General Beam-beam detuning formula and its tests
TRI-BN 2220

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ARTICLE HISTORY
Compiled June 20, 2022

1. Introduction

In previous reports [1], [2] an analytic formula describing the amplitude-dependant tune shifts $\Delta Q_{x,y}$ (detuning) generated by an individual long-range (l.r.) collision in the LHC was found to agree with the detuning found with MadX. This formula, first derived in [2], is valid only in the case of an exact anti-symmetry between the weak and strong beam rms sizes at the collision location:

$$\sigma_x^{\mathtt{str}} = \sigma_y^{\mathtt{wk}}, \quad \sigma_y^{\mathtt{str}} = \sigma_x^{\mathtt{wk}} \tag{1.1}$$

It provides a recipe to compute the two (horizontal, vertical) detunings:

$$\begin{aligned} &\Delta Q_x(a_x, a_y; \theta) \\ &\Delta Q_y(a_x, a_y; \theta) \end{aligned} \tag{1.2}$$

as a function of an array of optics parameters at the l.r. location $\theta = (d_x, d_y, r)$, defined in the next section. Only normalized parameters participate in θ. The formula was encoded in *Mathematica* [1], [2] and python.

In this paper Eqn 1.2 is modified so as it is valid also in case of a general collision, i.e. when

$$\sigma_x^{\mathtt{str}} \neq \sigma_y^{\mathtt{wk}}, \quad \sigma_y^{\mathtt{str}} \neq \sigma_x^{\mathtt{wk}} \tag{1.3}$$

The result found is as follows. With

$$A \equiv \frac{\sigma_x^{\mathtt{wk}}}{\sigma_y^{\mathtt{str}}}, \quad B \equiv \frac{\sigma_y^{\mathtt{wk}}}{\sigma_x^{\mathtt{str}}}. \tag{1.4}$$

the modified expressions 1.2, i.e. a general l.r. detuning formula is

$$
\begin{aligned}
&A^2\,\Delta Q_x(A\,a_x, B\,a_y;\theta')\\
&B^2\,\Delta Q_y(A\,a_x, B\,a_y;\theta')
\end{aligned}
\tag{1.5}
$$

where θ' now contains the two additional parameters $\sigma^{\mathtt{wk}}_{x,y}$.

Eqn 1.5 is here tested on a simple a MadX model that consists in some ring lattice counting a single l.r. collision.

2. Notes on derivation

The detunings are derivatives of the averaged Hamiltonian, i.e. the zero-index coefficient in its decompositions as Fourier series, [2]:

$$
\Delta Q_z(a_z;\theta) = \frac{2\xi}{a_z}\frac{\partial C_{00}(a_z;\theta)}{\partial a_z},\quad \text{where z = x or y} \tag{2.6}
$$

($\xi \equiv \frac{N_b r_0}{4\pi\gamma\varepsilon}$ is the beam-beam parameter). The Hamiltonian is the one that governs the Bassetti-Erskine kick:

$$
\begin{aligned}
&H(x,y;\theta) = \int_0^1 \frac{dt}{t g_r(t)}[1-e^{-t\left(P_x+P_y\right)}], \\
&P_x \equiv \frac{1}{2(\sigma^{\mathtt{str}}_x)^2}(x+\mathfrak{D}_x)^2, \qquad P_y \equiv \frac{1}{2g_r(t)^2(\sigma^{\mathtt{str}}_x)^2}(y+\mathfrak{D}_y)^2; \\
&g_r(t) \equiv \sqrt{1+(r^2-1)t},
\end{aligned}
\tag{2.7}
$$

where $r \equiv \frac{\sigma^{\mathtt{str}}_y}{\sigma^{\mathtt{str}}_x}$ is the strong-beam sigma aspect ratio at this location. Also $\mathfrak{D}_x = x^{\mathtt{wk}}_{\mathtt{CO}} - x^{\mathtt{str}}_{\mathtt{CO}}$ is the full separation – a difference between closed orbit offsets, and similar for $\mathfrak{D}_y$.

As a side note, it can be shown that an exchange of the all subscripts x and y, i.e. horizontal and vertical axes, preserves H, and this allows to derive expressions for IR1 from the ones in IR5, or reverse.

By definition of amplitudes $a_{x,y}$:

$$
\begin{aligned}
x &= \sqrt{2\beta^{\mathtt{wk}}_x J_x}\sin\phi_x = a_x\sigma^{\mathtt{wk}}_x\sin\phi_x\\
y &= \sqrt{2\beta^{\mathtt{wk}}_y J_y}\sin\phi_y = a_y\sigma^{\mathtt{wk}}_y\sin\phi_y,
\end{aligned}
\tag{2.8}
$$

and using the symmetry Eqns 1.1 we get

$$
\begin{aligned}
x &= r a_x\sigma^{\mathtt{str}}_x\sin\phi_x\\
y &= a_y\sigma^{\mathtt{str}}_x\sin\phi_y,
\end{aligned}
\tag{2.9}
$$

where $\beta^{\mathtt{wk}}_{x,y}$ are the weak-beam beta-functions $\sigma^{\mathtt{wk}}_{x,y} \equiv \sqrt{\varepsilon\beta^{\mathtt{wk}}_{x,y}}$ and $a_{x,y}$ are the horizontal and vertical amplitudes.

Clearly, when x, y from 2.9 are substituted in H, then θ contains only normalized (to sigmas) separations: $d_x = \frac{\mathfrak{D}_x}{\sigma^{\mathtt{str}}_x}$, $d_y = \frac{\mathfrak{D}_y}{\sigma^{\mathtt{str}}_x}$ and r. Note that θ can be chosen either $\theta = (d_x, d_y, r)$ (normalized to strong-beam sigma full separations and the aspect ratio), or $\theta = (d_x/r, rd_y, r)$ i.e. via normalized to weak-beam sigma separations.

The desired transforming coefficients A and B, Eqn 1.4, are the ratios between the right-hand sides of Eqns 2.8 and 2.9 (with taking into account the definition of r).

3. MadX model

The following scripts are used (download here `anybb.madx`):

(1) A thin-element ring `seqb1thin` containing marker IP and a kicked orbit in x plane at a location $s_{\mathtt{BB}}$ `sbb`
(2) Extract weak-beam parameters at l.r. location
(3) Choose strong beam parameters
(4) Track to create two dynaptune files w/o and with BB
(5) Compute $\Delta Q_{x,y}$

Code *MadX* 1 Load thin-element weak-beam ring seqb1thin containing marker IP and kicked orbit

```
call, file = "./seqb1thin";
nrj   =  700;
bg   =  nrj/pmass;
en   = 2.5e-06*200; ! changes
epsx = en/bg;
epsy = en/bg;
any=0.075;
Nb=1000*1.E11; !only changes ksi (and scales plots)
rp = 1.534698E-18;
ksi= (Nb*rp)/(4*Pi*en);
Beam, particle = proton, sequence=lhcb1, energy = nrj,
          sigt=any,bv = +1, npart=Nb, sige=      1.1e-4,
          ex=epsx,   ey=epsy;
```

Code *MadX* 2 Extract weak-beam parameters

```
create,table=bbpar,column=sepx,dx,sigxstr,sigystr,sigxwk,sigywk,r,xstr,xwk;
sbb=-4; ! lr bb location on left of IP
bbmk:marker;
seqedit,sequence=lhcb1;
install,element=bbmk,at=sbb,from=IP;
endedit;
use,sequence=lhcb1;
```

```
8 twiss;
9 sigxwk = sqrt(table(twiss,bbmk,sig11))*1000;
10 sigywk = sqrt(table(twiss,bbmk,sig33))*1000;
11 xwk = table(twiss,bbmk,X)*1000;
12 ywk = table(twiss,bbmk,Y)*1000; !assumed zero
13  };
```

Lines 9 to end extract the weak-beam parameters: sigmas: σ_x^{wk} σ_y^{wk} and orbit excursion x^{wk} at location s_{bb}. These are saved in table bbpar created on line 1.

Code *MadX* 3 choose strong beam parameters at the l.r.

```
1 ! exact asymmetry weak-strong
2 !sigxstr=sigywk;
3 !sigystr=sigxwk;
4 ! arbitrary sigma strong
5 sigxstr=1.2;
6 sigystr=2.1;
7
8 xstr=15; ! XMA parameter in beambeam
9 r=sigystr/sigxstr;
10
11 ! full separation = Dx in Hamiltonian
12 sepx=xwk-xstr;
13 dx=sepx/sigxstr;
14 value,sepx,dx,r;
15 fill,table=bbpar;
16
17 ON_BB=1; ! flag to switch off beam beam
18
19 bb_any : beambeam,
20 sigx = sigxstr/1000,
21 sigy = sigystr/1000,
22 xma  = xstr/1000,
23 yma = 0,
24 charge:= ON_BB;
25
26 seqedit,sequence=lhcb1;
27 install,element=bb_any,at=sbb,from=IP;
28 endedit;
29
30 write,table=bbpar,file=bbpar.dat;
```

Note the delayed assignment := on line 19.

Code *MadX* 4 tracking creates two dynaptune files: w/o beam-beam in dir temp_0 and with beam-beam in dir temp_1

```

use,sequence=lhcb1;
twiss, sequence=lhcb1,save,file=twiss_b1_bef_track;

! no bb
ON_BB=0;
use,sequence=lhcb1;
call,file="./MAKEFOOTPRINT.madx";
exec, MAKEFOOTPRINT($ON_BB);

!with bb
ON_BB=1;
use,sequence=lhcb1;
call,file="./MAKEFOOTPRINT.madx";
exec, MAKEFOOTPRINT($ON_BB);
```

Code *MadX* 5 MAKEFOOTPRINT

```
 MAKEFOOTPRINT(nam): macro = {
 system, "rm -rf temp\_nam";
 system, "mkdir temp\_nam";
set,    format="30.15f";
nturn=2000;
mtot=2;
amin=1;
amax=-sepx/sigxwk;
dn= amax/5;
small=0.05;
big=sqrt(1.-small^2);
delete,table=dynaptune;
delete,table=aini;
create,table=aini,column=xs,ys;
dpp=0;
track,deltap=dpp,dump;
n=amin;
m=0;
while (n <= amax)
{
angle = 90/mtot*m*pi/180;
if (m == 0) {xs=n*big; ys=n*small;}
elseif (m == mtot) {xs=n*small; ys=n*big;}
else
{
xs=n*cos(angle);
ys=n*sin(angle);
}
value,xs,ys;
fill,table=aini;
start,fx=xs,fy=ys;
m=m+1;
if (m == mtot + 1 ) { m=0; n=n+dn;}
}
run,turns=nturn;
dynap,fastune,turns=nturn;
endtrack;
write,table=aini,file="aini.dat";
write,table=dynaptune,file="temp_nam/dynaptune";
};
```

Code ***Mathematica 6***

```
A = sigxwk /sigystr;
B = sigywk/sigxstr;
- 2 ksi {
 A^2  DXC00[dx,0,r,A ax,B ay]/ax
   ,
 B^2  DYC00[dx,0,r,A ax,B ay]/ay
        };
```

The ax, ay are taken from file aini.dat.
The parameters dx, sigxstr, sigystr, sigxwk, sigywk, r are taken from the table saved in file bbpar.dat.
The DXC00, DYC00 are defined in [1].

4. Results

Parameters for the example shown on Figures 1, 2, 3 and 4 (all lengths in mm) :

sepx= – 27 sigxstr=1.2 sigystr=2.1 sigxwk=2.57 sigywk=2.55 dx= – 22.5 r=1.75

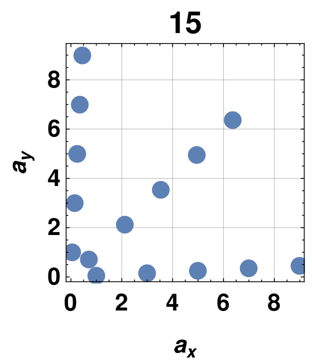

Figure 1. Weak-beam particle amplitudes.

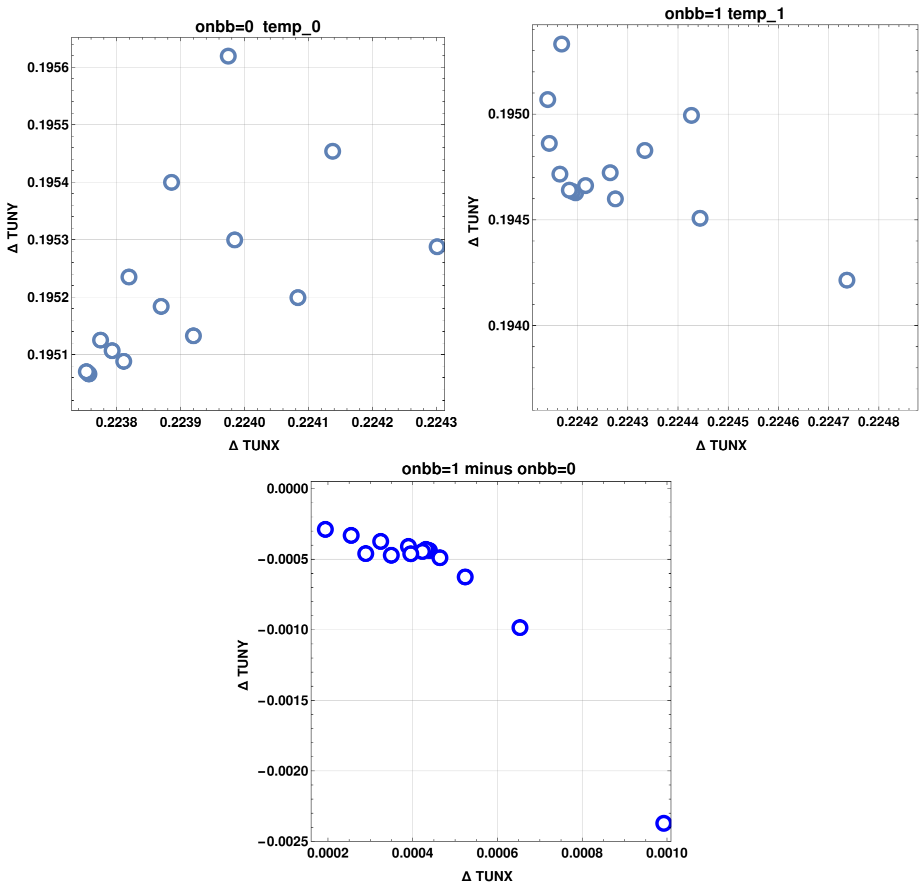

Figure 2. MadX detunings are the difference between the onbb=1 and onbb=0 detunings

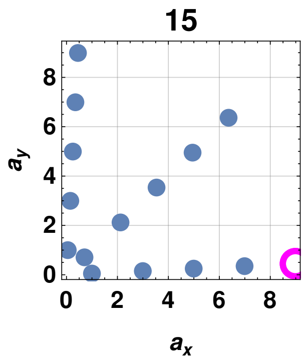

Figure 3. Divergent amplitude point near str beam core – magenta

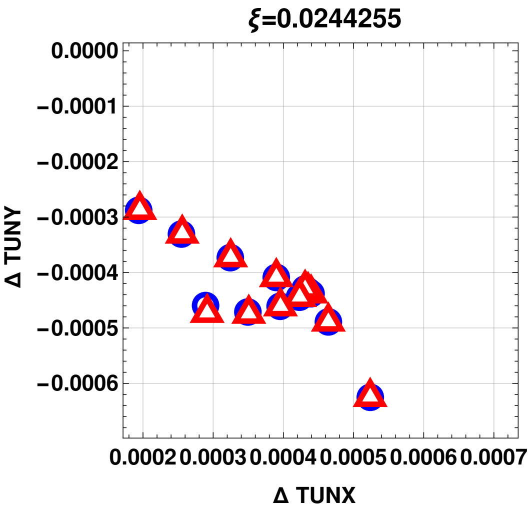

Figure 4. MadX detuning (blue) and analytic, Eqn 1.5 (red).

References

[1] D. Kaltchev, *Beam-beam detuning formula and its agreement with MadX. Individual collisions*, TRI-BN 2214, June 19, 2022, PDF

[2] D. Kaltchev, *Fourier Coefficients of Long-Range Beam-Beam Hamiltonian via Two-Dimensional Bessel functions* Proc. of IPAC 2018, Vancouver BC, Canada