is no singularity at $t = 0$ for any m, k (by using the asymptotic of I for small argument: $t \to 0$).

Check: Coefficients for single-plane head-on as in [7]

For collision point without offset, $d_x = 0$ in (20), from the delta property for zero argument $I_{m-2q}(0) = \delta(m-2q)$ it follows that only even m remain. Alternatively, one can use $\mathbf{I}_m(0,u_2) = I_{m/2}(u_2)$.

We get

$$\mathbf{Q}_m^{(x)}(a_x,0,t) = \begin{cases} -e^{-\frac{t}{4}a_x^2} I_0(-\frac{t}{4}a_x^2), & \text{if m=0} \\ -i^m e^{-\frac{t}{4}a_x^2} I_{m/2}(-\frac{t}{4}a_x^2), & \text{if m=even} \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

The corresponding single-plane expansion coefficient is

$$c_{m,0} = \begin{cases} \int_0^1 \frac{dt}{t}\left(1 - e^{-\frac{t}{4}a_x^2} I_0(-\frac{t}{4}a_x^2)\right), & \text{if m=0} \\ -i^m \int_0^1 \frac{dt}{t} e^{-\frac{t}{4}a_x^2} I_{m/2}(-\frac{t}{4}a_x^2), & \text{if m=even} \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

and may be found for example in the Lectures of A. Chao [7] [4].

Coefficients for single-plane long-range

By setting now $d_x \neq 0$ in (20) and using for the y-plane the fact that $\mathbf{I}_0(0,0) = 1$, the single-plane long-range expansion coefficient (this time in terms of $\mathbf{I}$) is

$$c_{m,0} = \begin{cases} \int_0^1 \frac{dt}{t}\,(1-K)\,, & \text{if m=0} \\ \int_0^1 \frac{dt}{t}\,(-K)\,, & \text{if} \neq 0 \end{cases} \tag{25}$$

where $K = i^m e^{-\frac{t}{4}a_x^2} e^{-\frac{t}{2}d_x^2} \mathbf{I}_{m,0}(t a_x d_x, -\frac{t}{4}a_x^2)$.

4 Footprint

For the footprint we need the partial derivative of $c_{0,0}$ over the actions, e.g. $\Delta Q_x = -\frac{1}{2\pi}\frac{\partial}{\partial J_x}c_{0,0}$, which is expressed through the derivative over amplitude by using $\frac{da_x}{dJ_x} = -\frac{1}{\epsilon a_x}$.

By setting $m = k = 0$ in (22) and also replacing the delta-product with unity, we get

$$\begin{aligned} c_{0,0} &= \int_0^1 \frac{dt}{t\,g(t)}\left[1 - \mathbf{Q}_0(a_x,d_x,t)\;\mathbf{Q}_0(a_y,d_y,t)\right]; \\ \frac{\partial}{\partial a_x}c_{0,0} &= -\int_0^1 \frac{dt}{t\,g(t)}\frac{\partial \mathbf{Q}_0(a_x,d_x,t)}{\partial a_x}\mathbf{Q}_0(a_y,d_y,t); \\ \frac{\partial}{\partial a_y}c_{0,0} &= -\int_0^1 \frac{dt}{t\,g(t)}\mathbf{Q}_0(a_x,d_x,t)\frac{\partial \mathbf{Q}_0(a_y,d_y,t)}{\partial a_y}, \end{aligned} \tag{26}$$

where (all arguments should be with bars for flat)

$$\begin{aligned} \mathbf{Q}_0(a_x,d_x,t) &= e^{u_2^{(x)}+u_3^{(x)}}\mathbf{I}_0^{(x)}(u_1^{(x)},u_2^{(x)}) = \\ &= e^{-\frac{t}{2}(a_x-d_x)^2}\mathbf{\Lambda}_0(u_1^{(x)},u_2^{(x)}) \end{aligned} \tag{27}$$

Recall that c_0 is in units $\frac{N_b r_0}{\gamma}$ and denote $\xi \equiv \frac{N_b r_0}{4\pi\gamma\epsilon}$. The tune-shifts are

$$\Delta Q_x = 2\xi \frac{1}{a_x}\frac{\partial}{\partial a_x}c_{0,0}, \quad \Delta Q_y = 2\xi\frac{1}{a_y}\frac{\partial}{\partial a_y}c_{0,0}. \tag{28}$$

The expressions (26), (27), (28) have been programmed in *Mathematica* and used to compute the footprint (with the derivative $\frac{\partial \mathbf{Q}_0}{\partial a_x}$ in (26) taken numerically).

[4] To compare with [7] use $\int_0^P (1-e^{-\alpha})\frac{d\alpha}{\alpha} = \int_0^1 (1-e^{-tP})\frac{dt}{t}$

The same derivative follows from the recursive property (4), which allows also to see which indices m participate in the footprint. The relations (4) become very simple for $m = 0$. Because of the symmetry property $\mathbf{I}_k = \mathbf{I}_{-k}$ for any integer k, we have $\frac{\partial}{\partial u_1}\mathbf{I}_0 = \mathbf{I}_1$ and $\frac{\partial}{\partial u_2}\mathbf{I}_0 = \mathbf{I}_2$. Then it is easily shown that the derivative in (26) depends on the first three coefficients (as advertised in the Introduction):

5 Tracking tests (footprint)

The expressions (28) have been compared with MadX (`dynaptune`) using HL-LHC settings with normalized emittance $\epsilon_{\text{norm}} = 2.5$ and a number of particles per bunch $N_b = 1 \times 10^{11}$ (tune-shift per IP ξ= 0.00488). The $\beta^{\star}$ is 15 cm. Tracking is on-momentum with sextupoles OFF.

5.1 Single Head-on collision

Figure 1 shows the comparison with MadX tracking for a singe head-on collision in IR5. Footprint parameters are: mtot=10, Amin=1, Amax=10, dA=1, where A is normalized amplitude.

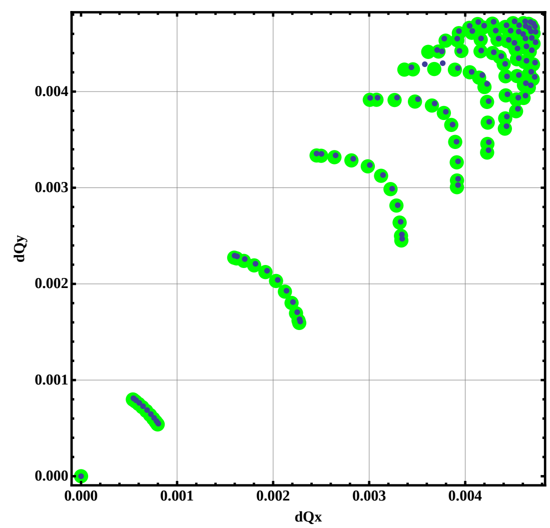

Figure 1: Green = analytic, Blue = MadX for the case: single Head-on at IP5. The IP5 X-ing angle is set to zero. ; Bessel series taken up to $q_{\max}$=15.

5.2 Single Long-range collision

Figure 2 shows the comparison with MadX tracking for a singe long-range collision. Footprint parameters: mtot=15, Amin=1, Amax=10, dA=1

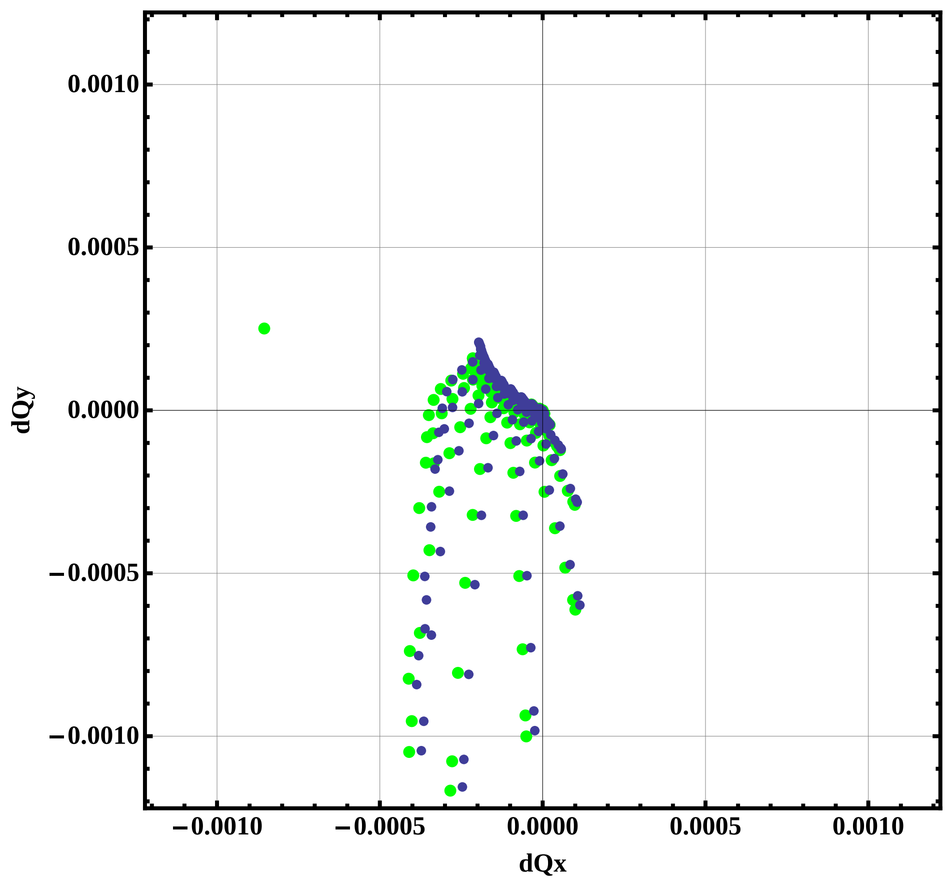

Figure 2: Green = analytic, Blue = MadX. A single LR in IR5, IP5 X-ing angle = 295/2, giving separation in x d_x=6.23 and round beam at this point $\sigma_x = \sigma_y = 0.1769$ mm; Bessel series up to $q_{\max}$=25

For the last case the particle initial coordinates are shown on Fig. 3.

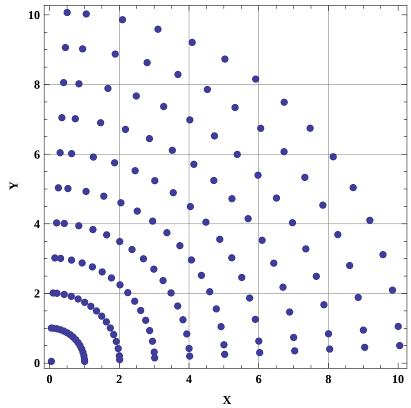

Figure 3: initial points in the X-Y quadrant for mtot:=15; Amin:=1; Amax:=10; dA:=1;

6 Appendix

We use

$$e^{-u_1 \ \sin\phi} = \sum_{q=-\infty}^{\infty} i^q e^{iq\phi} I_q(u_1),$$

$$e^{\,2u_2 \ \sin^2\phi} = e^{u_2} \sum_{k=-\infty}^{\infty} (-1)^k e^{2ik\phi} I_k(u_2).$$

For (16) we have:

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} e^{-im\phi} e^{-u_1 \ \sin\phi} e^{2\ u_2 \ \sin^2\phi + u_3} = \\ &= e^{u_2+u_3} \sum_{q,k=-\infty}^{\infty} (-1)^k i^q I_q(u_1)\ I_k(u_2)\ \delta(2k+q-m) = \\ &= i^m e^{u_2+u_3} \sum_{k=-\infty}^{\infty} I_{m-2k}(u_1) I_k(u_2) = \\ &= i^m e^{u_2+u_3} \mathbf{I}_m(u_1,u_2). \end{aligned} \tag{29}$$

Alternatively, (16) follows from the generating function:

$$e^{-u_1 \ \sin\phi + u_2(2\sin\phi^2 - 1)} = \sum_{k=-\infty}^{\infty} i^k \mathbf{I}_k(u_1,u_2) e^{ik\phi}.$$

Code *Mathematica* 1: Recursive calculation of generalized Bessel function of two variables `Itwo` using Eq. (3)

```
(*Define  Itwo*)
Itwo[u1_, u2_, m_] :=
 Sum[BesselI[m - 2 k, u1] BesselI[k, u2 ], {k, -kmax, kmax}]

 (*Choose arbitrary parameters*)
kmax = 10 ;
u1 = 5.678 ;
u2 = -3.456 ;

(* Solve (3) to arbitrary order m (here 10) for given *)
(* the first four Itwo 0,1,2,3                       *)
(* and  print the result in Table a                  *)

RecurrenceTable[
{-2 u2 a[m + 2] - u1 a[m + 1] ==
   2 m a[m] - 2 u2 a[m - 2] - u1 a[m - 1],

  a[0] ==  Itwo[u1, u2, 0]  ,
  a[-1] == Itwo[u1, u2, 1],
  a[-2] == Itwo[u1, u2, 2]  ,
  a[-3] == Itwo[u1, u2, 3]
  },
 a,
 {m, 0, 10}]

 (*Test some coefficients of order m>3 *)
Itwo[u1, u2, 4]
Itwo[u1, u2, 5]
Itwo[u1, u2,10]

{26.3313, 12.141, -10.9864, -15.0013, -2.32314, 6.23567, 3.77032,
```

```
 −0.974148,  −1.70009,  −0.231208,  0.445237}

−2.32314
 6.23567
 0.445237
```

References

[1] Clemente Cesarano and Claudio Fornaro, *GENERALIZED BESSEL FUNCTIONS IN TERMS OF GENERALIZED HERMITE POLYNOMIALS*, International Journal of Pure and Applied Mathematics Volume 112 No. 3 2017, 613-629 (see e.g. Eqn 27)

[2] H. J. Korsch, A. Klumpp , D. Witthaut, On two-dimensional Bessel functions, Technische UniversitaÌĹt Kaiserslautern D-67653, Germany

[3] Y. Papaphilippou, F. Zimmermann, Estimates of diffusion due to long-range beam-beam collisions, Phys Rev ST, VOLUME 5, 074001 (2002)

[4] D. Kaltchev, Analysis of long range studies in the LHC - comparison with the model, Beam-Beam ICFA mini workshop 2013, Geneva

[5] S. Fartoukh, A. Valishev, Y. Papaphilippou, D. Shatilov, Compensation of the long-range beam-beam interactions as a path towards new configurations for the high luminosity LHC, PRST 18, 121001 (2015)

[6] A.W. Chao, P. Bambade and W.T. Weng, Nonlinear beam-beam resonances, SLAC-PUB-3545 (1985). http://www.slac.stanford.edu/ cgi-wrap/getdoc/slac-pub-3545.pdf

[7] A.W. Chao, Truncated Power Series Algebra, lectures

[8] G. Lopez, Head-On and Long Range Beam-Beam Tune Shifts Spread in the SSC, SSCL-442, 1991

[9] CERN-ISR-TH/80-06