Near-Adiabatic Capture of Pendulum Phase Space

In Rising Potential Well

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Contents

Chapter 1

Near-Adiabatic Capture of Pendulum Phase Space

"As far as I see, all *a priori* statements in physics have their origin in symmetry", Hermann Weyl[1].

"When the rf voltage is turned on, the electrons are uniformly distributed along the equilibrium orbit. After the voltage has been built up, all electron phases will oscillate about zero phase with amplitudes which decrease adiabatically; thus the electrons tend to bunch", David Saxon and Julian Schwinger[2] (1946).

1.1 Prologue

Let t, x and $p = \dot{x}$ be time, position and momentum; and dots denote time derivatives. This article concerns the motion of a pendulum with a time-dependent restoring force, having the equation of motion $\ddot{x} + \omega^2(t) \sin x = 0$. The article attempts to answer "what happens to an ensemble of pendula (each having different initial conditions) as the restoring force is increased?" In the case of a true pendulum, $x \equiv \theta$ is the angular deviation from vertical. The pendula may spin about the pivot, or they may swing to-and-fro. The pendulum motion is analogous to that of a frictionless object traveling in a periodic potential well $U = V(t)(1 - \cos x)$. Particles with small kinetic energy (K.E.) are bound in a single local minimum of the potential, whereas particles with large K.E. stream across the top of the periodic well. If the height $V(t)$ of the well rises, then rotations may be transformed into confined oscillations; a process known as capture. Phase space is the manifold of configurations (x, p). A slow transformation process is usually described as being adiabatic. We introduce the term near-adiabatic to emphasize a deviation from the adiabatic regime.

The motivation for the study is the behaviour of charged particles in a synchrotron (or other ring type machine) where particles undergo synchronous acceleration by the voltage $V(t)$ while being confined by magnetic fields. Particles with momentum deviation p slip in position at a rate $A \times p$, and are confined by an effective potential $U(x, t)$. The slip constant is analogous to inverse of pendulum length $1/L$; and U is analogous to the gravitational potential $g \times (1 - \cos\theta)$. If the particles are initially uniform around the ring and streaming, raising the voltage will capture them into bunches. We may wonder "what voltage law will minimize the spread of positions and momenta?", i.e. how best to shape the final phase space of the particle ensemble.

1.1.1 Notation

- time t and dimensionless time $z = \omega_0 t$.
- derivatives $\dot{x} = dx/dt$ and $x' = dx/dz$; and $\dot{x} = \omega_0 x'$.
- position x; momentum p.

- angular frequency ω.
- perturbations in position α and momentum $\dot{\alpha}$ induced by $\dot{\omega}$.
- s or u is dummy parameter for integration.
- Hamiltonian H; adiabatic invariant I or J equal to H/ω; Jacobi m-parameter $m = H/\omega^2/2$.
- For rotation $m = [\sin(\hat{x}/2)]^2$ where $\hat{x}$ is the maximum excursion.
- potential energy U; spatial potential $U(x)$; voltage V; kinetic energy T.
- oscillation period for libration τ.
- ϵ (or ϵ_1) is dimensionless adiabaticity parameter $\epsilon = \dot{\omega}/\omega^2 = \omega'/\omega$.
- values subscripted 0, such as x_0, p_0 denote initial values $x(0), p(0)$.
- duration of voltage ramp T; fraction of the ramp duration $\tau = t/T$.
- values subscripted T, denote final values such as $H_T \equiv H(T)$.
- values subscripted c, denote values at the instant of capture; for example $V_c \equiv V(t_c)$.
- ϕ or q or Q all denote oscillation phases.
- hat, or circumflex, placed above a quantity $\hat{X}$ denotes the largest or maximum value.
- bar placed above a quantity $\bar{X}$ denotes time-average over a window; typically the window is a half-oscillation period.
- bra-kets placed around a quantity $\langle X \rangle$ denotes ensemble-average over the oscillation phases.
- a tilde placed above a quantity $\tilde{H}$ denotes the double-average: time-averaged over a sliding window and ensemble-averaged over oscillation phases.
- cn, dn and sn are the principal Jacobi elliptic functions; they have two arguments, the phase and the amplitude parameter m.
- $K(m)$ and $E(m)$ are the complete elliptic integrals of the first and second kinds, respectively.
- $\mathcal{A}(u, m)$ is the Jacobi amplitude, the inverse of the incomplete elliptic integral.

1.1.2 The structure of this article

Secs. 1.2 and 1.3 provides historical context, and introduces the adiabaticity parameter ϵ. Sec. 1.4 introduces pendulum motion, and the vocabulary of *capture*. Sec. 1.5 makes the connection to adiabatic capture in synchrotron particle accelerators, and introduces the concept of *averaging* over an oscillation. Sec. 1.7 presents some ideas from Hamiltonian mechanics. Sec. 1.8 uses the Hamiltonian formulation of *averaging* to find evolution equations for mean and deviation of the Hamiltonian during the pre- and post-capture eras starting from an initial value H_0. Sec. 1.9 demonstrates how to find the perturbed phase-space trajectories that are induced by the rising potential well. Sec. 1.10 applies the method of Sec. 1.9 to calculate the change of Hamiltonian during separatrix crossing for the most symmetric trajectory. Sec. 1.11 calculates the change of Hamiltonian across the separatrix for general (asymmetric) trajectories. Secs. 1.8 to 1.11 *are the key new results in this article.* Sec. 1.12 outlines how to use the results of Secs. 1.8 to 1.11 to construct the final distribution of Hamiltonian values of the bunched particle beam in terms of the original distribution prior to turning on the initial voltage. Sec. 1.13 elaborates some of the properties of the time-dependent separatrix. Sec. 1.16 is essentially an appendix that contains mathematical details for the calculation of the form factors used in Sec. 1.8.

1.2 What is adiabaticity?

For those readers unfamiliar with *adiabaticity*, a brief comment to set the stage. The mathematical description of a physical process performed with perfect adiabaticity is well known: there are characteristic properties of the system that are conserved and called invariants. For example, in classical mechanics, Hamilton's Action integral is an adiabatic invariant; and in quantum systems, the quantum number (which defines the state) is conserved. *But the description developed herein is for a near-adiabatic process*; one where the invariants are imperfectly or poorly preserved.

The word *adiabatic* derives from the Greek adiabatos meaning impassable, and in thermodynamics refers to a condition imposed on a system that prevents any passage of heat into or out of the system. In mechanics, it refers to a reversible process in which work is done but no heat is transferred. Reversible is synonymous with iso-entropic and the process being performed infinitely slowly. In mechanics, the stipulation about "heat" means there is no form of dissipation: no friction, no turbulence, no air resistance, etc.

A historical overview of the "meaning of adiabatic" is given by Laidler[3]. Generally, the word *adiabatic* is taken to mean a process that occurs slowly; and this means slow compared with the characteristic time of the system. The period and decay time of an oscillator are often encountered characteristic times, but are not the only ones[1]. However, oscillators are encountered everywhere; and adiabatic processes performed on them are a theme of study in classical and quantum mechanics. The stipulation of no dissipation implies no decay of oscillations. The ubiquity of the harmonic oscillator stems from the vast number of physical systems wherein the energy exists in two forms (and is periodically transferred from one to the other) and each type of energy is an (approximate) quadratic function of a dynamic variable. For example, kinetic and potential energy, or electric and magnetic energy.

In classical mechanics, since the 1870s, *adiabatic* is reserved for the case that evolution is examined as some characteristic parameter of the system is slowly varied, specifically the natural frequency. For the harmonic oscillator Hamiltonian $H = \frac{1}{2}Ap^2 + \frac{1}{2}Vx^2$, the parameters being varied could be A or V or both. In such a case $\omega(t)^2 = A(t)V(t)$. It is customary to define the adiabaticity parameter

$$\epsilon_1 \equiv \frac{d\omega/dt}{\omega^2} = \frac{\dot{\omega}(t)}{\omega(t)}\frac{\tau}{2\pi} \approx \frac{\Delta\omega}{2\pi\omega}\,. \tag{1.1}$$

$\Delta\omega$ is the change in angular frequency during one oscillation period τ. Notice that ϵ is synonymous with "small fractional change per oscillation period". As defined ϵ has two key properties: (i) it is dimensionless; and (ii) it contains within itself the time scale of the natural oscillation, $1/\omega$, and the rate of parameter variation $\dot{\omega}/\omega$. For counter example, $\dot{\omega}/\omega$ alone does not have those properties; and is not a suitable adiabaticity parameter. The *adiabatic condition* states that there is (almost) no change in the action $J = H(t)/\omega(t)$ provided that $|\epsilon| \ll 1$.

A property of a physical system that stays approximately constant when changes occur slowly is called an adiabatic invariant. By this it is meant that if a system is varied between two end points, as the time for the variation is increased to infinity, the change of an adiabatic invariant between the two end points goes to zero. In Hamiltonian mechanics, an adiabatic change is a slow deformation of H, where the fractional rate of change of the total energy is much slower than the natural oscillation[2] frequency ω. The function $J/(2\pi) = I(t) = H(t)/\omega(t)$ is almost constant during adiabatic changes. The action[3] J and invariant I differ trivially be 2π.

[1] There is, for example, a "relaxation time" defined in fluid mechanics, materials science and gas dynamics.

[2] In celestial mechanics, replace oscillation by orbital.

[3] In classical mechanics, the term "action" has several meanings. Here we specify J is the action of the generalized coordinate q. The corresponding canonical variable conjugate to J is its "angle".

1.2.1 Contribution of Rayleigh

The idea of "adiabatic invariant" in mechanics had existed since the 1870s or earlier, but is often credited to a 1902 paper of Lord Rayleigh[4]. Rayleigh's mission was to demonstrate the invariants of motion when pressure waves are trapped inside acoustic resonators with moving boundaries. He starts and ends with the example of a string and plumb-bob pendulum. Pendula, in general, have two types of motion: (i) *rotation* in which the bob spins around the pivot; and (ii) *libration* in which the bob oscillates from side to side. Libration owes its naming to the Greek *libra* for scales. In phase space (x, p) these motions (in a time-independent system) are divided by a boundary $p = \pm\omega_0\sqrt{2(1 + \cos x)}$ known as the separatrix.

Rayleigh only studies libration; in which the oscillator is was always captured by the gravitational potential well. The transition between rotation and libration is not considered. [Of course, to consider rotation, the string has to be replaced by a massless rigid rod.] In Rayleigh's example, work is done on the pendulum by an external agency that shortens the string (with respect to the suspension point). The work is returned if the string is lengthened. Using the Lagrangian formulation, the work ends by showing the deviation from adiabaticity scales as the integral over time of the difference between kinetic and potential energy (all multiplied by the derivative of the time-varying parameter). For a harmonic oscillator, such as an acoustic resonator, this time-average difference is zero over a single cycle of the motion; and the invariant is proven. *However, for the pendulum oscillator, the time-averaged kinetic and potential energies are unequal.* Thus, for the pendulum, the assumptions of Rayleigh's proof are violated.

1.2.2 Contribution of Ehrenfest and antecedents

During the period circa 1860-1900, the classical and statistical theories of thermo-dynamics were being crafted by Rudolf Clausius and Ludwig Boltzmann, respectively (and others). One of Clausius' many contributions was the realization that heat in gases could be described by mechanics, that heat and its transfer are a mechanical problem. Boltzmann's brilliant insight was that heat transfer is described by the dynamics of statistical populations. Both saw the world as mechanical; that *thermal* is based in mechanics. Thus both wrote papers and books[5, 6] on thermo-dynamics that are heavily invested in classical mechanics.

Paul Ehrenfest[7, 8] introduced the concept of "adiabatic invariant" into the early discussion of quanta over the period 1911-1916. The focus of the time was quantization of light inside a cavity, as a resolution of the "ultra-violet catastrophe" associated with the Rayleigh-Jeans law for black-body radiation. The time-averaged electrical and magnetic energy in an optical or radio-frequency resonator are equal; and satisfy perfectly the conditions in Rayleigh's proof.

Ehrenfest made a second contribution[9]: to point out the over-looked work of Clausius[10, 11], Szily[12] and Boltzmann[6]. He credits them all with developing the idea of adiabatic invariant for non-linear oscillations in which the time-averaged K.E. and P.E. are unequal. The three authors[4] say that invariant H/ω is replaced by $\overline{T}/\omega$ where T is the kinetic energy and the average is performed over a cycle of the motion. Thus far, I have not located the elements of the derivation. Without details, it is unclear under what circumstances and conditions this result applies.

Journal referencing to this period of 1860-1900 can be confusing. "Annalen der Physik" changed its title from time to time, and some ranges of volume numbers are known by the editor-in-chief of the time. For example, Annalen der Physik is known as Poggendorff's "Annalen der Physik und Chemie" (abbreviated Pogg. Ann.) after its editor Johann Christian Poggendorff. Obviously, Annalen der Physik is published in German. For the time period discussed, English translations are found in the Philosophical Magazine. Further, the structure of the volumes and series have altered over time; leading to consistent but confusing numbering. Citations herein use the modern numbering adopted by Annalen der Physik in the 21st century.

[4] Boltzmann is aware that the pendulum motion has a separatrix, but does not consider crossing.

1.3 Variation of Adiabatic Invariant

Our interest is the deviations from perfect adiabaticity. We begin with the example of a harmonic oscillator such as a mass and spring. For this simple system, the “particle” is always captured; there is only libration. The potential energy is $U = V(t)x(t)^2/2$ and kinetic energy $T = Ap(t)^2/2$. For a harmonic oscillator with constant V, the time average values of U and T are equal. Hamilton’s equations are:

$$H(t) = T + U\,, \qquad \dot{x} = \partial H/\partial p = Ap(t)\,, \qquad \dot{p} = -\partial H/\partial x = -x(t)V(t)\,. \tag{1.2}$$

$$\text{Leading to} \quad \ddot{x} = -AV(t)x \quad \text{and} \quad \dot{H} = \frac{1}{2}x(t)^2\dot{V}(t)\,. \tag{1.3}$$

Suppose $V(t)$ varies slowly enough that the natural frequency $\omega/(2\pi)$ is defined for an oscillation. We denote amplitude a, initial phase ϕ and $\omega = \sqrt{AV(t)}$. The approximate solution is

$$x(t) \to a\sin(\phi + \omega t)\,, \qquad p(t) \to a\sqrt{V/A}\cos(\phi + \omega t)\,, \qquad H \to V(t)a^2/2\,. \tag{1.4}$$

The action is defined to be the area swept out in phase-space during one cycle τ:

$$J \equiv \int_0^\tau p(t)\dot{q}(t)dt = V(t)\int_{-\tau/2}^{+\tau/2} a^2\cos^2(\phi + \omega t)dt \;\to\; a^2\pi V/\omega = 2\pi H/\omega\,. \tag{1.5}$$

The change in the action is

$$\Delta J = \int_0^\tau \dot{J}dt \quad \text{where} \quad \frac{1}{2\pi}\frac{dJ}{dt} = \frac{\dot{H}}{\omega} - \frac{H\dot{\omega}}{\omega^2} = [U(t) - T(t)]\frac{\dot{\omega}}{\omega^2}\,. \tag{1.6}$$

This is a delicate integral to calculate. Let $\epsilon \equiv \dot{\omega}/\omega^2$. Certainly the integral is very small if ϵ is small; but it is not zero unless $\dot{\omega} = 0$. If $|\epsilon| > 0$ is removed from the integral, as being slowly varying, and the unperturbed motion is substituted then the integral is zero: $\Delta J \approx \epsilon \times 0$. Customarily, it is stated that $|\epsilon| \ll 1$, for all time t, is a sufficient condition for the change in action to be small; and this is true. But often it is interpreted to mean the change in action is of order ϵ; and that is false. Let us pursue this further.

The reasoning that if ΔJ is small, then $U(t)$ and $T(t)$ can be replaced by their unperturbed values, in which case the integral is almost zero, is a self consistent argument. But the circular reasoning does not help us to actually calculate the value of the integral. To quantify “how small?” is ΔJ, we have to work from the perturbed functions $x(t)$ and $p(t)$. We calculate the integral four times according to differing assumptions or imposed conditions. In all cases, to first order in ϵ the integral is zero; so we have to calculate up to order ϵ^2.

<u>The first case</u>: ϵ is constant over the interval $t = [-\tau/2, +\tau/2]$, and $x(t)$ is the unpertubed motion. The integral is identically zero:

$$\Delta J = -\frac{a^2 V}{2}\epsilon\int_{-\tau/2}^{+\tau/2}\cos 2(\phi + \omega t)dt = 0\;. \tag{1.7}$$

<u>The second case</u> is to make time-linear expansions of the voltage and the oscillation frequency: $V(s) = V(t) + \dot{V}(t)\times s$ and $\omega(s) = \omega(t) + \dot{\omega}(t)\times s$; but insert the unperturbed motion for $x(t)$.

$$\Delta J \approx \frac{a^2(\dot{\omega})^2}{A\,\omega}\int_{-\tau/2}^{+\tau/2}\cos 2(\phi + \omega t)tdt = \frac{a^2\pi V}{\omega}\epsilon^2\sin 2\phi = \epsilon^2 J\sin 2\phi\,. \tag{1.8}$$

Note, we have omitted terms from the integrand that integrate to zero.

The third case is to make time-linear expansions of all parameters: $V(s) = V(t) + \dot{V}(t)s$ and $\omega(s) = \omega(t) + \dot{\omega}(t)s$ and $x(s) = [a(t) + \dot{a}(t)s]\sin[q + s\omega(t)]$. The relation $\dot{a} = -a\dot{\omega}/\omega$ is found from the condition $\ddot{x} = -AV(t)x$. And $\omega = \sqrt{AV}$ leads to $\dot{V} = 2\omega\dot{\omega}/A$. The action integral becomes:

$$\Delta J \approx \frac{a^2(\dot{\omega})^2}{A\omega}\int_{-\tau/2}^{+\tau/2}[\cos 2(\phi+\omega t) + \omega t \sin 2(\phi+\omega t)]\, t dt = 2\frac{a^2\pi V}{\omega}\epsilon^2 \sin 2\phi\,. \tag{1.9}$$

Note, we have omitted terms from the integrand that integrate to zero. This is double the previous case; so including the effect of $\dot{V}$ on the motion is important.
The fourth case stems from the intuition that the integral will be smaller if $\epsilon(t)$ is actually a constant. We retain the amplitude variation $[a(t) + \dot{a}(t)s]$. The action integral becomes:

$$\Delta J \approx \frac{a^2(\dot{\omega})^2}{A\omega}\int_{-\tau/2}^{+\tau/2}\left[-\frac{1}{2}\cos 2(\phi+\omega t) + \omega t \sin 2(\phi+\omega t)\right] t dt = \frac{1}{2}\frac{a^2\pi V}{\omega}\epsilon^2 \sin 2\phi\,. \tag{1.10}$$

Note, we have omitted terms from the integrand that integrate to zero. This is a quarter the previous case; so setting ϵ equal to a constant produces a significant improvement. To summarise: if the adiabaticity parameter $|\epsilon| \ll 1$, the change in action per oscillation period is of order ϵ^2; and is reduced if $\dot{\epsilon} = 0$. Variations for which $\dot{\epsilon} = 0$ are called *iso-adiabatic.*

1.3.1 Evolution of deviation $J(\phi)$

Each of the integrals Eqns. (1.8, 1.9, 1.10) is of the form

$$\Delta J(\phi) = R \times J(t)\epsilon^2(t)\sin 2\phi \quad \text{with scaling factor} \quad R = 1,\ 2,\ 1/2\,. \tag{1.11}$$

The ensemble average over oscillation phases is

$$\langle \Delta J \rangle = \frac{1}{2\pi}\int_{-\pi}^{+\pi}\Delta J(\phi)d\phi = 0\,. \tag{1.12}$$

Hence the average value $\langle J \rangle$ does not change; or $d\langle J \rangle/dt = 0$. Now $\frac{1}{2\pi}\langle J \rangle = H/\omega$; this leads to the relation

$$\frac{H(t+T)}{\omega(t+T)} = \frac{H(t)}{\omega(t)} \quad \text{or} \quad H(t+T) = H(t)\sqrt{\frac{V(t+T)}{V(t)}}\,. \tag{1.13}$$

$\Delta J(\phi)$ is the phase-dependent deviation during an oscillation period. Hence if ϵ is slowly varying, there is the approximate time derivative:

$$\overline{\frac{d}{dt}\Delta J(\phi)} = J_0\frac{\epsilon^2(t)}{\tau(t)}R\sin 2\phi\,. \tag{1.14}$$

This may be integrated for a known time-variation of ϵ^2/τ, and initial action J_0. Further, the formalism lends itself to computing the evolution of n-th moments $\langle \Delta J^n \rangle$.

1.3.2 Second adiabaticity parameter

Now let us understand the implication of treating $\dot{\omega}/\omega^2$ like a constant, and removing it from the integrand.

$$\epsilon_2 \equiv \dot{\epsilon} = \frac{d}{dt}\left(\frac{\dot{\omega}}{\omega^2}\right) = \frac{\ddot{\omega}}{\omega^2} - 2\frac{(\dot{\omega})^2}{\omega^3}\,. \tag{1.15}$$

If ϵ_2 is identically zero, second order effects are reduced. Setting Eqn. (1.15) to zero, and then solving for $\omega(t)$ leads to $\omega(t) = c_2/(t+c_1)$ where c_1, c_2 are constants of integration. The corresponding value of $\epsilon = -1/c_2$. Introducing the boundary conditions at start $\omega(t{=}0) = \omega_0$ and finish $\omega(t{=}T) = \omega_T$, yields the *iso-adiabatic* frequency variation:

$$\omega(t) = \frac{T\omega_0\omega_T}{T\omega_T + (\omega_0 - \omega_T)t}\,. \tag{1.16}$$

We may rewrite the same conditions in terms of the parameter that is directly controlled in a synchrotron, namely the voltage ramp $V(t)$. Let A be the slip-rate constant and $\omega = \sqrt{AV(t)}$, then

$$\epsilon_1 \equiv \frac{\dot\omega}{\omega^2} = \frac{\dot V}{2V\sqrt{AV}} \quad \text{and} \quad \epsilon_2 \equiv \frac{d\epsilon_1}{dt} = \frac{-3(\dot V)^2 + 2V\ddot V}{4V^2\sqrt{AV}}\,. \tag{1.17}$$

1.3.3 $\Delta J(\phi)$ versus voltage laws

A variety of voltage ramping laws have been used for capture and bunching of charged particle beams, with a range of success, but only one of them (the linear ramp) a clear failure. We now investigate what the two adiabaticity parameters, ϵ_1 and ϵ_2 can indicate about the relative merits of these voltage ramps. Strictly speaking, this comparison can address only the post-capture process of *bunching*; because the harmonic oscillator has no analog of streaming. We use $\omega = \sqrt{A \times V(t)}$ and set the slip constant $A = 1$. Let the initial and final voltage values be V_0 and V_T, the ramp duration T, and $\Delta V = V_T - V_0$.

Linear ramp: $V(t) = V_0 + (t/T)\Delta V$. We find at the start $t = 0$, the parameters $\epsilon_1 = (\Delta V/T)/(2V_0^{3/2})$ and $\epsilon_2 = -3\Delta V/(2TV_0)\epsilon_1$. At later times, $\epsilon_1 \approx \sqrt{T/\Delta V}/(2t\sqrt{t})$ and $\epsilon_2 \approx -3/(2t)\epsilon_1$. The action deviation at the end of the voltage ramp is $\Delta J(\phi)/J_0 \approx R(V_T/V_0/2)\sin 2\phi$ with $R \approx 2$. The growth is large, and due to the large value of ϵ_1 at early times.

Quadratic ramp: $V(t) = V_0 + (t/T)^2\Delta V$. We find at the start $t = 0$, the parameters $\epsilon_1 = 0$ and $\epsilon_2 = \Delta V/(T^2V_0^{3/2})$. At later times, $\epsilon_1 \approx T/(t^2\sqrt{\Delta V})$ and $\epsilon_2 \approx -(2/t)\epsilon_1$. The growth of deviation at $t = T$ is $\Delta J(\phi)/J_0 \approx R\sqrt{V_T/V_0}\sin 2\phi$ with $R \approx 1$. This value is significantly smaller than its predecessor, because ϵ_1 is smaller at early times.

Exponential ramp: $V(t) = V_0 \times C^{(t/T)}$ where the ratio $C \equiv V_T/V_0 \gg 1$. The adiabaticity parameters are $\epsilon_1 = \ln C/(2T)/\sqrt{V(t)}$ and $\epsilon_2 = -\epsilon_1 \times (\ln C)/(2T)$. For long duration, $\epsilon_2 \to 0$. The growth of deviation at $t = T$ is $\Delta J(\phi)/J_0 \approx (3/2)R(1 - 1/\sqrt{C})(\ln C)\sin 2\phi$ with $R \approx 1$. This value is smaller than for the quadratic ramp.

Iso-adiabatic ramp: $V(t) = T^2V_0V_T/[(T-t)\sqrt{V_T} + t\sqrt{V_0}]^2$. The adiabaticity parameters are constant: $\epsilon_1 \approx 1/(T\sqrt{V_0})$ and $\epsilon_2 = 0$. The action deviation at the end of the voltage ramp is given by the expression for the exponential ramp, but with $R = 1/2$.

To conclude, evolution of the adiabatic invariant suggests that the iso-adiabatic ramp will be superior, and the linear ramp inferior, to all others for the process of bunching.

Our interest is in motion where there is both libration and rotation. Rotation is not possible for the harmonic oscillator; it has a potential well of infinite height. So, we shall not pursue this model any further. However, it served to introduce several features of the problem: any growth of the action is a second order effect, and difficult to calculate; (ii) the time-average over a quickly varying quantity; (iii) the ensemble average over oscillation phases. We shall encounter all of these again in the context of the pendulum oscillator.

1.4 Pendulum with rising potential well

To prepare a context, we make a sketch of the pendulum motion. What is termed adiabatic capture is in fact three processes. (1) Pre-capture: a near-adiabatic transformation of almost uniform streaming trajectories (i.e. rotation) into ones with strong modulations. During this period, the particle Hamiltonian is much greater than that on the separatrix. (2) Capture: a non-adiabatic transformation of rotation into libration. This is often called separatrix crossing. During this process, the particle Hamiltonian is approximately equal to that on the separatrix. (3) Post-capture: a near-adiabatic transformation of oscillating trajectories (i.e. libration) into ones with smaller amplitude and aspect ratio $(\hat{x}/\hat{p})$. During this period, the particle Hamiltonian is typically much smaller than that on the separatrix. For the majority of initial particle coordinates, the pre- and post-capture periods last much longer than the capture. The exception is coordinate pairs close to the unstable fixed points $(x_0, p_0) = (\pm\pi, 0)$. For a single particle, the three processes are sequential. For an ensemble of particles, with a wide range of initial Hamiltonian values, the processes become concurrent; albeit for different particles.

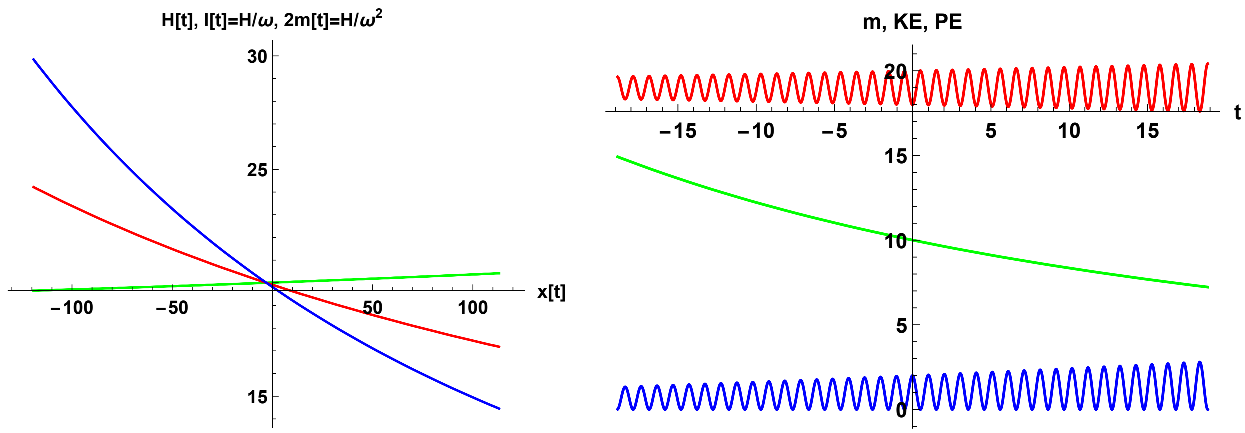

Figure 1.1: Rotation with $m_0 = 10$. Left: Hamiltonian (green), action $= H/\omega$ (red), Jacobi m (blue); all versus position. Right: kinetic energy (red), potential energy (blue), m-parameter (green); all versus time.

The following series of figures sketch the values of Hamiltonian (H), adiabatic invariant $I = H/\omega$, Jacobi m-parameter $H/\omega^2/2$, and kinetic energy T and potential energy U as a function of time t or position $x(t)$ for a pendulum with linear increasing natural frequency $\omega(t) = \omega_0(1 + \beta t)$. Here the slew constant $\beta = \epsilon \times \omega_0$. The three quantities H, I, m change most slowly at the position-extrema of motion (separated by a half-cycle); we may think of these locations as being where H, I, m are best defined, particularly so for m in Fig. 1.2. The sketches assume particles enter with positive momenta, streaming rightwards. There are analogous plots for negative momenta, streaming leftwards; these plots are double mirror symmetric about $p = 0$ and about $x = 0$.

The first example, Fig. 1.1, is taken from the pre-capture period, with Jacobi $m = 10$. During this time, the K.E. is much greater than P.E.; and the supposed invariant H/ω is not. The most slowly changing function is H, which evolves as $H(t) = H(0) + V(t)\langle U(x)\rangle - V(0)$ where $H_0 - V_0 = p_0^2/2 = T_0$.

The second example, Figs. 1.2 & 1.3, is taken from the capture process, with $m \approx 1$. During this process, there is a time-period where the P.E. is much greater than the almost zero K.E. The reader will see, Fig. 1.3, that H and I both change noticeably while the trajectory loiters in the vicinity of the fixed point $x = -\pi$. There is an equal effect for negative momenta entering at $x = +\pi$.

The final example, Fig. 1.4, is taken from the post-capture period, with Jacobi $m = 1/2$. During this time, the K.E. and P.E. become progressively more equal (while $U > T$); and the invariant H/ω is the most slowly changing function.

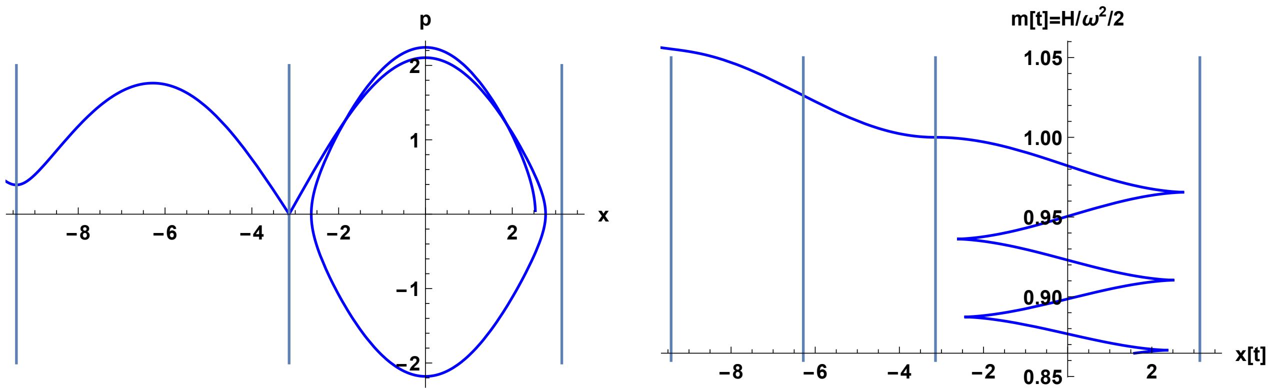

Figure 1.2: Capture. Left: momentum versus position. Right: Jacobi m-parameter versus position.

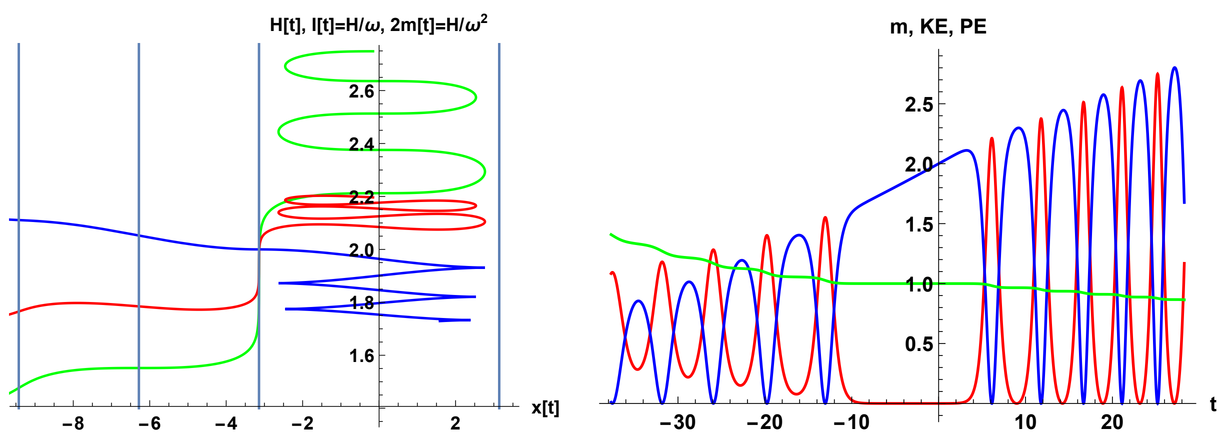

Figure 1.3: Capture at $m_0 = 1$. Left: Hamiltonian (green), action $= H/\omega$ (red), Jacobi m (blue) all versus position. Right: kinetic energy (red), potential energy (blue), m-parameter (green) all versus time.

1.4.1 Work performed on the pendulum

Later discussions will develop the pre- and post-capture processes, including the transition from rotation to libration. The mathematical description is complicated, but the physical process is simple. Suppose the height of the potential well is rising. For rotation, Hamilton's action (J) is not a conserved quantity; essentially because there is an excess of kinetic energy. So it is more straight forward to consider changes of the Hamiltonian. Work W is the product of force multiplied by distance traveled. In the rising potential, this is $W = V(t) - V(0)$. This work is almost totally transferred to potential energy. Eventually, the time-average kinetic and potential energy become equal; and the motion is prepared to make the change from rotation to libration (the maximum swing of the pendulum becomes horizontal).

Once the pendulum is captured into libration, the action becomes an almost conserved quantity. The work done by the rising potential continues to be force multiplied by distance traveled, but the distance shrinks because the amplitude falls; and this results in a square-root dependence of H on $V(t)$. Moreover, the new work is almost equally shared between kinetic and potential energy.

We may elaborate in terms of energy changes. In the rotation regime, the particle K.E. is roughly constant but the time-average P.E. slowly rises toward the K.E. When the peak values of K.E. and P.E. become equal, the particle has reached the separatrix. In the libration regime, the peak values of P.E. and K.E. are equal, but not the time-averaged values. The particle P.E. and K.E. both rise; but the time-average K.E. is initially smaller and rising slightly faster. When the average values become equal, the particle has reached the stable fixed point at the centre of the phase space.

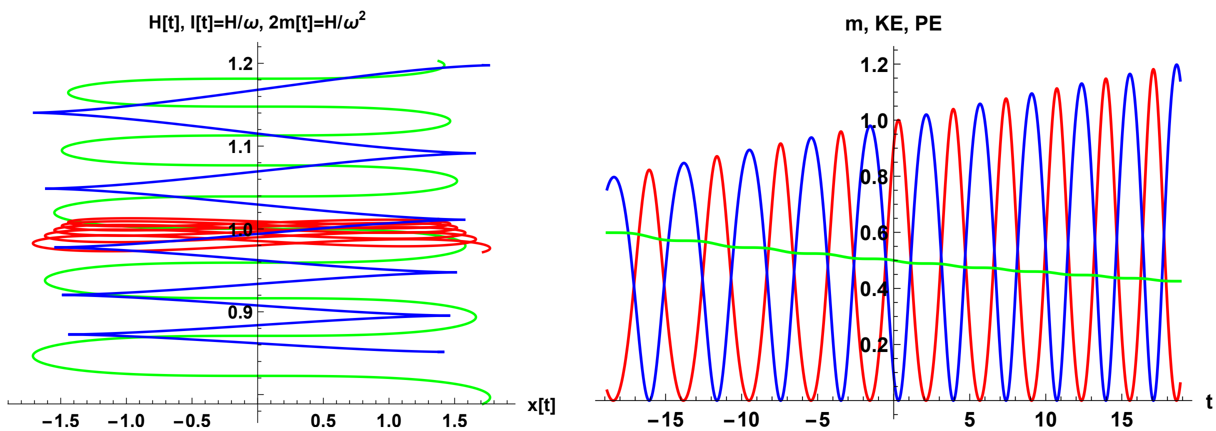

Figure 1.4: Libration at $m_0 = \frac{1}{2}$. Left: Hamiltonian (green), action $= H/\omega$ (red), Jacobi m (blue); all versus position. Right: kinetic energy (red), potential energy (blue), m-parameter (green); all versus time.

1.4.2 Unequal PE and KE for pendulum

We now demonstrate the inequality of time-averaged kinetic and potential energy for the pendulum oscillator. The general form of pendulum Hamiltonian is $H = p(t)^2/2 + \omega^2[1 - \cos x(t)]$. Passing to dimensionless time $z = \omega t$, this becomes $H = p^2/2 + [1 - \cos x(z)]$ and has the value $H = 2m$.

Consider first, the unbounded motion (rotation) for which Jacobi $m > 1$. The peak and time-averaged K.E. are both much greater than their P.E. counterparts when $m \gg 1$. The motion is given by: $x(z) = 2 \arcsin[\mathrm{sn}(\sqrt{m}z, 1/m)]$ and $p(z) = 2\sqrt{m}\,\mathrm{dn}(\sqrt{m}z, 1/m)$. $T(z) = 2m\,\mathrm{dn}^2(z, 1/m)$

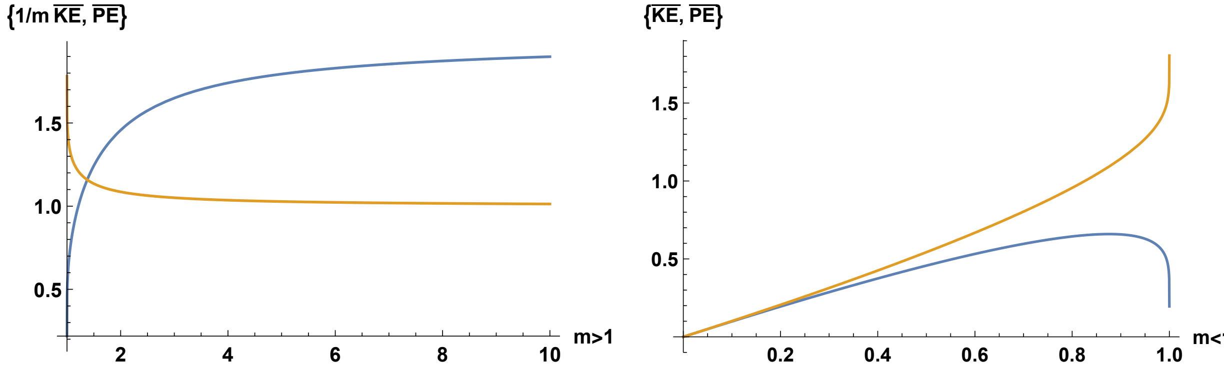

Figure 1.5: Time-averaged kinetic energy (blue) and potential energy (gold) versus Jacobi m-parameter for pendulum oscillator. Left: rotation, $m > 1$. Right: libration, $m < 1$.

and $U(z) = 2\,\mathrm{sn}^2(z, 1/m)$. The rotation period is $\tau = 2K(1/m)/\sqrt{m}$. The time-average values are

$$\overline{T}(m) = \frac{1}{\tau}\int_{-\tau/2}^{+\tau/2} T(z)dz = 2m\frac{E(1/m)}{K(1/m)} \quad \text{and} \quad \overline{U}(m) = \frac{1}{\tau}\int_{-\tau/2}^{+\tau/2} U(z)dz = 2m\left[1 - \frac{E(1/m)}{K(1/m)}\right] .$$

The functions are shown in Fig. 1.5-left. The time-averaged K.E. is $\approx 2m$ times the P.E.. The values become equal at $m = 1.2105$. During (adiabatic) precapture, Jacobi m moves from values much greater than 1 toward unity. The limiting values are $[T/m, U] = [2, 1]$ as $m \to \infty$; and $[T, U] = [0, 2]$ at $m = 1$.

Consider now, the bounded motion (libration) for which Jacobi $m < 1$. The motion is given by $x(z) = 2\arcsin[\sqrt{m}\,\mathrm{sn}(z, m)$ and $p(z) = 2\sqrt{m}\,\mathrm{cn}(z, m)$. $T(z) = 2m\,\mathrm{cn}^2(z, m)$ and $U(z) = 2m\,\mathrm{sn}^2(z, m)$. The libration period is $\tau = 4K(m)$. The time-average values are

$$\overline{T}(m) = \frac{2}{\tau}\int_{-\tau/4}^{+\tau/4} T(z)dz = 2\left[m - 1 + \frac{E(m)}{K(m)}\right] \quad \text{and} \quad \overline{U}(m) = \frac{2}{\tau}\int_{-\tau/4}^{+\tau/4} U(z)dz = 2\left[1 - \frac{E(m)}{K(m)}\right] .$$

The functions are shown in Fig. 1.5-right. During (adiabatic) post-capture, Jacobi m moves $1 \to 0$. The time-averaged P.E. is greater than K.E. for $m < 1$; and does not reach perfect equality until $m = 0$. But *near-equality* is reached rapidly with respect to decreasing m for values $m < 0.9$. The limiting values are $[T, U] = [0, 2]$ at $m = 1$; and $[T, U] = [0, 0]$ at $m = 0$; and are explained as follows. As $m \to 1$, the particle spends progressively more of its time in the vicinity of the peak of the potential well. At $m = 1$, the particle gets stuck at the top of the potential (a fixed point of motion). Contrastingly, $m = 0$ is the limit of infinitesimal oscillation amplitude; so the particle is stuck at the bottom of the well.

1.4.3 Symmetry and the method of averaging

The reader is reminded of Weyl's observation on the role of symmetry at the head of this article. For the pendulum oscillator, the time-averaged kinetic and potential energies are generally unequal. Nevertheless, there is an adiabatic regime because of two properties. (1) There is a separation of time scales: slow variation of the potential versus the (comparatively) fast oscillation period; and (2) an underlying *symmetry*: the motion is periodic. Therefore, whatever work is performed by the slow external parameter variation during one half-cycle is almost exactly canceled during the next half-cycle of motion. This leads to slowly evolving changes for the oscillator amplitude and aspect ratio (and also phase). Further, if the potential well is symmetric about its minimum, then the motion is symmetric within a half-cycle; and the same conclusion of near-cancellation is made for the half-cycle. Such is the case of the unbiased pendulum oscillator.

These observations are collected under the rubrique *method of averaging.* Averaging has a long history rooted in celestial mechanics, associated with Gauss, Fate, Delone-Hill and others. Essentially the idea is to integrate rapidly varying cyclic functions over time, reducing them to their average effect. Since the 1930's the term *method of averaging* has a specific technical meaning: the method introduced by Krylov and Bogoliubov in 1935 for finding the asymptotic behaviour of non-linear differential equations with time-varying coefficients. The method is elaborated in Bogoliubov's book[13, 14]. The present work does not adopt the Bogoliubov formulation. Instead, the averaging integral for adiabatic capture arises in a natural and obvious[5] way in Hamiltonian mechanics; see Sec. 1.8.

The method of averaging tacitly assumes there is no change in the nature of the mathematical function describing the underlying oscillatory motion; its parameters may change, but the form does not. The method fails at the separatrix; because the period of oscillation becomes infinite as the motion tends to the unstable fixed points, the averaging integral becomes undefined and possibly very large as its limits tend to infinity. Despite the mathematical difficulty presented by the capture (i.e. separatrix crossing), it should be noted that the vast majority of the time, particles are either outside or within the instantaneous separatrix; and so the cumulant effects of the pre- and post-capture eras compete with (or even dominate over) the separatrix crossing process that is, comparatively speaking, more like a fleeting moment.

Recently, the neo-adiabatic theory[15, 16, 17, 18, 19, 20, 21] of probabilistic transport[6] has been developed to estimate the change of the adiabatic invariant when separatrix-crossing phenomena occur in slowly modulated one-degree-of-freedom Hamiltonian systems. The review[18] by Neishtadt is a useful introduction to the topic. The theory provides explicit formulas for the trapping probabilities in a resonance region[16], for the change of the adiabatic invariant due to separatrix crossing[17], and for the error estimate defining the regions of validity in phase space. However, the theory is not intuitive; and is limited to linear variation of the external parameter.

We shall use the method of averaging in Hamiltonian form. The present work differs in three other respects. (1) The voltage law is not limited to linear time variation, thereby facilitating the comparison of different capture schemes. (2) Our calculation of the integrals that average over the fast motion are performed in terms of Jacobi elliptic functions, rather than action-angle variables. (3) The method is extended to separatrix crossing. We shall find the perturbation to the motion induced by the parameter variation $\omega(t)$. This has negligible effect on the form of the averaging integral; but utterly transforms the limits of integration, leading to finite increments ΔH.

Finally, it must be noted that longitudinal beam dynamics in a synchrotron is founded on (transverse) averaging. The particles execute both longitudinal (synchrotron) and transverse (betatron) oscillations, but the periods are vastly different: a fraction of a synchrotron oscillation per turn, but many betatron oscillations per turn. So the transverse oscillations may be averaged.

1.5 Near-Adiabatic Capture in Synchrotron

In accelerator parlance, the separatrix is named the rf-bucket; rotation is called "outside the bucket" and libration is referred to as "inside the bucket". The term bucket was coined by the MURA[7] group, in reference to "bucket lift". The bucket is an area in phase space that is confined and transported in energy. The bucket boundary is simply the instantaneous separatrix. The prefix rf- denotes radio-frequency. Usually, the accelerator literature reserves the name *bunching* for the post-capture process wherein the aspect ratio (ratio of width Δx to height Δp) of the phase-space

[5] Indeed, both the present author[31] in his Ph.D Thesis pgs. 65-66 and Robert Gluckstern (University of Maryland, personal communication) conceived of it without knowledge of the K.B. formulation.

[6] The method of averaging is used inside and outside the separatrix, but the treatment of separatrix crossing is probablistic.

[7] The Mid-Western Universities Research Association, 1953-1967. An account of the wonderous MURA activities was given by Frank Cole[22] in 1994.

orbits decreases and a hill-like line density emerges. Nevertheless, some slight modulation of the line density also occurs during pre-capture. Oscillations in the periodic potential provided by the rf-waveform are termed synchrotron oscillations.

In accelerator parlance, the occupied area in phase space is called *emittance*. By custom, "Adiabatic Capture" in a synchrotron charged-particle accelerator, is the process wherein a uniform, coasting, particle beam is captured into a slowly rising-voltage $V(t)$ rf bucket while controlling emittance growth. But perfect adiabaticity demands infinite duration; so, in practise, the process inevitably departs from idealized adiabaticity and is "near adiabatic".

Typically the beam is injected from a linear accelerator, is accumulated in the synchrotron and then accelerated using RF cavity resonators. The cavity voltage is zero (or very small) during the beam injection; and then it is ramped to form a bunched beam with all particles oscillating within an effective potential well. The first mention of this process is that of Saxon at the head of this article. Almost always, the magnetic bending field and the average beam energy is constant during the injection and capture. When capture is complete, the magnetic field rises and the beam is accelerated in synchronism with the bending field. However, in some cases [23, 24, 25, 26] the electrical power circuits energizing the magnetic field law are not compatible with a long or flat bottom.[8] When the time-duration is constrained, part of the capture may take place on a slowly rising magnetic field with a varying synchronous beam energy; but we shall not address that case here.[9] Further, in the case of fast-cycling synchrotrons the magnetic field and voltage both ramp quickly and there is no possibility of adiabatic capture; for example Refs. [30, 31, 32, 33].

Although the magnetic field law can rarely be altered, the voltage ramp is less constrained and several forms have been attempted. The published literature of comparative experimental studies is sparse. Mohite[34], Chap. 6, describes ESME[35] simulations and beam measurements made at the GSI-SIS (heavy-ion synchrotron) wherein linear, quadratic and iso-adiabatic voltage ramps are compared and for different ramp durations and also the effect of varying the initial voltage step. Results are reported both for single and dual-harmonic operation. Ng[36] reports a comparative study (using simulations) of the iso-adiabatic law and the Kang[32] near-linear voltage law: at constant $V_T/V_0 = 75$, the duration of the ramp is extended from the minimum iso-adiabatic value to twelve times that value. Significantly, when the duration is limited the iso-adiabatic law generates clearly smaller emittance; but when the duration is extended to the maximum, the two voltage laws generate indistinguishable final occupied longitudinal phase space. Feng[37] reports a similar study for iso-adiabatic and Kang-cubic ramps. Koscielniak and Zeno[39] report comparative experiments and computations for quadratic and iso-adiabatic voltage laws at the BNL-AGS Booster, including variation of the voltage step. Recently, Kelliher[40] reports a comparison of linear and iso-adiabatic ramps, and concludes that the phase space area enclosing the final particle ensemble is smaller for the linear ramp. Kelliher is interested in particle losses, and they come from the periphery of the occupied phase space. It is probable that if root-mean-square emittances were compared, those developed by the iso-adiabatic ramp would be found the smaller of the two.

Ideally, a theory of adiabatic capture would predict the final distribution of particles in phase space, and provide means to optimize the voltage law for given constraints. Until now, this has only been achieved through particle tracking simulations (i.e. computer experiments); for example Refs. [23, 24, 37]. In particular, Rasmussen[41, 42, 43] reports sophisticated simulations in which

[8] Sometimes called an injection porch.

[9] Except by way of a footnote. In the case of injection near the bottom of a slow sinusoidal magnet ramp, an approximation to a flat injection porch may be made by deliberative radial steering prior to field minimum. The radio-frequency is kept constant (not synchronised to the magnetic field law) and the beam moves radially so that its momentum is constant. In this way the duration of adiabatic capture can be lengthened, but not substantially. Later, the radial-loop of the LLRF control system will bring the beam back to the central orbit. The efficacy of the scheme depends on the radial aperture, which may be compromised by an injection orbit bump. Tricks of this type have been used at the RAL-ISIS and CERN PS-Booster (private communication Ian Gardner & Grahame Rees, Rutherford Appleton Laboratory circa 1986; and Steven Hancock CERN-PS circa 1996). The most successful implementation appears to be that of Bhat[27, 28, 29] at Fermilab.

particles emanate from the final separatrix and are tracked backward in time to find the regions of phase space that will be captured when time runs forward. Adiabatic capture was first investigated at MURA[44]. Based on very limited[10] computer simulations, and impressive insight and inspiration, they abandoned linear voltage ramp in favour of their *iso-adiabatic* voltage law; which has been championed and employed by various adherents[45, 46, 47]. There is little reported of other voltage laws in the literature, though anecdotal evidence that quadratic, exponential, etc, variations have been tried with some success. All sources agree the linear voltage ramp has a poor outcome. The RF bucket area is proportional to $\sqrt{V}$. The quadratic law results in "acceptance" rising linear in time; the iso-adiabatic law results in "acceptance" rising linearly each synchrotron period.

Our overall plan is to find the mapping between the values of the Hamiltonian $H(0)$ at the start, and at the terminus of capture $H(T)$. Associated with each $H(0)$ there is an initial oscillation phase q. We shall find that for each value $H(0)$ there is an average value $\langle H(T)\rangle$, and a spread generated by three processes. (1) The sudden voltage turn on at $t \leq 0$ generates a spread ΔH_0. (2) Slow cumulative growth before and after capture arising from the rapidity of the rising voltage $V(t > 0)$ generates an additional spread $\Delta H(q)$. (3) One-time increments that occur at the moment of capture. Process (1) results in instant capture for a small faction of the beam, and has a separate mathematical treatment. Process (2) is near-adiabatic, while (1) and (3) are non-adiabatic.

Let x and p be RF-phase and momentum deviation, respectively. (x is the analogue of the deviation angle θ for a true pendulum.) At the start of the process, $t = 0$, the beam is a rectangle in (x, p) matched to $V = 0$; but this corresponds to $\omega = 0$, and therefore adiabaticity cannot be perfect. And so $V(t)$ must start with a small step, at the cost of a small (but important) mismatch. Outside the bucket, particles stream in RF phase; and inside they circulate in (x, p). The transition of a pendulum from rotation to libration is often called "separatrix crossing". The term correctly identifies the dynamics occurring at that time as unusual, but is potentially misleading: there is no impenetrable separatrix in the time-dependent system; no separatrix is crossed.

Nevertheless the switch from rotation to libration (both of which are periodic oscillations) is not adiabatic because particles move glacially slowly in the immediate vicinity of the astable fixed points. Moreover, any mathematical formulation that takes an infinite-period motion as its basis will break down at the separatrix. This is the reason for the failure (at the separatrix) of the "method of averaging", and the motivation for the "neo-adiabatic" theory. However, we repeat, "separatrix crossing" by a single particle receives a disproportionate amount of attention; it is but a fleeting moment in the capture of a particle beam.

1.6 Voltage ramps

There are several commonly used voltage ramps; we list them along with their corresponding adiabaticity parameters. In Secs. 1.10, 1.11 it is shown that the increment of Hamiltonian across the separatrix is proportional to ϵ_1 at the time of crossing.

1.6.1 MURA ramp

Lilliequist and Symon[44] (abbreviated to L&S) were the first[11] to adddress adiabaticity in longitudinal capture. Although seminal, this work has two infelicities. Rather than start from the consequences of $|\epsilon_1| \ll 1$, they write an equation for the instantaneous acceptance (i.e. area S) of the RF-bucket for a rising voltage. The area S (eV.sec) is proportional to $\sqrt{V(t)}$. $\tau = 2\pi/\sqrt{AV(t)}$ is the synchrotron oscillation period. The principle of equal fractional increments in equal fractions

[10] A consequence of the inadequate computing resources of the time.

[11] First in the Western world, but there may be similar work in Russian journals.

of a period, leads to:

$$\frac{1}{S}\frac{dS}{dt} = \frac{\alpha}{\tau} \ . \tag{1.18}$$

$\alpha = 2\pi\epsilon_1$ is the L&S dimensionless adiabaticity parameter, which differs from the conventions above. Integration yields a prescription for the voltage law

$$V(t) = \frac{(2\pi)^2}{A(\alpha\tau + \pi c_1)^2} \ . \tag{1.19}$$

τ is set to be the value either at start or finish. The presence of the slippage factor A implies that "adiabaticity is built in". The single adjustable coefficient c_1 can satisfy only one boundary condition. Hence the parameters α, duration T, and voltages $V(0) = V_0$ and $V(T) = V_T$ cannot be chosen independently. L&S choose α and V_T and T, and let V_0 be derived. If α is chosen small, and duration T is severly constrained by the magnet excitation, the initial voltage can become quite large. But equally, one could set T and V_0, V_T, and find the resulting parameter α:

$$\alpha T = \frac{2\pi}{A}\left[\frac{1}{\sqrt{V_0}} - \frac{1}{\sqrt{V_T}}\right] \ . \tag{1.20}$$

L&S report computer simulations that confirm capture is adiabatic provided that $\alpha < 1$ [i.e. $\epsilon_1 < 1/(2\pi)$]. This formulation is equivalent to $\epsilon_1(t)$ is equal to a constant, in which case $\epsilon_2 = 0$.

1.6.2 Linear ramp

We set $V(t) = V_0 + \Delta V\tau$ with $\Delta V = (V_T - V_0)$ and the fractional time $\tau = t/T$, and evaluate the adiabaticity parameters.

$$\epsilon_1 = \frac{\Delta V}{2T\sqrt{A}[V_0 + \tau\Delta V]^{3/2}} \quad \text{and} \quad \epsilon_2 = -\frac{3\Delta V^2}{4T^2}\frac{1}{\sqrt{A}[V_0 + \tau\Delta V]^{5/2}} \ . \tag{1.21}$$

These parameters vary dramatically. The ratios from start to finish $\epsilon_1(0)/\epsilon_1(T) = (V_T/V_0)^{3/2} \gg 1$ and $\epsilon_2(0)/\epsilon_2(T) = (V_T/V_0)^{5/2} \gg 1$, imply violent changes in the areas swept out by the particles at the start of capture; therefore considerable initial emittance growth is expected for the beam core. However, it must be noted that ϵ_1 diminishes; and the final value is $\epsilon_1(T) = \Delta V/(2T\sqrt{A}V_T^{3/2}) \approx 1/(2T\sqrt{AV_T})$ is significantly smaller than for the iso-adiabatic law where $\epsilon_1 \approx 1/(T\sqrt{AV_0})$. This implies that the last particles to be captured by the linear ramp will suffer a smaller emiitance increase than for the iso-adiabatic ramp. The last particles captured form the boundary of the occupied phase space, and this will be smaller for the linear ramp.

1.6.3 Quadratic ramp

We set $V(t) = V_0 + \Delta V \times \tau^2$, and evaluate the adiabaticity parameters:

$$\epsilon_1 = \frac{\tau\Delta V}{T\sqrt{A}[V_0 + \tau^2\Delta V]^{3/2}} \quad \text{and} \quad \epsilon_2 = \frac{(V_0 - 2\tau^2\Delta V)\Delta V}{T^2\sqrt{A}[V_0 + \tau^2\Delta V]^{5/2}} \ . \tag{1.22}$$

Notably, $\epsilon_1(0) = 0$ and $\epsilon_1(T) = \Delta V/(TV_T\sqrt{AV_T})$; and $\epsilon_2(0) = \Delta V/(T^2V_0\sqrt{AV_0})$. So there is no emittance growth at the start of capture. However ϵ_1 quickly rises to a value of $2/(3\sqrt{3})\sqrt{\Delta V/A}/(TV_0)$ at $t = (T/\sqrt{2})\sqrt{V_0/\Delta V}$ before decaying toward $\epsilon_1 \approx 1/(T\sqrt{AV_T})$. Some some emittance growth (greater than for the iso-adiabatic ramp) is anticipated.

The linear voltage ramp begins extremely non-adiabatic; but becomes extremely adiabatic towards the end, after all the damage has been done. It has also the disadvantage of being much slower than necessary toward the end. The quadratic voltage ramp has the appealing feature that only first and second derivatives are non-zero; so the chain of evolution equations (see Sec. 1.8) terminates at $k = 2$.

1.6.4 Kang's cubic ramp

Kang[32] introduced a near-linear ramp that is continuous in value and 1st and 2nd derivatives. Kang never intended this voltage law for adiabatic capture at constant magnetic field; but he labelled the ramp as *adiabatic*, and it propagated as such. We set $V(t) = V_0 + \tau^2[3 - 2\tau]\Delta V$ and $\tau = t/T$ is the fractional time, and evaluate the adiabaticity parameters:

$$\epsilon_1 = \frac{3\Delta V(1-\tau)\tau}{T\sqrt{A}[V_0 + \Delta V(3-2\tau)\tau^2]^{3/2}} \quad \text{and} \quad \epsilon_2 = -\frac{3\Delta V[V_0(2\tau-1) + \Delta V\tau^2(6 - 10\tau + 5\tau^2)]}{T^2\sqrt{A}[V_0 + \Delta V(3-2\tau)\tau^2]^{5/2}} \ .$$

These parameters vary wildly. The values of ϵ at start and finish are $\epsilon_1(0) = \epsilon_1(T) = 0$; and $\epsilon_2(0) = 3\Delta V/[T^2\sqrt{A}V_0^{3/2}]$ and $\epsilon_2(T) = -3\Delta V/[T^2\sqrt{A}V_T^{3/2}]$. This looks promising, although $\epsilon_2(0)$ is large if $V_T \gg V_0$. However, ϵ_1 rises quickly to a peak and then slowly decays as time progresses. This behaviour is similar to the pure linear ramp, but the outcome may be worse because time is wasted at the start and end of the ramp. In detail, $\epsilon_1(\tau)$ has extrema at the zeros of ϵ_2 which is a quartic in τ. There are four solutions: two real and two complex. To simplify the appearance of the roots, we write $(1+a) \equiv V_T/V_0 = C$ which defines $a = C - 1$. We are interested in the positive real root:

$$\tau = -\frac{\sqrt{5}}{10}\left[-\sqrt{5} + \sqrt{1+Z} - \sqrt{2 - Z + \frac{2\sqrt{5}(2+a)}{a\sqrt{1+Z}}}\right] \quad \text{where} \tag{1.23}$$

$$Z \equiv \frac{2^{1/3}Y}{(aY)^{2/3}} + \frac{2\times 2^{2/3}a}{(aY)^{1/3}} \quad \text{and} \quad Y \equiv 5 + 5a + 4a^2 + \sqrt{5}\sqrt{5 + 10a + 13a^2 + 8a^3} \ .$$

The root is substituted in ϵ_1 to yield a complicated expression for the peak value; fortunately, this is well approximated by the simpler form:

$$[T\sqrt{AV_0}]\,\hat{\epsilon}_1 \approx 0.6709\,a^{0.4991} - 0.19806 \quad \text{for } 1 \le a \le 200 \ . \tag{1.24}$$

The value on the right is of order $\sqrt{V_T/V_0} \gg 1$ and is comparatively large. Contrastingly, the peak and average value of ϵ_1 for the iso-adiabatic ramp is $[T\sqrt{AV_0}]\epsilon_1 = 1 - \sqrt{V_0/V_T} \le 1$. Evidently, for the Kang ramp, there will be significant increase in Hamiltonian (and emittance) for particles that are captured at early times; and, of course, such particles constitute the core of the particle beam. Similar remarks apply to the pure linear ramp.

1.6.5 Iso-Adiabatic ramp

This is, essentially, the ramp of L&S; but formulated to be consistent with the other ramps. $\omega = \sqrt{AV}$ and $V(t)$ is the solution of $\epsilon_2 = 0$.

$$V(t) = \frac{c_2}{(t + 2c_1)^2} = \frac{T^2 V_0 V_T}{[\sqrt{V_T}T + (\sqrt{V_0} - \sqrt{V_T})t]^2} \ . \tag{1.25}$$

This differs from the L&S expression by the absence of slippage-factor A. This has the consequence that voltages V_0, V_T and duration T can be freely chosen, but without a "built-in" constraint of adiabaticity. The degree of adiabaticity is recovered when we form the parameter ϵ_1. Of course, $\epsilon_2 \equiv 0$.

$$\epsilon_1 = \frac{\alpha}{2\pi} = \frac{\sqrt{V_T} - \sqrt{V_0}}{T\sqrt{AV_0V_T}} \ . \tag{1.26}$$

1.6.6 Exponential ramp

We set $V(t) = V_0 C^\tau$ where $C = V_T/V_0$ and the fractional time $\tau = t/T$; and evaluate the adiabaticity parameters:

$$\epsilon_1 = \frac{\ln(C)}{2T\sqrt{AV_0C^\tau}} \quad \text{and} \quad \epsilon_2 = -\epsilon_1 \frac{\ln(C)}{2T} \,. \tag{1.27}$$

These parameters vary considerably. $\epsilon_1(t)$ diminishes provided that $V_T > V_0$. At the end points, $\epsilon_1(0) = \ln C/[2T\sqrt{AV_0}]$ and $\epsilon_1(T) = \epsilon_1(0)/\sqrt{C}$.

1.6.7 Adiabatic family of ramps

There is a family of voltage laws $V(t)$ that are the solutions of

$$V(t)\ddot{V} = [1 + 1/N](\dot{V})^2 \,.$$

$N > 0$ is the power law index. The case $N = 2$ gives the so-called iso-adiabatic[44] law. Let the process start and stop at $t = 0$ and $t = T$, respectively. The boundary condition $V(T) = C \times V(0)$ leads to the solution:

$$V(t) = \left[\frac{T}{t + B(T - t)}\right]^N V(T) \quad \text{where } B \equiv C^{\frac{1}{N}} \,. \tag{1.28}$$

The cases $C > 1$ gives capture and bunching; $C < 1$ gives debunching and release; and $C = 1$ gives a constant voltage $V(t) = V(T)$. In the limit of index $N \to \infty$, the adiabatic law becomes an exponential: $V(t) = C^{(t/T)}V(0)$. The adiabaticity parameters are:

$$\epsilon_1 = \frac{(B-1)N}{2\sqrt{AV_T}} \sqrt{\frac{[t + B(T-t)]^{N-2}}{T^N}} \quad \text{and} \quad \epsilon_2 = -\epsilon_1 \times \frac{(B-1)(N-2)}{2[t + B(T-t)]} \,. \tag{1.29}$$

When $N = 2$, the parameter ϵ losses its time dependence and becomes iso-adiabatic. The values of ϵ at start and finish are of interest:

$$\epsilon_1(0) = \frac{(B-1)N\sqrt{B^N}}{2TB\sqrt{AV_T}} \quad \text{and} \quad \epsilon_1(T) = \epsilon_1(0)\frac{B}{\sqrt{B^N}} \,.$$

If $C > 1$ and $N > 2$, then $\epsilon_1(0) > \epsilon_1(T)$ and the process becomes more adiabatic as time progresses. As a general trend, as N increases the voltage ramps are less adiabatic at early times and more adiabatic at late times. Note also that at $C > 1$ is increased, and $V(0)$ falls, the moment of capture is progressively skewed toward later times. Hence it is possible for voltage ramps with $N > 2$ to provide better outcomes (i.e. less emittance growth) than the iso-adiabatic ramp; and particularly so if the duration T must be decreased by external constraints.

MURA ramp versus adiabatic family

If V_0 and V_T are freely chosen, then the MURA voltage law implies a minimum ramp duration T. If the available time is shorter than T, then the initial voltage must be raised to respect adiabaticity; which results in an emittance increase due to the sudden voltage turn on. The adiabatic family of ramps provides, in principle, the freedom to choose the ratio V_T/V_0 for given duration T by adjusting the power law index N. This opens the opportunity to trade-off the sources of emittance growth (sudden versus gradual) to achieve a lower over all occupied phase space area by the time the procedure is completed. The adjustable index also offers, in principle, to manipulate the core versus the periphery of the ensemble phase space.

1.6.8 Particle tracking

The default method of investigating capture and evaluating the merits of different voltage laws is tracking an ensemble of particles with a variety of initial momenta and positions. In the summer of 2022, a systematic study[38] was made of the adiabatic family of voltage ramps and the exponential ramp. The slip constant was set to $A = 1$. The power law index N, the ratio $C = V_T/V_0$ and ramp duration T were sampled over the ranges $N = [1, 50]$ and $C = [50, 1000]$ and $T = [125, 2000]$. Including 20 different exponential ramps, the total number of cases was in excess of 150. The number of particles was typically a thousand, leading to poor statistics. The particles were initialy uniformly distributed in x and p. The final data (x, p) were processed with a view to finding, empirically, a relation between the initial H_0 and final H_T values of Hamiltonian and their respective distributions $\rho(H)$. The simulations pointed to several features of a completed capture: (1) that the final density $\rho(\tilde{H}_T) \approx \sqrt{H_0}$ with weak dependence on parameters; (2) that there is lift $L_T \approx \sqrt{V_0 V_T}$ and initial spread due to the sudden voltage turn on; and (3) that spreads $\Delta H(H_0)$, about the mean $\tilde{H}_T$, evolve and are somewhat sensitive to the parameters N, C, T.

1.7 Hamiltonian mechanics

Let $q, \dot{q}$ be position and velocity. We assume the motion is conservative, and that forces can be derived from a potential. The Lagrangian is $T - U$, the difference of kinetic and potential energy. The conjugate (or canonical) momentum is $p(q, \dot{q}) \equiv \partial L/\partial \dot{q}$. Hamilton's description of dynamics uses the energy function $H = T + V$ written in terms of p, q. The conjugate coordinates (q, p) evolve according to:

$$p = dq/dt = \partial H/\partial p\,, \qquad dp/dt = -\partial H/\partial q\,, \qquad dH/dt = \partial H/\partial t\,. \tag{1.30}$$

An example of conjugate coordinates are energy and time. *Phase space* is the manifold of all possible configurations of the system. Nolte[48] gives a fascinating account of the history of this terminology.[12] For 2N-dimensional motion, it is the space with coordinates $(\mathbf{q},\mathbf{p})$ where each vector has N elements. For one-dimensional motion, phase space is simply the graphical plot of all flow lines $x(t), p(t)$ for all relevant time. In this "picture" we gain global insight into an entire trajectory or bundle, but discard "time". The loss of time has the demerit that we may lose track of how quickly or slowly the particle moves through different portions of the phase space.

Hamiltonian dynamics has one fundamental property: the convective derivative (or total derivative) of the particle density in phase space is constant; this is Liouville's theorem. A consequence of Liouville's theorem is that the area occupied in phase space is constant. If Hamiltonian $H(q, p)$ is time-independent, there are additional properties: (1) flow lines never cross; and (2) the area swept out in phase space by cycles of periodic motion is constant. The latter is encapsulated in the statement that the area J enclosed by one cycle of periodic motion $J = \oint p\dot{q}dt = \oint pdq = 2\pi H/\omega$ is an adiabatic invariant. If H and ω are time dependent and slowly changing, nevertheless $dJ/dt \approx 0$ provided the motion is cyclic in phase space. J is called the "action of a generalized coordinate". If H has the dimension of energy, then the action is an area having units of energy $\times$ time. Strictly linear proportionality between J and H holds only for the harmonic oscillator. For the pendulum, the proportionality is nonlinear; but is still monotonic increasing in H.

If particles coordinates fill some bounding trajectory (q, p), then the area occupied in phase space is given the integral $\oint pdq = \oint qdp$. This has the same value as the action J, and is proportional to the Hamiltonian. Hence changes in the Hamiltonian may be used as a proxy for changes in the phase space area or emittance, even though J and H have different units.

[12] The term was intoduced in the late 19th century by Boltzman and Jacobi, but is often attributed (incorrectly) to Joseph Liouville because B&J profiled his work. However, the phase-space density conservation theorem is Liouville's, albeit in a different context. A counterpart to this tale is that Liouville championed the work of Evariste Galois, the founder of Group Theory.

If the Hamiltonian is constant, flow lines cannot cross; and one flowline cannot cross itself. If H is time dependent, flow lines may cross; but only in a restrictive sense. Consider two trajectories which are initially separate. One trajectory (at a later time) may pass over a phase-space point occupied by the other trajectory at an earlier time, or it may cross its own historical path. This type of behaviour is the mainstay of near-adiabatic capture.

During the rotation era, there is little or no crossing of trajectories until capture is approached. During the capture, and immediately prior, trajectories may cross one another. During the libration era, trajectories cross themselves, but not one another; so "what is toward the inside, stays inside".

1.7.1 Synchrotron coordinates

Let ϕ be the phase of the rf-wave responsible for acceleration, T_{rev} be the revolution period around the synchrotron circumference, and δE be energy deviation from the reference value E_s. Let η be the slip factor such that $\delta T/T_{\text{rev}} = \eta\delta p/p$. Let the harmonic number h be the ratio of the radio-frequency to the revolution frequency ($= 2\pi/T_{\text{rev}}$). e is the particle electric charge, and $\beta = v/c$. Let n be the number of turns made around the circumference of the synchrotron. The longitudinal motion is approximated by the equations:

$$\frac{d}{dn}\delta E = e.V\sin\phi \quad \text{and} \quad \frac{d}{dn}\phi = \frac{2\pi h\eta}{E_s\beta_s^2}\delta E \equiv A\delta E \tag{1.31}$$

ϕ is dimensionless, and A has units of 1/energy [or 1/(e.Volt)]. The square of number of oscillations per turn is $a_s^2 = A.eV$. The associated Hamiltonian is $H = A\delta E^2/2 + eV(1-\cos\phi)$, and has units of energy; while A has units of inverse e.Volt. The phase space area has units of rf-phase×energy. Strictly speaking, rf-phase and energy deviation are not a conjugate pair. The true conjugate coordinates are arrival time and energy deviation. This distinction becomes important when the revolution (or orbital) frequency changes in a cyclic accelerator, such as a synchrotron.

The standard form of the penulum Hamiltonian is $H = \frac{1}{2}Ap^2 + V\times(1-\cos x)$. x is dimensionless, and $p = \dot{x}$ has units of Hz. The oscillation frequency squared is $\omega^2 = AV$. The slip-rate constant A has the units of H/p^2 or Volt/Hz^2. Comparing the two forms of H, we make the identifications $\phi \to x$ and $\delta E \to p$ and $d/dn \to d/dt$.

1.7.2 Time-dependent separatrix

When the voltage V is constant, the separatrix is frozen and has an obvious meaning: the two curves $p = \pm\omega_0\sqrt{2(1+\cos x)}$ in phase space (x, p). But there is a subtlety, by discarding time we lost sight of the fact that the separatrix is an "infinite time object"; to construct the complete separatrix from a trajectory starting at $x = 0$ would take infinite time because of the glacially slow motion in the neighbourhood of the unstable fixed points.

If we think of the separatrix as the trajectory of a certain particle traced until it has just been captured, then we are faced with tracking it backward in time. This exercise is performed in Fig. 1.6 for three select points that fall on the curves $p = \pm\omega_0\sqrt{2(1+\cos x)}$ at zero time. We could make a plot of this kind at any time, and it would look similar. Consider the left figure, which follows trajectories receding into the distant past (very large values of $|x|$). Particles between the (red) streamlines at upper left (UL) and lower right (LR) were captured earlier; and previously gave rise to a similar plot. Particles above and below, respectively, the (green) streamlines at UL and LR will be captured in the future; and will produce a similar plot later. The long series of modulations prior to capture are not useful for discussing or visualizing separatrix crossing. Instead, we need a local-time description that traces out the separatrix trajectory for one or two synchrotron oscillation periods; and attempts to show what range of particle trajectories were captured during that time, as shown in the righthand figure. This type of picture was introduced by Sayasov and Melnikov[49, 50]. The sketch on the right of Fig. 1.6 is a detailed view of a portion of the figure to its left for the short timescale around the moment of capture. When the voltage varies, a tempting picture is that separatrix expands and trajectories in its vicinity slide through. At any moment, capture of a small range of momenta is inevitable; and as time progresses higher ranges are captured. We shall elaborate a modified version of this picture that is the ensemble (or family) of trajectories that are captured during a local time window of one-half synchrotron oscillation period. The picture has to be constantly refreshed each synchrotron period. However, apart from a shift of time and a scale change of the momenta, the pictures are essentially identical; so a generic stands in place of them all. Sayasov assumed that the upper bounding trajectory (which captures all positive momenta below it) terminates infinitesimally close to the right hand astable fixed point; but that is incorrect. The final coordinates (at zero time) in Fig. 1.6 are given by $x = 2\arcsin[\tanh\phi]$ and $p = \pm 2\omega_0 \,\mathrm{sech}\phi$. The local-time picture suggests that the bounding trajectories terminate at $|\phi| \gg 1$ (or $x \to \pm\pi$) like the red and blue curves. But the distant-past picture shows (correctly) that the momenta are bounded in the far past by the green curves that terminate at $\phi \approx \pm\pi/3$ and $\mathrm{sign}(p) = \mathrm{sign}(\phi)$. But we caution the reader: this picture is only good for short time durations. When $V(t)$ varies, there are no trajectories that terminate on the hyperbolic fixed points ($x = \pm\pi$) until after the variation stops and V becomes a constant.

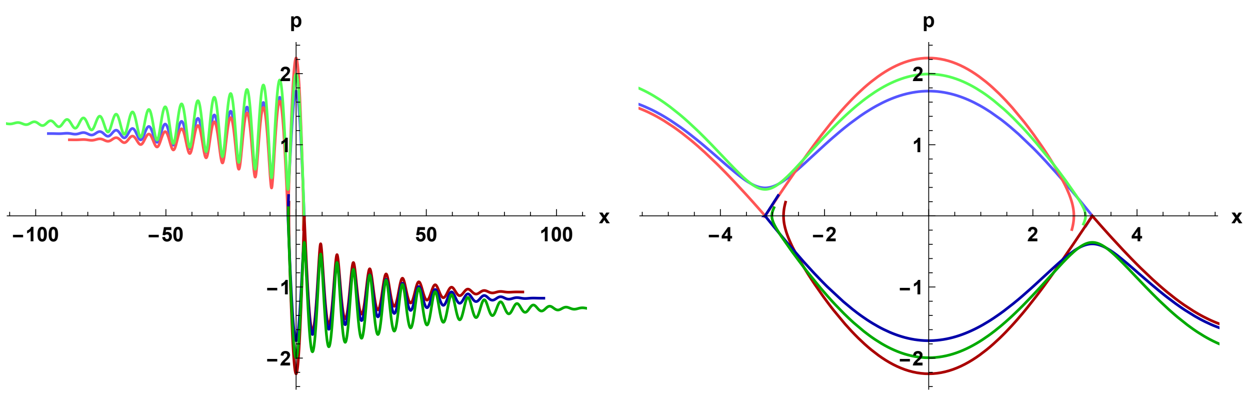

Figure 1.6: Left: distant-past separatrix. Right: local-time separatrix. Any particles above/below the light/dark green curve are captured at a later time. Any particles below/above the light/dark red curve were captured previously. For $p > 0$ the initial phases are $\phi = -4\pi$ & $+4\pi$ light red & blue; $\phi = \pi/3$ light green. For $p < 0$ the initial phases are $\phi = -4\pi$ & $+4\pi$ dark blue & red; $\phi = -\pi/3$ dark green.

1.7.3 Liouvillian processes

The equations of motion do not know anything about the presence or absence of particles. Therefore they apply equally well to empty phase space as they do to occupied phase space. This is the insight behind the technique, invented by the MURA group, of acceleration by sweeping empty RF-buckets from above (in energy) to below. The conservation of density implies the particle beam is displaced upwards (in energy). It is also the insight that led Simon van der Meer[51, 52] to invent stochastic cooling. He realized that Liouville's theorem can be exploited to manipulate phase space: that clumps of occupied phase space can be aggregated, leaving the voids behind – if one can build an active system[13] sufficiently sensitive to small time scales to detect the clumps and actuators sufficiently fast to respond to them.

A picture of empty and occupied phase space is to think of two immiscible, incompressible liquids of equal density and zero viscosity, but of two different colours such as black (filled) and white (empty). The liquids, like oil and water, must have no molecular affinity. The local density of

[13] Van der Meer was fortunate to have an outstanding technical team[53] to realize his brilliant idea.

each liquid is constant as it moves. Although they can be macroscopically mixed, like two colours of oil paint stirred in a paint can, they cannot be microscopically mixed. The mixing, if continued, leads to long thin filaments of paint that are interlaced. Likewise, during non-adiabatic processes, empty and occupied phase space form interlaced loops, whorls and arches like fingerprint patterns. This is the mechanism behind "emittance growth".

1.7.4 Required final voltage

Here we derive the final voltage $V(t = T)$ such that the rf-bucket contains exactly (and no more) the phase-space area of the unbunched beam. Due to Liouville's conservation theorem, the derivation is trivial; as many authors have noted since the time of MURA-106[54]. The Hamiltonians of the initial un-bunched and final bunched beams are $H = \frac{1}{2}Ap^2$ and $H = \frac{1}{2}Ap^2 + V \times (1 - \cos x)$, respectively. Let $\pm\hat{p}$ be the largest momenta in the unbunched beam, and the corresponding Hamiltonian value $\hat{H}_0 = \frac{1}{2}A\hat{p}^2$.

The initial phase space area corresponding to one rf-period is $4\pi\hat{p}$. Particles in the bunched beam execute synchrotron oscillations. If the maximum oscillation amplitude is $x = X$, the occupied area is

$$\text{Area} = 4\sqrt{2V/A}\int_0^X \sqrt{\cos x - \cos X}\,dx = 8\sqrt{2V/A}\sqrt{1-\cos X}\,E[X/2, 1/m] \tag{1.32}$$

where $m = (\sin X/2)^2$. If the rf-bucket is filled, then $X = \pi$ and the area is $16\sqrt{VA}$. Equating the initial and final areas yields the final voltage:

$$V_T = (\pi/4)^2 A\hat{p}^2 = \hat{H}_0\pi^2/8\,. \tag{1.33}$$

At the start ($t = 0$) the range of Jacobi-m is very large, extending up to $\hat{H}_0/(2V_0) \gg 1$. At the end ($t = T$) with ideally all particles inside the bucket, the range of Jacobi parameter is $m = [0, 1]$. In practise, V_T is taken slighly larger than the theoretical value to allow for emittance growth.

1.7.5 Sudden voltage turn-on

Prior to voltage turn-on, there is a uniform coasting beam; particles have Hamiltonian values $H(0) = \frac{1}{2}Ap^2$. We suppose the turn-on is so fast that particles do not have time to move. So the particle distribution is still uniform immediately after turn-on. However, the sudden voltage turn-on lifts all values of the Hamiltonian at $t = 0$, and adds a spread $\Delta H(0, x)$ to all values. We write the value before (H_0^-) and the value after (H_0^+) the voltage turn on:

$$H_0^+(x) = H_0^- + V_0(1 + \cos x)\,. \tag{1.34}$$

The ensemble average is $$H_0^+ \equiv \langle H_0^+(x)\rangle = H_0^- + V_0\int_{-\pi}^{+\pi}(1 - \cos x)dx = H_0^- + V_0\,. \tag{1.35}$$

The average lift increment is V_0. If the range of $H_0^- = [0, \hat{H}_0]$, then the range of $H_0^+ = [V_0, \hat{H}_0 + V_0]$. We may write the post-turn-on Hamiltonian $H_0^+(x) = H_0^+ + \Delta H_0(x)$ where $\Delta H_0(x) = -V_0 \cos x$ is the spread (or deviation about the mean) that is common to all H_0^+ values. We integrate ΔH^2 over x to find the variance. The common r.m.s spread immediately after turn on is $V_0/\sqrt{2}$. $H(t, x)$ can also be written in terms of an oscillation phase $H(t, x(q))$; this is discussed in Sec. 1.8.5.

Usually the spread is small because V_0 is small. Nevertheless, at early times, ΔH_0 is the dominant spread because other sources have not yet accumulated appreciable amounts. Further, if the duration of the voltage ramp duration is constrained to be short and the initial voltage is raised to give a small adiabaticity parameter then the effect[14] can be noticeable at the end of the ramp. Such is the case at PIMMS[25, 26, 39].

[14] There is a slight irony given that the step is introduced to guarantee adiabaticity at the start of the ramp.

The initial lift and spread will evolve over the course of the entire voltage ramp, as described below. The details depends on whether the value H_0 is inside or outside the initial RF bucket at $t = 0$.

1.8 Method of Averaging

Let the potential function be a product: $V(x,t) = U(x)V(t)$ where dimensionless $U(x)$ is the potential well centred at $x = 0$. The RF phase-slippage parameter A is a constant, and has the units of Hz^2/volt. The Hamiltonian is

$$H(x,p,t) = A\,p(t)^2/2 + U[x(t)]V(t)\ . \tag{1.36}$$

The small amplitude synchrotron frequency is $\omega = \sqrt{AV(t)}$. The Hamilton equations $\dot{x} = \partial H/\partial p$ and $\dot{p} = -\partial H/\partial x$, lead immediately to the time derivative:

$$dH/dt = U[x(t)]dV/dt\ . \tag{1.37}$$

Here $x(t)$ is itself affected by $\dot{H}$. We assume this effect an be written as the sum of the unperturbed motion x and the deviation α induced by $\dot{H}$. Thus $x(t) \to x(t) + \alpha(t)$.

The motion of the particles is fast compared with the rise of the potential well. Therefore, we may time average Eq. 1.37 over a sliding window. Because the potential is symmetric, one period of rotation, or a half-period of libration is all that is needed. Need only up-and-down (or visa versa) the well. Do not need to-and-fro (one period of libration). Rotation is periodic, but not cyclic; whereas libration returns to the same point after one cycle. Thus up-and-down the well is a full period (τ_{rot}) of rotation, but a half period ($\tau_{\mathrm{lib}}/2$) for libration. Near the separatrix $\tau_{\mathrm{rot}} \approx \tau_{\mathrm{lib}}/2$.

Let τ be the oscillation period for libration. If $V(t)$ varies slowly enough compared with the oscillation period, we may replace the instantaneous value $\dot{H}$ with its average effect over one $\frac{1}{2}$-period. This effect depends on the initial oscillation phase q of the particle, so we shall write $H = H(t,q)$. The average, denoted by a bar, is made by the moving window integration:

$$\overline{\dot{H}(t,q)} \equiv \frac{1}{\tau/2}\int_{-\tau/4}^{+\tau/4} \dot{H}(t+s,q)ds = \frac{1}{\tau/2}\int_{-\tau/4}^{+\tau/4} U[x(t+s)+\alpha(t+s)]\dot{V}(t+s)ds\,. \tag{1.38}$$

Essentially this is the nett work done on the particle by the changing potential as it moves (once) up and down in that potential. [The corresponding integral for rotation has limits $\pm\tau_{\mathrm{rot}}/2$.] The implementation of averaging is as follows. (1) To drop the term in α as negligible.[15] (2) To expand $\dot{V}(t)$ as a Taylor series, and retain only low order terms because $V(t)$ changes little during an oscillation period. (3) To assume there is no change in the integration limits[16], because τ is not affected by α. Thus, the $\frac{1}{2}$-period average becomes:

$$\overline{\dot{H}(t,q)} = F_0(q)\dot{V}(t)\ + F_1(q)\ddot{V}(t)\ + \tfrac{1}{2}F_2(q)\dddot{V}(t) + \ldots \tag{1.39}$$

where the form factor

$$F_k(q) = \frac{1}{\tau/2}\int_{-\tau/4}^{+\tau/4} U[x(s,q)]s^k ds \tag{1.40}$$

depends on the value of the Hamiltonian and differs significantly for rotation versus libration. Eq. (1.39) may be ensemble averaged over the oscillation phases q at time t. The ensemble average operation is denoted by $\langle \ldots \rangle$;

$$\overline{\dot{H}(t)} \equiv \left\langle \overline{\dot{H}(t,q)} \right\rangle = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \overline{\dot{H}(t,q)}dq\ . \tag{1.41}$$

[15]It transpires that α is of order the adiabaticity parameter ϵ.
[16]This assumption breaks down on the separatrix.

Notice that $\dot{H}(t)$ with a single argument (q is absent) means the ensemble average. We define the deviation and the M^{th} moment:

$$\Delta\overline{\dot{H}(t,q)} = \overline{\dot{H}(t,q)} - \overline{\dot{H}(t)} \quad \text{and} \quad \left\langle \left(\Delta\overline{H(t,q)}\right)^M \right\rangle . \tag{1.42}$$

We take these phases to be always (i.e. for all time t) uniform distributed, which implies that the Hamiltonian flow lines are uniformly populated. This seemingly trivial assumption concerning initial phases has profound consequences and deserves further discussion; see Sec. 1.8.5. Note that while q ranges over $\pm\pi$, the argument Q of Jacobi functions ranges over $\pm 2K(m)$; thus $q = Q \times \pi/(2K)$. Inserting the initial oscillation phase q, the motion is shifted to $x(s + \tau(q/2\pi))$; leading to the double-average integral

$$\langle F_k(q)\rangle = \frac{1}{2\pi}\int_{-\pi}^{+\pi} dq\, \frac{1}{\tau/2}\int_{-\tau/4}^{+\tau/4} U[x(s+\tau(q/2\pi))]s^k ds\,.$$

The properties of a periodic oscillation in a symmetric potential are such that this integral is

$$\langle F_k(q)\rangle = \frac{F_0(0)}{\tau/2}\,\frac{s^{k+1}}{(k+1)}\bigg|_{s=-\tau/4}^{s=+\tau/4} .$$

This is zero unless $k = 0, 2, 4, \ldots$ is even. Thus for the symmetric potential, $\ddot{V}$ is immaterial to $\langle\dot{H}\rangle$. Hence we may write

$$\overline{\dot{H}(t)} \equiv \left\langle \overline{\dot{H}(t,q)} \right\rangle = F_0(0)\dot{V}(t) \; + F_0(0)\left(\frac{\tau}{4}\right)^2 \dddot{V}(t) \tag{1.43}$$

Time-integration of Eqn. (1.43) enables us to find the average evolution of some particular Hamiltonian value that starts from H_0 at $t = 0$. Going forward, we shall assume $|\dot{V}| \gg (\tau/4)^2|d^3V/dt^3|$ and that the third derivative may be neglected; this is consistent with $\alpha(t)$ remaining a small quantity.

Time-integration of Eqn. (1.42) allows to find the evolution of the deviation about the average (for that particular H_0); and from that, the progress of the moments about the mean. The first moment ($M = 1$) is zero. The second moment ($M = 2$) is $\left\langle (\overline{H(t,q)})^2 \right\rangle - (\overline{H(t)})^2$, with the understanding that the expression applies to some specific value H_0 chosen from the initial spectrum.

Jacobi elliptic functions

The Jacobi functions satisfy the pendulum equation of motion, when the confining potential is constant. The functions are listed in sec. 1.15. They are parametrised by $m = H/(2V)$ which is taken to be a constant of motion. During the entire capture process, the value of m changes from $m \gg 1$ to $m \ll 1$. But if $V(t)$ varies slowly enough, we may hope that Jacobi functions with $m(t)$ are still valid solutions for at least one half-period. In Sec. 1.9 it is demonstrated that if $x(t) = 2\,\text{arcsin}[\sqrt{m}\,\text{sn}(\omega t, m)]$ then the change in motion is $\alpha \simeq \epsilon\sqrt{m} \times (\omega t)^2$ for $|t| \leq \tau/4$ or $|\alpha| \leq \epsilon\sqrt{m}$. The unperturbed motion is of order $\sqrt{m}$. So the Jacobi functions are a valid basis provided that $|\epsilon| \ll 1$.

1.8.1 Pendulum form factors

Thus far, U has been a general, symmetric potential. We turn now to the specific case of the pendulum. The form factors are derived in detail in Sec. 1.16. Here we excerpt a few results that enable us to continue the derivation of the evolution equations. Let m be the Jacobi amplitude parameter. Let a primed quantity F' denote rotation, and the absence of the prime denote libration. For libration $F_k(q) = 2m\,\mathcal{I}_k(m,q)$; and for rotation $F'_k(q) = 2\mathcal{I}_k(1/m,q)/\sqrt{m^k}$; where

$$\mathcal{I}_k(m,q) = \int_{-K}^{+K} \left[\frac{u}{\omega}\right]^k \mathrm{sn}^2(u+Q,m)du \Bigg/ \int_{-K}^{+K} du \ . \tag{1.44}$$

The quantity $\langle F_0(q)\rangle = F_0(0) = \langle\overline{U}\rangle$ has a simple intepretation as the double average of motion in the potential.

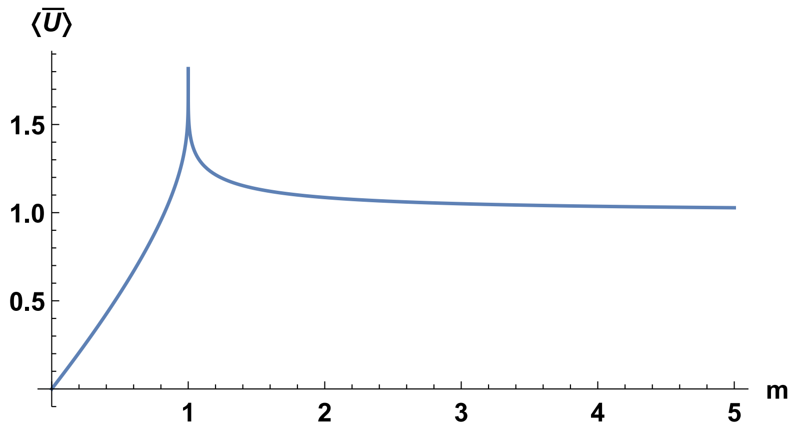

Figure 1.7: Form factor $\langle\overline{U}\rangle = 2m\mathcal{I}_0(m)$ for $m \leq 1$ and $\langle\overline{U}\rangle = 2\mathcal{I}_0(1/m)$ for $m \geq 1$.

The function $\mathcal{I}_0(m)$ is given explictly in Eq. (1.45), and the corresponding form factors plotted in Fig. 1.7.

$$\mathcal{I}_0(m) = \frac{1}{m}\left[1 - \frac{E(m)}{K(m)}\right] = \left[1 + \frac{m}{8} + \frac{m^2}{16} + \ldots\right] \tag{1.45}$$

The form factors are well approximated by simple expressions: $\langle\bar{U}\rangle \approx 1$ for $m \geq 2$, and $\langle\bar{U}\rangle \approx m$ for $m \leq \frac{1}{2}$. The form factor is undefined and extremely large at $m = 1$. However, the immediate vicinity of $m = 1$ is excluded because this is the domain of separatrix crossing, and this process restores finite values to the change of Hamiltonian.

1.8.2 Evolution of $\tilde{H}(t)$

The double-average[17] evolution of some particular Hamiltonian value that starts from H_0 is given by integration of Eq. (1.46) with conditions (1.45& 1.47):

$$\overline{\dot{H}}(t) = \langle\overline{U}\rangle\dot{V}(t) \tag{1.46}$$

$$\text{For rotation}: \quad \langle\overline{U}\rangle = 2\mathcal{I}_0(1/m) \qquad \text{For libration}: \quad \langle\overline{U}\rangle = 2m\mathcal{I}_0(m) \tag{1.47}$$

Here, in keeping with the method of time-averaging, m is shorthand for $\overline{m} = \overline{H(t)}/V(t)/2$. For most of the pre-capture era $m \gg 1$ (outside the bucket); and for most of the post-capture era $m \ll 1$ (inside the bucket); and this suggests it is legitimate to truncate $\mathcal{I}_0$ to low order in Jacobi parameter m. If the series is truncated to $\mathcal{I}_0 = 1$, there are exact differentials:

$$\text{For rotation}: \quad \dot{H} = \dot{V}, \qquad H(t < t_c) = H_0 - V_0 + V(t) \tag{1.48}$$

$$\text{For libration}: \quad d/dt[\ln(H)] = \frac{1}{2}d/dt[\ln(V)], \qquad H(t > t_c) = H(t_c)\sqrt{V(t)/V(t_c)}\ . \tag{1.49}$$

[17]Time-averaged over the sliding window and ensemble-averaged over oscillation phases.

To be precise, H_0 appearing in Eq. 1.48 is H_0^+, the value immediately after $t = 0$. Expression (1.48) has a simple interpretation: that the kinetic energy averaged over a cycle, $\overline{T}$, is a constant of motion during the rotation era. This notion is at odds with Ehrenfest who reports $\overline{T}/\omega$ to be the invariant, but is confirmed in Fig. 1.6 where the cycle-average momentum $\bar{p}$ is (almost) constant. Expression (1.49) is identical with that derived from the adiabatic invariant Eq. (1.13). But the combination of Eqs. (1.48, 1.49) leads to a very different result.

Let us make a rudimentary check that the evolution equations yield expected results. Equal initial and final phase-space areas impy that $\hat{H}_0 \approx V_T$. This value of H_0 is captured at $t_c = T$, at which time the Hamiltonian has increased to $\hat{H}_T = 2V_T$. A smaller value of H_0 is captured at an earlier time; and then rises to $H_T^* = H(t_c)\sqrt{V_T/V_c}$ where $H_c = 2V_c$. In the limit $t_c \to T$, $H_T^* \to 2V_T = \hat{H}_T$. So the pre- and post-capture evolution equations make consistent predictions for $\hat{H}_T$. The area of the original beam rectangle is $4\pi\hat{p}_0 = 4\pi\sqrt{\hat{H}_0} \approx 4\pi\sqrt{V_T}$. The beam final almond-shape area is roughly $(4\pi/\sqrt{2})\hat{p}_T = (4\pi/\sqrt{2})\sqrt{\hat{H}_T} = 4\pi\sqrt{V_T}$. So the calculation is consistent.

Captured at sudden turn on

Particles that are captured by the initial voltage step V_0 at $t = 0$, satisfy $H_0^+(x) \leq 2V_0$, that is $m_0 \leq 1$. They immediately enter the libration era. Accordingly we set $t_c = 0$, $t = T$ and $H(t_c) = H_0$ in Eq.1.49; to find the double-average $\tilde{H}_T = H_0^+\sqrt{V_T/V_0} = (H_0^- + V_0)\sqrt{V_T/V_0}$. Here the captured range is $H_0^+ = [V_0, 2V_0]$, or $H_0^- = [0, V_0]$.

The effect of the sudden turn on is to produce both a spread and a lift. The lift effect is already included in $\tilde{H}_T$. The smallest value $H_0^- = 0$ is lifted to V_0 at $t = 0$; and that lift evolves to $L_T = V_0\sqrt{V_T/V_0} = \sqrt{V_0 V_T}$. Above and below the average value, $\tilde{H}_T(H_0^- = 0)$, there is a spread at $t = 0$ extending to $\Delta H = \pm V_0$ which evolves (as per libration) to $\pm L_T$ at $t = T$.

Captured later

Particles that satisfy $H_0^+(x) > 2V_0$, that is $m_0 \geq 1$, become captured after $t = 0$. Let us suppose that the pre-capture (rotation) era and the post-capture (libration) era are both much longer than the capture process (separatrix crossing), and that they dominate changes of the Hamiltonian. The capture condition is $H(t) \leq 2V(t)$. Almost all particles start their careers outside the nascent RF bucket with $H_0 \gg V_0$, and evolve according to Eq. 1.48. Contrastingly, the value of the Hamiltonian on the instantaneous separatrix ($m = 1$) rises at the rate $H_{\text{sep}}(t) = 2V(t)$, and so capture is inevitable. At the moment of capture t_c, $H(t_c) = 2V(t_c)$. Hence $V(t_c) = H_0^+ - V_0$ is the difference of initial values. The precise time of capture depends on the inital Hamiltonian value H_0 of each particle. Post-capture (inside the bucket) the particle Hamiltonian evolves as Eqn. 1.49. Substituting $H(t_c) = 2V(t_c)$ yields $H(t \geq t_c) = 2\sqrt{V(t_c)V(t)}$. Further, substituting the rotation outcome $V(t_c) = H_0^+ - V_0$, we find $H(t \geq t_c) = 2\sqrt{(H_0^+ - V_0)}\sqrt{V(t)}$. Taking into account the lift, due to sudden turn on, $H_0^+ = H_0^- + V_0$, the energy value at the end of the libration process is: $\tilde{H}(T) = 2\sqrt{H_0^- V(T)}$.

Universal function

$\tilde{H}(T)$ is a universal function of $H(t < 0)$ independent of the shape of the voltage ramp, provided that the ramp obeys the adiabaticity condition $\epsilon \ll 1$.

$$\tilde{H}_T = \begin{cases} (H_0^- + V_0)\sqrt{V_T/V_0} & \text{for } 0 \leq H_0^- < V_0 \\ 2\sqrt{H_0^- V_T} & \text{for } V_0 \leq H_0^- \leq \hat{H}_0 \end{cases} \tag{1.50}$$

The piecewise defnition is continuous at $H_0^- = V_0$.

To summarise, there is a two-step process for an individual particle. Step (1), almost linear in $V(t)$ growth of Hamiltonian (from $t = 0$) until capture at $t = t_c$. Step (2), square root in $V(t)$ growth of the Hamiltonian from capture until the voltage ramp is complete at $t = T$. For an ensemble with a range of energies and capture times, the processes are concurrent.

Each of the integrals (1.48, 1.49) $H(0 \leq t \leq t_c)$ outside, and $H(t_c \leq t \leq T)$ inside the bucket, is independent of the details of the voltage law between its end points. The exact moment of capture t_c depends on the shape of the ramp, but not the value $H(t_c) = 2V(t_c)$. Cascading the functions, to give the nett effect from $t=0$ to $t=T$, is also independent of the voltage law. Those properties reflects that truncating $\mathcal{I}_0(m)$ to unity has eliminated some of the non-linear effects that compromise adiabaticity.

Effect of capture

We can extend this simple argument leading to Eq. (1.50) to include the ensemble-average effect of the capture derived in Secs. 1.10 & 1.11. Let $\Gamma(\epsilon) \equiv [-4 + 2\ln(8/\epsilon)]$. The increase in Hamiltonian at separatrix crossing is given by $\Delta H/H(t_c) = \epsilon(t_c)\Gamma(\epsilon)$. Now $H(t_c) = 2V(t_c) = 2H(0)^-$; and thus

$$\begin{aligned} \tilde{H}(T) &= H(t_c)\,[1 + \epsilon(t_c)\Gamma(\epsilon)]\,\sqrt{V(T)/V(t_c)} = 2\,[1 + \epsilon(t_c)\Gamma(\epsilon)]\,\sqrt{V(T)V(t_c)} \\ &= 2\,[1 + \epsilon(t_c)\Gamma(\epsilon)]\,\sqrt{V(T)H(0)^-} \end{aligned} \quad (1.51)$$

Here t_c depends on the value $H(0)$ and the details of the voltage law. In Sec. 1.6, we derive formulae for $\epsilon(t)$ for a variety of common voltage laws. Eq. (1.51) gives some insight as to why the linear ramp generates large emittance growth. For the linear ramp, captures occur earlier than for the quadratic ramp; and at those times the adiabaticity parameter ϵ is larger. For the iso-adiabatic ramp, $\epsilon(t_c)$ is in fact a constant independent of the exact time of capture, leading to an un-complicated formula for H_T.

Influence of voltage law on capture time

We have glossed over the details of the time of capture t_c of a particular initial value H_0. This time depends on the form of the voltage law. For the linear ramp, the times t_c rise linearly with H_0; if the density $\rho(H_0)$ was uniform, then the t_c would be spread uniformly over the ramp duration. Contrastingly, for the iso-adiabatic ramp the moments of capture are initially sparse and become progressively more concentrated toward the end of the ramp. For the linear ramp, the adiabaticity parameter varies from very large to very small. For the iso-adiabatic ramp, the adiabaticity parameter is constant and small. If the duration of the ramps is so much curtailed that adiabaticity is generally compromised, then the linear ramp will fare less well; because more of the captures occur at early times when ϵ is comparatively larger.

Our argument has ignored the collective effects, such as the longitudinal space-charge force. The Coulomb-Faraday electro-magnetic force between moving charged-particles is always repulsive. In synchrotrons for which the slip factor A is positive, this force opposes bunching and is proportional to the spatial derivative of the bunch shape. The emergence of the bunch shape, from the initial uniform line density, will arise differently for dissimilar voltage ramps. The bunching starts earlier for the linear ramp, and this may imply that there is a longer period available for the internal motions inside the bunch to equilibriate in response to the growing space-charge forces. This reasoning is speculative, but suggests the linear ramp may have some merit when longitudinal space-charge effects are significant.

1.8.3 Nonlinear evolution of $\tilde{H}(t)$

Retention of terms m^k In Eq. (1.45) leads to a nonlinear equation linking the path $V(t)$ to the resulting $H(t)$; the mathematical model gains realism, and adiabaticity becomes imperfect. To first order in $\bar{m}$ the evolution equations are

$$\frac{d}{dt}H(t) = \dot{V}(t) + \frac{V(t)\dot{V}(t)}{4H(t)} \qquad \text{for rotation} \tag{1.52}$$

$$\frac{d}{dt}H(t) = \frac{H(t)}{2V(t)}\left[1 + \frac{H(t)}{16V(t)}\right]\dot{V}(t) \qquad \text{for libration .} \tag{1.53}$$

Rotation

Outside the bucket, the integral $H(t \leq t_c)$ is defined by the implicit equation:

$$4\ln\left[\frac{V[t]}{V_0}\right] + \frac{c_1}{\sqrt{2}}\ln\left[\frac{c_1 - 4H[t]/V[t]}{c_1 - 4H_0/V_0}\right] + \frac{c_2}{\sqrt{2}}\ln\left[\frac{c_2 + 4H[t]/V[t]}{c_2 + 4H_0/V_0}\right] = 0\ . \tag{1.54}$$

$c_1 = 2(\sqrt{2}+1)$ and $c_2 = 2(\sqrt{2}-1)$. Evidently, Eq. 1.54 can be re-written in terms of the ratios $m(t) = \frac{1}{2}H(t)/V(t)$ and $m_0 = \frac{1}{2}H(0)/V(0)$. The range of initial values is $2V_0 \leq H_0 \leq \hat{H}_0 = (8/\pi^2)V_T$. The quantity $H[t]$ is the ensemble average over oscillation phases, and satisfies $H[t]/V[t] \leq H_0/V_0$; the equality occurs at $t = 0$. Eq. (1.54) is of the form $a\ln A + b\ln B + c\ln C = 0$, and could be written $B^bC^c = 1/A^a$ where $A = V(t)/V_0$ and $a = 4$; but the powers b and c are irrational numbers, and so the alternate form is not useful. However, Eq. (1.54) can be reduced to a simpler form for $V(t)/V_0$. The logarithms are expanded as infinite series, and the terms are reordered and resummed; we do this below for the special case $m(t_c) = 1$.

We want the voltage at time of capture, $V(t_c)$, in terms of the initial Hamiltonian value H_0. Despite the formidable form of Eqn. 1.54, we may find this relationship; and, moreover, it is almost linear. $V(t_c)$ is a monotonic increasing function of H_0, so the stratification of Hamiltonian flow lines is unchanged. To start, we substitute the capture condition $H(t_c) = 2V(t_c)$. For brevity, replace H_0 by $m_0 = H_0/(2V_0)$ where $m_0 \geq 1$ is the initial value of Jacobi parameter. Eqn. 1.54 becomes:

$$(2+\sqrt{2})\ln\left[\frac{-4m_0 + x}{-3+\sqrt{2}}\right] + (-2+\sqrt{2})\ln\left[\frac{3+\sqrt{2}}{4m_0 + y}\right] + 4\ln\left[\frac{V(0)}{V(t_c)}\right] = 0\ . \tag{1.55}$$

where $x = \sqrt{2}+1$ and $y = \sqrt{2}-1$. Typically $m_0 \gg 1$; thus x and y can be considered small quantities. Making an expansion in x, y to fourth order, we find the approximate form:

$$\ln\left[256 \times 7^{(-2-\sqrt{2})}(3+\sqrt{2})^{2\sqrt{2}}\right] + 4\ln[m_0] + 4\ln\left[\frac{V(0)}{V(t_c)}\right] - \frac{2}{m_0} - \frac{5}{8m_0^2} - \frac{1}{4m_0^3} - \frac{29}{256m_0^4} + \ldots \tag{1.56}$$

Eqn. 1.56 is set to zero and solved for $V(t_c)$. Truncating the expansion leads to an error. To restore the trivial condition $V(t_c = 0) = V(0)$, we shall need to add a constant.

To zeroth order in m_0, we have the approximate condition

$$\ln\left[\frac{A_vH_0}{V(t_c)}\right] = 0 = \ln(1) \qquad \text{where } A_v = 2 \times 7^{(-\frac{1}{2}-\frac{1}{2\sqrt{2}})}(3+\sqrt{2})^{\frac{1}{\sqrt{2}}} \approx 1.08561$$

which has the solution $V(t_c) = A_vH_0$. Eqn. 1.55 has the property that if $m_0 = 1$, then $V(t_c) = V(0)$ and $t_c = 0$. To recover that property, we need an additive constant.
Thus $V(t_c) = A_vH_0 + (1-2A_v)V_0 \approx 1.08561H_0 - 1.17122V_0$.
To first order in m_0, we have the approximate condition:

$$V(t_c) = A_v\exp\left[\frac{-1}{2m_0}\right]H_0 + [1 - 2A_v\exp(-1/2)]\,V_0 \approx 1.08561\exp\left[\frac{-1}{2m_0}\right]H_0 - 0.316912V_0\ .$$

To second order in m_0, we have the approximate condition:

$$\begin{aligned} V(t_c) &= A_v \exp\left[-\frac{5+16m_0}{32m_0^2}\right] H_0 \; + [1-2A_v\exp(-21/32)]\, V_0 \\ &\approx 1.08561 \exp\left[-\frac{5+16m_0}{32m_0^2}\right] H_0 \; - 0.126414 V_0 \, . \end{aligned}$$

The series expansion in m_0 appearing in Eq. 1.56 can be summed to infinity:

$$\sum_{n=1}^{\infty} \frac{1}{n(4m)^n}[(1-\sqrt{2})^n(-2+\sqrt{2})-(1+\sqrt{2})^n(2+\sqrt{2})] = 2\ln\left[1-\frac{1}{2m}-\frac{1}{16m^2}\right]+\sqrt{2}\ln\left[\frac{4m-1-\sqrt{2}}{4m-1+\sqrt{2}}\right]$$

to give an exact and explict (but unwieldy) form for the capture voltage:

$$V(t_c) = 2A_v\sqrt{1-\frac{1}{2m_0}-\frac{1}{16m_0^2}}\left[\frac{4m_0-1-\sqrt{2}}{4m_0-1+\sqrt{2}}\right]^{\frac{1}{2\sqrt{2}}} m_0 V_0 \, . \tag{1.57}$$

In practise, the linear form $V(t_c) = A_v H_0 + (1-2A_v)V_0$ is an excellent approximation; and we shall adopt it going forward. In either case, the effect of separatrix crossing may be incorporated following the manner of Eq. 1.51.

Libration

Inside the instantaneous bucket, $H(t)$ is the integral of Eq. 1.53; namely

$$H(t > t_c) = [16V(t)\sqrt{V_c}]/[7\sqrt{V(t)} + \sqrt{V_c}] \, . \tag{1.58}$$

The explicit function $V(t_c)$, Eqn. 1.57, can be cascaded with $H(t > t_c)$ to yield a single function that links $H(T)$ to H_0. Alternatively, the linear form $V(t_c) = A_v H_0 + (1-2A_v)V_0$ can be cascaded with $H(t>t_c)$ to give the approximate nett effect.

Captured at sudden turn on

Particles that are captured by the initial voltage step V_0 at $t = 0$, satisfy $H_0^+(x) \le 2V_0$, that is $m_0 \le 1$. They immediately enter the libration era. We integrate Eq. 1.53 starting from H_0^+ at $t = 0$ to $t = T$. The result is

$$\tilde{H}_T = \frac{16 H_0^+ V_T \sqrt{V_0}}{H_0^+\sqrt{V_0} - (H_0^+ - 16V_0)\sqrt{V_T}} \tag{1.59}$$

We may replace H_0^+ by $H_0^- + V_0$.

Captured later

Particles that satisfy $H_0^+(x) > 2V_0$, that is $m_0 \ge 1$, become captured after $t = 0$. To find the final energy, we set $t = T$ and $V(t_c) = A_v H_0^+(1-2A_v)V_0$ into Eq. 1.53; leading to

$$\tilde{H}_T = \frac{16 V_T \sqrt{A_v(H_0^+ - 2V_0) + V_0}}{7\sqrt{V_T} + \sqrt{A_v(H_0^+ - 2V_0) + V_0}} \tag{1.60}$$

We may replace H_0^+ by $H_0^- + V_0$.

Universal function

Eqs. 1.59 & 1.60 imply that the final value of Hamiltonian depends only on the final value V_T and the initial values H_0 and V_0. Thus $\tilde{H}(T)$ is almost a *universal function* of $H(0)$ independent of the shape of the voltage ramp, provided that the ramp obeys the adiabaticity condition $\epsilon_1 \ll 1$. The piecewise definition of $\tilde{H}_T$, provided by Eqs. 1.59 & 1.60, is continuous at $H_0^- = V_0$.

The evolution Eqns. 1.52 and 1.53 were truncated to first order in $\bar{m}$; and so are in error. The saving grace is that most of the time $m \ll 1$ inside and $m \gg 1$ outside the bucket. So the error, from discarding second and higher order terms, arises only when $m \to 1$, during the transition from rotation to libration.

1.8.4 Evolution of deviation $\Delta H(t, q)$

Time-integration of Eqn. (1.42) allows to find the evolution of the deviation. Again we shall appeal to the property that for most of the time during pre-capture $1/m \ll 1$ and during post-capture[18] $m \ll 1$, respectively; and therefore we may employ low order expansions in Jacobi parameter.

Rotation era

Outside the bucket form factors are denoted with a prime, F'. The phase-dependent deviation evolves as:

$$\frac{d}{dt}\Delta H(t,q) \approx [F_0(m,q) - F_0(m)]'\,\dot{V}(t) + [F_1(m,q) - F_1(m)]'\,\ddot{V} + \frac{1}{2}[F_2(m,q) - F_2(m)]'\,\dddot{V} \quad (1.61)$$

We expand the form factors in powers of $1/m$, where typically $m \gg 1$. To first order, the expression is

$$\frac{d}{dt}\Delta H \approx \frac{\sin 2q}{\sqrt{m}}\frac{\ddot{V}}{2\,\omega} + \frac{\cos 2q}{4m}\left[-\dot{V} + \frac{\dddot{V}}{\omega^2}\right] .$$

For consistency with the previous section, we drop the third time-derivative as being small compared to the first. What remains is a delicate competition between second order effects: $\dot{V}/m$ versus $(\ddot{V}/\omega)/\sqrt{m}$. We substitute $\omega = \sqrt{AV}$ and $m = H/V/2$.

$$\frac{d}{dt}\Delta H \approx \sin 2q\frac{\ddot{V}}{\sqrt{2AH(t)}} - \cos 2q\frac{V(t)\dot{V}}{2H(t)} .$$

Now for the double average motion $\tilde{H}(t) \approx H_0 + V(t)$; this substitution results in an evolution equation for $\Delta H(t, q)$ in terms of the $V(t)$ and the slip constant A. Further progress, and determining which terms dominate, cannot be made without inserting a specific voltage law. The main features of the behaviour of ΔH are displayed under the quadratic voltage law $V(t) = V(0) + \Delta V \tau^2$, with fractional time $\tau = t/T$ and $t \le t_c$ and $\Delta V = [V(T) - V(0)]$. The integral becomes:

$$\Delta H(t,q) \approx \sin 2q\frac{\sqrt{2\Delta V}}{T\sqrt{A}}\,\mathrm{arctanh}\sqrt{\frac{\Delta V\tau^2}{H_0 + \Delta V\tau^2}} - \frac{1}{2}\cos 2q\left[\Delta V\tau^2 + (H_0 - V_0)\ln\left[\frac{H_0}{H_0 + \Delta V\tau^2}\right]\right]$$

This is to be evaluated at the respective capture time t_c for each value of $H_0 \equiv H_0^+$. For short time-scale, Taylor expansion gives the further approximation:

$$\Delta H(t,q) \approx \Delta V\tau\frac{\sqrt{2}\sin 2q}{T\sqrt{AH_0}}\left[1 - \frac{\Delta V\tau^2}{6H_0}\right] - \Delta V V_0\tau^2\frac{\cos 2q}{2H_0}\left[1 + \frac{\Delta V(H_0 - V_0)\tau^2}{2H_0V_0}\right] \quad (1.62)$$

[18] It is unfortunate that the condition applies better to the end rather than beginning of post-capture.

This is dimensionally correct, because A has dimensions of $\mathrm{Hz}^2/\mathrm{volt}$. It becomes clear from Eq. 1.62 that this contribution to ΔH falls as $1/\sqrt{H_0}$ or faster. For most of the particles during their pre-capture era, $H_0 \gg V_0$ and $\Delta V/H_0 \geq 1$. Evidently the quantity AT^2 is a control parameter; if it is large, then the terms in $\sin 2q$ are reduced. In contrast, the terms in $\cos 2q$ depend solely on V_0/H_0 and V_T/H_0 and cannot easily be altered. Hence the duration of the complete capture must satisfy the adiabaticity condition $AT^2 \gg 1$ per volt.

Libration era

Inside the bucket, the phase-dependent deviation evolves as:

$$\frac{d}{dt}\Delta H(t,q) \approx [F_0(m,q) - F_0(m)]\dot{V}(t) + [F_1(m,q) - F_1(m)]\ddot{V} + \frac{1}{2}[F_2(m,q) - F_2(m)]\dddot{V} \quad (1.63)$$

To first order in Jacobi m, Eqn. (1.63) becomes:

$$\frac{d}{dt}\Delta H = \frac{m}{2\omega}\left[\ddot{V}\sin 2q + \frac{1}{2}\frac{\dddot{V}}{\omega}\cos 2q\right] . \quad (1.64)$$

Notably, at this order, there is no term in $\dot{V}$. We drop the third time derivative in comparison to the second (i.e. $|\ddot{V}| \gg |\dddot{V}|/\omega$). Continuing the expansion to second order in m, and retaining only the leading (i.e. largest) terms yields:

$$\frac{d}{dt}\Delta H(t,q) \approx \frac{m}{2\omega}\ddot{V}\sin 2q - \frac{m^2}{4}\dot{V}\cos 2q . \quad (1.65)$$

The expression contains competing second order effects. We substitute $\omega = \sqrt{AV}$ and $m = H/V/2$. t_c is the time of capture of a particular particle with individual Hamiltonian value H_0. For the post-capture double-average motion we may substitute $\tilde{H}(t > t_c) = 2\sqrt{V(t_c)V(t)}$, yielding:

$$\frac{d}{dt}\Delta H(t,q) \approx \sin 2q\frac{\sqrt{V(t_c)}}{2\sqrt{A}V(t)}\ddot{V} - \cos 2q\frac{V(t_c)}{4V(t)}\dot{V} . \quad (1.66)$$

To understand the main features of the evolution $\Delta H(t,q)$ after t_c, we substitute the quadratic voltage law, and integrate over time. For brevity, let $\Delta V/V_0 \equiv a \gg 1$ and $\tau = t/T$ and $\tau_c = t_c/T$.

$$\begin{aligned}\Delta H(t > t_c, q) \;&\approx\; \Delta H(t_c,q) - \frac{1}{4}\cos 2q \ln\left[\frac{1 + a\,\tau^2}{1 + a\,\tau_c^2}\right] V(t_c)\\ &\quad + \sin 2q\frac{\sqrt{a\,V(t_c)}}{T\sqrt{A}}\left\{\arctan\left[\tau\sqrt{a}\right] - \arctan\left[\tau_c\sqrt{a}\right]\right\} .\end{aligned}$$

$\Delta H(t_c)$ is the deviation that accrues before t_c. Our interest lies with the final value, at the end of capture, so insert $t \to T$. There is no way to reduce the term in $\cos 2q$; but the term in $\sin 2q$ can be reduced by the increasing the control parameter AT^2. Both terms tend to zero as the moment of capture $t_c \to T$; but this is different for every particle and is not under our control[19].

It might be surprising to see the trigonometric terms $\sin 2q, \cos 2q$, etc, appearing in both expressions for $\Delta H(q)$ in the very different regimes of rotation and libration; in one case particles are streaming and in the other they are oscillating to-and-fro. However, changes in the Hamiltonian are effected only by motions in the potential well where work can be performed on the particles. Thus the trigonometric terms arise from the nett work performed by the rising well, and that depends on the arrival time (which is related to phase q).

[19]Except by adjusting the voltage law so that capture occurs later; which is suggestive of the iso-adiabtic law.

1.8.5 Assumption of "all oscillation phases"

We have introduced two equations that were not sufficiently precise concerning the particle distribution that is ensemble averaged. Eq. 1.41 is more correctly written:

$$\left\langle \overline{\dot{H}(t,q)} \right\rangle = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \rho(q) \overline{\dot{H}(t,q)} dq \; ,$$

where $\rho(q)$ is the phase distribution. The lift Eq. 1.35 could have been written:

$$\Delta H_0 = V_0 \int_{-\pi}^{+\pi} \rho(x)(1-\cos x)dx \; = V_0 \int_{-\pi}^{+\pi} \rho(q)[1-\cos x(q)]dq$$

and must produce the same result, V_0, whether the average is over position x or phase q. We glossed over the step of setting $\rho(q) \equiv 1$.

In performing the ensemble average, we assume that at any time (i.e. at all times) the oscillation phases are uniformly distributed (or at least symmetric in q) and all values present. This deserves some thought, particularly so because a weakness of the theory presented here is that we do not have an evolution equation for the oscillation phases q.

Pre-capture, $m > 1$

It is straight forward to reason that the initial oscillation phases are almost uniformly distributed at each value of $m = H/(2V)$; and that the departure from uniformity is greater for smaller values of Jacobi m. At the outset, the particle distribution is uniform in position $x = [-\pi, +\pi]$. Hence the line density $\rho(x) = 1/(2\pi)$ is constant. For the streaming motion (i.e. rotation era), the relation between position and oscillation phase is $x(q) = 2\arcsin[\mathrm{sn}(\sqrt{m}q, 1/m)]$. The transformation of densities is $\rho(q,m) = \rho(x(q))|dx/dq| = (\sqrt{m}/\pi)\,\mathrm{dn}(\sqrt{m}q, 1/m)$ with the range of q equal to $\pm\hat{q} \equiv \pm K(1/m)/\sqrt{m}$. The density in phase $\rho(q,m)$ is that which will make the line density $\rho(x)$ constant when integrated over $\rho(m)$. The density $\rho(q)$ is itself almost constant: the minimum value is $\rho(\hat{q}) = \sqrt{(1\,-\,1/m)}\sqrt{m}/\pi$, and the maximum is $\rho(0) = \sqrt{m}/\pi$ where typically $m \gg 1$.

We may think of these densities as arising from the underlying kinematics of the particles. Now $q = \omega t$, and for compactness we set $\omega = 1$. $p(q) = \dot{x} = dx/dq$ is the instantaneous speed of a particle with phase q. $p(t)$ and $p(q)$ are synonymous. Let X be the particular position of the particle with phase q. Thus $q(X) = \mathrm{arcsn}[\sin(X/2), 1/m]/\sqrt{m}$ where arcsn is the inverse of the Jacobi function. The dwell function quantifies where, during its motion, the particle spends more of its time. The dwell function $D(x)$ is proportional to the inverse of speed $1/p(q(X))$, or explicitly

$$D(X) \propto \frac{\sqrt{m}\,\mathrm{dn}\left[\mathrm{arcsn}[\sin(X/2), 1/m], 1/m\right]}{-1 + 2m + \cos X} \; . \tag{1.67}$$

The dwell function is almost constant. The minimum value is $D(0) \propto 1/(2\sqrt{m})$. The maximum value is $D(\pm\pi) \propto 1/(2\sqrt{m-1})$. This slight modulation of D exactly cancels the deviation of $\rho(q)$ from precise uniformity, to yield the constant line density $\rho(x) = 1/(2\pi)$.

Adiabatic transformation of $\rho(q)$

Consider now that the height of the potential well slowly increases. Then, as the initial distribution of oscillation phases evolves we propose it does so adiabatically; so that it stays approximately uniform after the streaming motion has been converted to to-and-fro motion confined inside the potential well (i.e. the libration era). Adiabatic evolution of the Hamiltonian, implies adiabatic evolution of the particle trajectories; and because the particle trajectories are intimately connected to their individual oscillation phases, so it follows that the distributions of q at each value of $m(t)$ also evolve in an adiabatic or near-adiabatic manner.

Post-capture, $m < 1$

Let us take up now the theme of density distributions in the post-capture (or libration) era. And let us assume that the distribution of initial oscillation phase has adiabatically transformed such that this new distribution is uniform in q for each value of $m < 1$. This means the trajectory in phase space, which is approximately an ellipse, is uniformly populated with particles along its annular length. The half-period of oscillation is $\tau/2 = 2K[m]$, and the density $\rho(q) = 2/\tau$. Each initial q value maps to a position value $x = 2\arcsin[\sqrt{m}\,\mathrm{sn}(q,m)]$. The density in position space is $\rho(x) = \rho(q)|dq/dx| = 1/(2K(m))/|dx/dq|$, or explicitly

$$\rho(x) = \frac{1}{4K(m)\sqrt{m}\,\mathrm{cn}[q(x),m]} = \frac{1}{4K(m)\sqrt{m}\,\mathrm{cn}\left[\mathrm{arcsn}[\sin(x/2)/\sqrt{m},m]\right]} \; .$$

For a particular value of m, the range of $x = [-X, +X]$ where $X = 2\arcsin\sqrt{m}$. The shape of $\rho(x,m)$ is like a steep-sided bowl, with infinite cusps at the extrema of motion: $\rho(x = 0) = 1/(4\sqrt{m}K(m)$ while $\rho(X) \to \infty$. The vast majority of phases q contribute to the line density at the extrema of motion, generating infinite cusps. Nevertheless, summing over a continuum of m values gives a hill-shaped line density with the maximum centred at $x = 0$. In other words, summing over m yields a particle bunch.

Capture or separatrix crossing

But there is an exception to these assumptions of initial uniformity in q and adiabatic evolution: the separatrix where at the outset, $\rho(q)$ is not uniform; and nor does it evolve appreciably toward uniformity. At the very start and early on, the line density $\rho(x) = 1/(2\pi)$ is uniform. Immediately that the voltage is established, $x = 2\arcsin[\tanh(q)]$ at the instantaneous separatrix; and the corresponding density of phase is given by $\rho(q) = \rho(x)|dx/dq| = (1/\pi)\,\mathrm{sech}(q)$. The range of $\pm q \to \infty$. The infinite range is a mathematical artefact that corresponds to there being particles at $(x,p) = (\pm\pi, 0)$. The phase density varies from $\rho(0) = 1/\pi$ to $\rho(\pm\infty) = 0$. This variation exactly cancels, the kinematic dwell time to yield a line density $\rho(x)$ that is uniform. The speed $\dot{x} = p(t)$ is synonymous with dx/dq. The dwell function is $D = 1/p$ which we may write as a function of t or q or x; $\sqrt{2}D(x) = 1/\sqrt{1 + \cos x}$. $\rho(q = 0)$ and $D(0) = \frac{1}{2}$ are both finite. Although $D(\pm\pi) \to \infty$, there are no particles present at $\rho(q = \pm\infty)$ and so the divergence of D cannot be manifested. If $\rho(q)$ at the separatrix were to evolve toward a uniform distribution, these precise cancellations (resulting in finite values for $\rho(x)$) would not operate. Indeed it is probable that the reverse may occur, that during the evolution of the separatrix its corresponding distribution of initial phase becomes more deeply modulated. On a static separatrix, the range of $q = \pm\infty$. Mechanistically, particles have to advance to these extreme phases. It is difficult to imagine that in a dynamical process occurring in a finite time duration T, that an infnite range of oscillation phases is established. Heuristically, the largest value is limited to order $\int_0^T \omega(t)dt$. So it might be that the phase distribution becomes further depopulated at large $|q|$. This has implications for ensemble-averaging (over q) the separatrix crossing process, possibly canceling divergences and leading to finite results.

1.8.6 Density function $\rho(\Delta H)$

The moments introduced in Eqn. (1.42) are much less useful than the density function $\rho(\Delta H)$ that tells us how often some particular deviation occus [about some particular double-average value $\tilde{H}(t)$]. So we need to find the relation between $\rho(\Delta H)$ and the distribution of oscillation phases $R(q) = 1/(2\pi)\ \ \forall\ |q| \leq \pi$. The transformation of densities obeys:

$$\rho(\Delta H) = R(q(\Delta H)) \left| \frac{d}{d\Delta H} q(\Delta H) \right|$$

In the preceding section, we found deviation to be of the form $\Delta H = a\cos(2q) + b\sin(2q)$ where $a^2 + b^2 = (\widehat{\Delta H})^2 \equiv c^2$ is the square of the maximum deviation. In this case,

$$2q = \arctan[a\Delta H \pm b\sqrt{c^2 - \Delta H^2},\ b\Delta H \mp a\sqrt{c^2 - \Delta H^2}]$$

and ΔH in this context has become a simple parameter running over the range $-c \leq \Delta H \leq +c$. The normalisation integral is $\pi^2/4$; and the density:

$$\rho(\Delta H) = \frac{1}{\pi\sqrt{c^2 - \Delta H^2}}\ . \tag{1.68}$$

The transformations simplify if a single term dominates; if either a or b can be set to zero. For example, if $a = 0$ then $q = (1/2)\mathrm{ArcSin}[\Delta H/b]$ and $\rho(\Delta H) = 1/(\pi\sqrt{b^2 - \Delta H^2})$.

Eqn. (1.68) is analogous to the dwell function of a harmonic oscillator. The dwell function is largest at the extremeties of motion, and the density is greatest at the extreme values of $\Delta H = \pm c$. a, b both depend on the initial value of Hamiltonian; and other parameters such as A and T, etc. The shape of ρ does not change as the process time T is increased, but the maximum deviation falls as $1/T$; and so the distribution becomes narrower. In reality, the situation is more complicated: there are additional trigonometric terms with arguments $4q$, $6q$, etc.; and their super-position will change the shape of the density function. The function (1.68) is the antithesis of commonly encountered point-spread functions, such as the Gaussian. This has the consequence that the third and fourth moments, the normalized skewness and kurtosis, may be large compared with the second moment (the variance).

1.9 Perturbed motion

Thus far, we have not attempted to find how the form of motion $[x(t), p(t)]$ is changed in response to the parameter variation $\omega(t)$; we simply assumed that the change is small. We now rectify this failing, find the change; and verify that it is small. The method will enable us to find the perturbed time-dependent separatrix (which is actually a family curves, depending on initial conditions).

The time-dependent Hamiltonian is $H(t, x, p) = p(t)^2/2 + \omega^2(t)[1 - \cos x(t)]$ where $p = \dot{x}$. The pendulum equation of motion $\dot{x}[\ddot{x} + \omega^2(t)\sin x(t)] = 0$, where dx/dt is generally non-zero, is generated by the setting derivative $dH/dt = [1 - \cos x(t)]d\omega^2/dt$. It follows that the first integral is:

$$\Delta H = H(t, x, p) - H(0) \equiv 2\int_0^t [1 - \cos x(s)]\omega(s)\dot{\omega}(s)ds\ . \tag{1.69}$$

Suppose that the parameter variation $\omega(t)$ neither changes the motion x nor the period τ of one cycle; and substitute $x = 2\arcsin[\sqrt{m}\,\mathrm{sn}(\omega_0 t, m)]$. For compactness we shall write $\omega(0) = \omega_0$ and $\dot{\omega}(0) = \dot{\omega}_0$, and $z = \omega_0 t$. $H(0) = 2m\omega_0^2$. The H increment is:

$$\Delta H = 4m\int_0^t \mathrm{sn}^2(\omega_0 s, m)\omega(s)\dot{\omega}(s)ds \approx 4\dot{\omega}_0\,[z - E[\mathcal{A}(z, m), m]] \approx \frac{4}{3}\dot{\omega}_0 m z^3\left[1 - (1+m)z^2 + \ldots\right]\ . \tag{1.70}$$

This confirms that H increases if $\dot{\omega} > 0$. Time-averaging $\dot{H}$ over a half period leads to $\overline{\dot{H}} = \Delta H/(\tau/2) = 4\omega_0\dot{\omega}_0[1 - E(m)/K(m)]$, which diverges as $m \to 1$. The adiabatic invariant is $I \equiv [H_0 + \Delta H(t)]/(\omega_0 + \dot{\omega}_0 t)]$. Let $\dot{\omega} = \epsilon\,\omega^2$. To low order in elapsed oscillation phase z, the invariant evolves as:

$$I(z) = 2m\omega_0\left[1 - \epsilon z + (\epsilon z)^2 + \epsilon(2/3 - \epsilon^2)z^3 + \ldots\right]\ .$$

Expression (1.69) can be viewed as an equality between two different ways of writing ΔH. We may use this equality to discover the effect of $\dot{\omega}$ on the unperturbed motion; and also the change in cycle limits, and a revised estimate of ΔH. For the former, we shall have to insert a trial solution and demonstrate self-consistency. For the latter, we need the integration limits t. For motion sufficiently far from the separatrix, these are trivial: $\pm\tau/4$. But on the separatrix, the cycle limits depart vastly from unperturbed values. To find the correct $x(t)$, we insert into (1.69) the trial solution:

$$x(t) = 2\arcsin\left[\sqrt{m}\,\mathrm{sn}(Z)\right] + \alpha(\omega_0 t) \;\;\&\;\; p(t) = \omega(t)2\sqrt{m}\,\mathrm{cn}(Z) + \omega_0\alpha'(\omega_0 t) \;\;\&\;\; Z = \int_0^t \omega(u)du \quad (1.71)$$

where α is a function to be determined, and $\alpha' = d\alpha(z)/dz$. The introduction of phase Z implies that the trial function has some character of the perturbation even in the absence of α and α'. Here we are not interested in computing ΔH. Rather we are interested in what the equality between $H(t, x, p) - H(0)$ and the integral have to tell us about $\alpha(t)$. We expand the trigonometric functions to first order in α and its derivative. (It will be shown that α is small, of order ϵ.) We suppose $\dot\omega_0\tau \ll \omega_0$, and replace $\omega(t)$ by $\omega_0 + \dot\omega_0 t$.
The equality (1.69) becomes:

$$\begin{aligned} & 2\sqrt{m}\omega_0\left[\mathrm{dn}(\omega_0 t)\mathrm{sn}(\omega_0 t)\alpha \times (\omega_0 + 2\dot\omega_0 t) + \mathrm{cn}(\omega_0 t)\alpha' \times (\omega_0 + \dot\omega_0 t)\right] \\ \equiv\; & 4\omega_0\dot\omega_0\sqrt{m}\int_0^t \mathrm{sn}(\omega_0 t)\left[\sqrt{m}\mathrm{sn}(\omega_0 s) + \mathrm{dn}(\omega_0 s)\alpha(\omega_0 s)\right] ds \;-\; 4m\omega_0\dot\omega_0 t \,. \end{aligned} \quad (1.72)$$

α is of order $\epsilon = \dot\omega/\omega_0^2$ smaller than other terms on the right, and is dropped from the integral. The righthand side is:

$$\begin{aligned} 4m\omega_0\dot\omega_0\left[\int_0^t \mathrm{sn}^2(\omega_0 s)ds \;-\; t\right] &= -4\dot\omega_0\left\{E[\mathcal{A}(\omega_0 t, m), m] + (m-1)\omega_0 t\right\} \\ &\approx -4m\dot\omega_0\left[z - z^3/3 + (1+m)z^5/5 + \ldots\right] \end{aligned}$$

In a final (but not essential) step we drop terms $\dot\omega_0 t$ in comparison to ω_0. (This implies $\dot\omega_0\tau \ll \omega_0$, which might not be justified for trajectories emanating from coordinates near the unstable fixed points.) The equality (1.69) has been manipulated into a first order differential equation for α:

$$\sqrt{m}\omega_0^2\left[\mathrm{dn}(z, m)\mathrm{sn}(z, m)\alpha(z) + \mathrm{cn}(z, m)\alpha'(z)\right] = -2\dot\omega_0\left[E[\mathcal{A}(z, m), m] + (m-1)z\right] \,. \quad (1.73)$$

Now, $\mathrm{dn}(z)\mathrm{sn}(z)$ is the derivative of $-\mathrm{cn}(z)$; and so this equation is of the form:
$-f'(z)y(z) + f(z)y'(z) = g(z)$ which has the formal solution: $y(z) = f(z)\int_0^z g(s)/f(s)^2 ds$.
The explicit form of the solution for $\alpha(z)$ is unwieldy, but is amenable to Taylor series expansion:

$$\alpha(z) \approx -\epsilon\sqrt{m}z^2\left[1 \;-\; z^2/6 \;+\; (3+28m)z^4/360 \;-\; (1+108m-64m^2)z^6/5040 + \ldots\right] \,. \quad (1.74)$$

If the terms $\dot\omega_0 t$ are not dropped, the expansion becomes $\alpha(z) \approx$

$$-\epsilon\sqrt{m}z^2\left[1 \;-\; (2/3)\epsilon z \;-\; (1-3\epsilon^2)z^2/6 \;+\; (1-6\epsilon^2)\epsilon z^3/15 \;+\; (3+28m-10\epsilon^2+120\epsilon^4)z^4/360 + \ldots\right]$$

1.9.1 Separatrix Family

In Sec. 1.7.2 we introduced two ideas: (1) the time-dependent separatrix is useful only for a few (or less) oscillations; and (2) that each snapshot lasting a couple of oscillations looks similar except for a change of the vertical scale proportional to $\sqrt{V(t)}$. Thus we are led to visualizing the separatrix in a sliding window of time $[t-\tau, t+\tau]$ centred on t. In what follows, we present formulae as if the window were centred at $t = 0$; but t is a dummy variable, it is not the start of the entire capture process.

When $\dot\omega = 0$, the separatrix (that divides the libration and rotation domains) is a single trajectory that connects the fixed points at $x = \pm\pi$. When $\dot\omega \neq 0$, the separatrix decomposes into a family of trajectories. If the frequency $\omega(t)$ is rising, every point on the instantaneous separatrix $(x_0, p_0 = \pm\sqrt{1+\cos x_0})$ at time $t = 0$ will captured - with the exception of the fixed points. The family constitutes all the trajectories emanating from those points at $t = 0$. It is possible that the family is bounded by one of it trajectories (to give a capture manifold), but this is not guaranteed. Coordinate pairs on this same curve at a different initial time will behave differently because there are additional "forces" arising from the differing values of ω.

We may use the analogues of Eqns. (1.69, 1.72) to construct this family; as in Secs. 1.10 & 1.11. Because the effect of the perturbation (due to $\dot{\omega}$) is most prominent in the vicinity of the astable fixed points, it is sometimes inferred that particles enter the separatrix only at those locations. This notion is incorrect; particles may enter at any value of (x_0, p_0), except $(\pm\pi, 0)$.

A more profound viewpoint is that the notion of separatrix in a time-dependent system should be abandoned. There is no definite curve that is crossed; and attempting to define such a curve and relying upon crossing of it to determine capture is probably doomed to inconsistency and failure. The only real criterion for capture is $H(t)/V(t) < 2$.

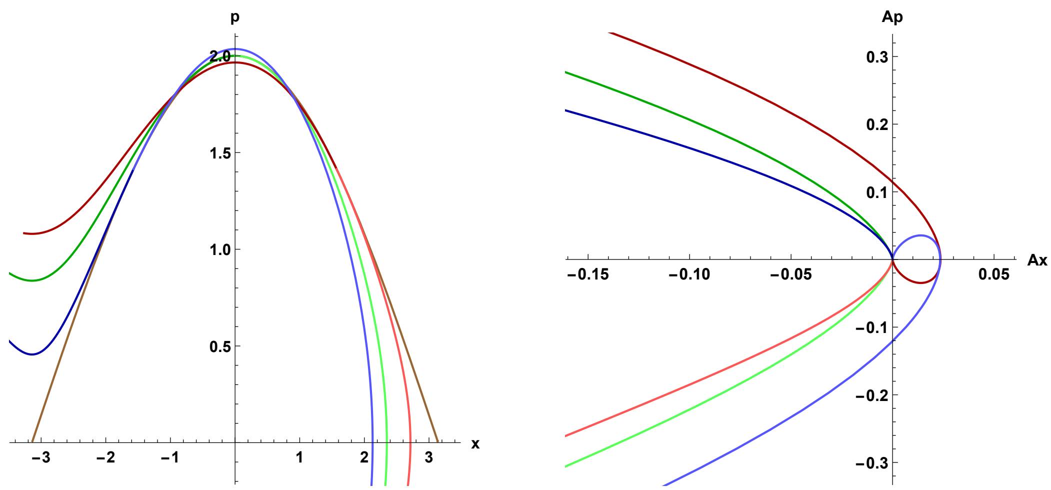

Figure 1.8: Left: Three trajectories $[x(z), p(z)]$: emanating from $x_0 = 0$ (green), $x_0 = -\pi/2$ (blue) and $x_0 = +\pi/2$ (red). The brown curve is the instantaneous separatrix. Right: the corresponding trajectories $[Ax, Ap] \equiv [\alpha(z), \alpha'(z)]$. The tone, darker or lighter, indicates before or after $t = 0$.

1.10 Symmetric trajectory

The trajectory emanating from $(x_0, p_0) = (0, 2)$ is particularly easy to find. It is both the simplest and least important trajectory; unimportant because it is atypical: the absolute majority of trajectories are asymmetric. Suppose for a moment that we substitute unperturbed separatrix trajectory $x(t) = 2\arcsin[\tanh(w_0 s)]$ into expression Eqn. (1.69). The change in Hamiltonian is the integral $\Delta H(z) = 4[z - \tanh(z)]\dot{\omega}_0$ with $z = \omega_0 t$; for large z values this becomes $\Delta H = 4z\dot{\omega}_0$. The unperturbed period $\tau \to \infty$ because the motion becomes frozen as the particle approaches the fixed point. This would imply that ΔH is unbounded. This is incorrect; and implies we must introduce the perturbation $\alpha(z)$.

The starting point is the expression for ΔH, Eq. (1.69). We substitute the trial motion

$$x(t) = 2\arcsin\left[\tanh(Z)\right] + \alpha(\omega_0 t) \;\;\&\;\; p(t) = 2\,\mathrm{sech}(Z) + \omega_0\alpha'(\omega_0 t) \tag{1.75}$$

where α is an unknown function to be determined by the equivalence Eq. (1.69) between $H(t) - H(0)$ and an integral. Z is the cumulant phase, as above. On the separatrix $H_0 = 2\omega_0^2$ at $t = 0$. We expand the trigonometric functions, and expand $\omega(t) \approx \omega_0 + \dot{\omega}_0 t$, and retain only first order terms in α and α' and $\dot{\omega}_0$. The change in Hamiltonian is:

$$H(t) - H(0) = 2\left[2z\dot{\omega}_0 + \text{sech} z \tanh z\, \alpha(z)(\omega_0^2 + 2z\dot{\omega}_0) + \text{sech} z\, \alpha'(z)(\omega_0^2 + z\dot{\omega}_0)\right] . \tag{1.76}$$

The integral form for the change is:

$$\Delta H = 4\omega_0\dot{\omega}_0 \int_0^t \text{sech}(\omega_0 s)\tanh(\omega_0 s)[\sinh(\omega_0 s) + \alpha(\omega_0 s)]ds . \tag{1.77}$$

α can be neglected because it is of order $\epsilon = \dot{\omega}_0/\omega_0^2$ (as we find below). Equating the two forms for ΔH yields a differential equation for α.

$$\sinh(z)\alpha(z) \times (1 + 2\epsilon z) + \cosh(z)\alpha'(z) \times (1 + \epsilon z) = -\epsilon \sinh(2z) . \tag{1.78}$$

In a final (but not essential) step we discard the terms ϵz, which are assumed to be small quantities. Now $d\cosh z/dz = \sinh z$, and so the equation is of the form $d/dt[f(t)y(t)] = g(t)$ which has the formal solution $y(t) = \int_0^t g(s)ds/f(t)$. Thus the response to the perturbation is the known function:

$$\alpha(z) = -\epsilon \sinh z \tanh z \quad \text{and} \quad \alpha'(z) = -\epsilon[\sinh z + \text{sech} z \tanh z] . \tag{1.79}$$

Technically, this function diverges for large $|z|$ values; but the range of z needed to describe a half-cycle excludes such values.

1.10.1 ΔH crossing separatrix

Our overall objective is to find the average rate of change of Hamiltonian over the half-cycle of motion as the particles crosses the separatrix. The first step is to find the moments in time when the particle is at the position-extrema of its motion. These values are then inserted into the limits of the integral. We suppose the particle to have positive momentum, and to be moving from left to right. The leftward extrema is the solution of $x(t) = -\pi$ or the equivalent $\sin[x(t)] = 0$. We substitute $x(z) = 2\arcsin[\tanh z] + \alpha(z)$. To first order in ϵ, there results the equation

$$\tanh z[\epsilon \sinh z + 2\,\text{sech} z(1 - \epsilon \tanh z)] = 0 . \tag{1.80}$$

We are interested in the solution for $z \simeq -\pi$. Now for $|z| > 2$, $\sinh z \to \text{sign}(z) \times \cosh z$, and therefore we may make the substitution $\tanh z \to -1$. The resulting equation, $2(1+\epsilon)+\epsilon\cosh z \sinh z = 0$, has an exact solution. The steps are: set $y = \exp(z)$ and $\cosh z = (y + 1/y)/2$ and $\sinh z = (y - 1/y)/2$; and set $X = y^2$ and solve a quadratic equation for X. Then invert the process: $z = \ln y = \ln[\sqrt{X}] = (1/2)\ln X$. We shall asume $\epsilon > 0$. The instant in time of the left extrema is:

$$\check{z} = \frac{1}{2}\log_e\left[\frac{-4 - 4\epsilon + \sqrt{16 + 32\epsilon + 17\epsilon^2}}{\epsilon}\right] \approx \frac{1}{2}\ln\left[\frac{\epsilon}{8}\right] - \frac{\epsilon}{2} + \frac{31\epsilon^2}{128} + \ldots \tag{1.81}$$

So the first approximation is $\check{z} = (1/2)\ln(\epsilon/8) < 0$. We may find the momentum value at the time $\check{z}$.

The rightward extrema is the solution of $p(t) = 0$. We substitute $p(z) = \omega_0[2\text{sech} z + \alpha'(z)]$. We are interested in the solution for $z \simeq +\pi$. Again we set $\sinh z \to \text{sign}(z)\cosh z$, and therefore we may make the substitution $\tanh z \to +1$. The resulting equation, $(2 - \epsilon) - \epsilon\cosh z \sinh z = 0$, has an exact solution; found by the steps above. The instant in time of the right extrema is:

$$\hat{z} = \frac{1}{2}\log_e\left[\frac{4 - 2\epsilon + \sqrt{16 - 16\epsilon + 5\epsilon^2}}{\epsilon}\right] \approx \frac{1}{2}\ln\left[\frac{8}{\epsilon}\right] - \frac{\epsilon}{4} - \frac{7\epsilon^2}{128} + \ldots \tag{1.82}$$

So the first approximation is $\hat{z} = (1/2)\ln(8/\epsilon) > 0$. We may find the position value at the time $\hat{z}$. For brevity, we define

$$\xi(\epsilon) \equiv \frac{4 - 2\epsilon + \sqrt{16 - 16\epsilon + 5\epsilon^2}}{-4 - 4\epsilon + \sqrt{16 + 32\epsilon + 17\epsilon^2}} \,, \tag{1.83}$$

such that $(\hat{z} - \check{z}) = \frac{1}{2}\ln(\xi)$. ξ is positive for $\epsilon \neq 0$.

The half period is the duration between the two position extrema of motion, namely:

$$\omega_0 \times \frac{\tau}{2} = \frac{1}{2}\log_e[\xi(\epsilon)] \approx \ln\left[\frac{8}{\epsilon}\right] + \frac{\epsilon}{4} - \frac{19\epsilon^2}{64} + \ldots \tag{1.84}$$

This half period is actually the separatrix crossing duration (for a trajectory starting from $x(0) = 0$). Note, when $\dot{\omega} = 0$ the half period on the separatrix is $\tau/2 \to \infty$ because the motion gets stuck at the fixed points. So the logarithmic divergence when $\dot{\omega} \neq 0$ is both a vast improvement and realistic.

We remind the reader that the previous and following results are specifically for a particle emanating from $x(0) = 0$, and the attending symmetry of motion that results from that condition. We find the change in Hamiltonian, by substituting the values of $z = \omega_0 t$ at the position extrema into the integration limits of $\Delta H(z) = 4\dot{\omega}[z - \tanh z]$. Now, we consider $|z| \simeq \pi$ and so replace $\tanh z$ by $\mathrm{sign}(z)$. The individual increments entering and leaving the separatrix are

$$\Delta H(\check{z} \to 0) \;=\; -\Delta H(0 \to \check{z}) = 2\dot{\omega}\left[-2 + \ln(8/\epsilon) + \epsilon - (31/64)\epsilon^2 + \ldots\right] \tag{1.85}$$

$$\Delta H(0 \to \hat{z}) \;=\; 2\dot{\omega}\left[-2 + \ln(8/\epsilon) - \epsilon/2 - (7/64)\epsilon^2 + \ldots\right] \tag{1.86}$$

Here we used the identity $\ln(1/a) = -\ln(a)$. The two increments are almost equal, as reflects the symmetry of the initial condition $x_0 = 0$. The total change of Hamiltonian is $\Delta H(\check{z} \to \hat{z}) = -4\dot{\omega}_0[2 + \check{z} - \hat{z}]$, or explicitly

$$\Delta H = 2\dot{\omega}_0\left(-4 + \log_e[\xi(\epsilon)]\right) \approx \dot{\omega}_0\left[-8 + 4\ln\left[\frac{8}{\epsilon}\right] + \epsilon - \frac{19}{16}\epsilon^2 + \ldots\right]$$

Now on the separatrix $H = 2\omega_0^2$; and thus $\Delta H/H \approx \epsilon[-4 + 2\ln(8/\epsilon)]$.
The time-averaged rate of separatrix crossing is

$$\begin{aligned}\overline{\dot{H}} = \frac{\Delta H}{\tau/2} &= \frac{\Delta H(\hat{z}) - \Delta H(\check{z})}{\hat{z} - \check{z}} = 4\omega_0\dot{\omega}_0\left[1 - \frac{2}{\hat{z} - \check{z}}\right] \\ &= 4\omega_0\dot{\omega}_0\left[1 - \frac{4}{\ln[\xi(\epsilon)]}\right] \approx 4\omega_0\dot{\omega}_0\left[1 - \frac{2}{\ln[8/\epsilon]} + \frac{\epsilon}{2 \times (\ln[8/\epsilon])^2} + \ldots\right]\end{aligned}$$

The limit of $\epsilon \to 0$ (i.e. infinitely slow) yields $\overline{\dot{H}} = 4\omega_0\dot{\omega}_0$ which has a simple interpretation. $H = 2m(t)\omega^2(t)$ and $\dot{H} = 4m\omega\dot{\omega} + 2\omega^2\dot{m}$. So the limit is the derivative of H at constant $m = 1$. The functions $\Delta H/H$ and $\overline{\dot{H}}/(H\omega_0)$ are shown in Fig. 1.9 versus $\epsilon = \dot{\omega}_0/\omega_0^2$. The plots tell us that fractional changes are approximately linear in ϵ.

It is possible that the reader may have expected the increment ΔH to be negative, signalling that the particle has "dropped into the well". However, all increments $\Delta H(q)$ are positive if $\epsilon > 0$. It must be recalled that all trajectories (and their Hamiltonian) are inflated as the voltage rises. Capture occurs because the Hamiltonian of particles trajectories rises less quickly than the Hamiltonian on the separatrix $H = 2V(t)$.

1.11 Asymmetric trajectories

Thus far, we have explored a single member of the separatrix family of trajectories. We now consider the asymmetric case where particles emanate from $x(0) = \tanh(q)$ when $\omega(t) = \omega(0) = \omega_0$. q is the

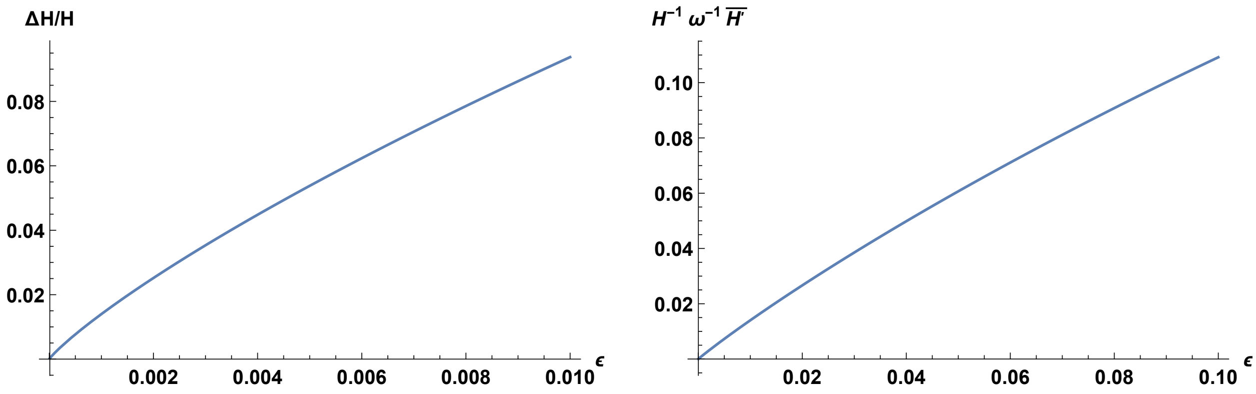

Figure 1.9: Plot of $\Delta H/H$ (left) and $\overline{\dot{H}}/(H\omega)$ (right) versus adiabaticity parameter ϵ.

initial phase of the oscillation. The starting point is the equality (1.69). We substitute the trial motion

$$x(t) = 2\arcsin\left[\tanh(q+Z)\right] + \alpha(\omega_0 t) \;\;\&\;\; p(t) = 2\,\text{sech}(q+Z) + \omega_0\alpha'(\omega_0 t) \;\;\&\;\; Z = \int_0^t \omega(u)du \quad (1.87)$$

The working is similar to above, but complicated by the initial oscillation phase q. Retaining small quantities only to first order, the differential equation for $\alpha(z)$ is

$$d/dz[\cosh(q+z)\alpha(z)] + 2\epsilon\cosh(q+z)\text{sech}q\sinh z = 0\;. \quad (1.88)$$

The solution is the perturbation

$$\begin{aligned}\alpha(z) &= \epsilon\left[-\cosh z\,\text{sech}q + \text{sech}(q+z)[1+z\tanh q]\right]\\ \alpha'(z) &= \epsilon\omega_0\left\{-\text{sech}q\sinh z - \text{sech}(q+z)[\tanh(q+z)+\tanh q[-1+z\tanh(q+z)]]\right\}\;.\end{aligned}$$

The next step is to find the moments in time corresponding to the position-extrema, and construct the change in Hamiltonian and the duration of the half cycle.

The intercept with the momentum axis occurs at $x(z) = -\pi$ or $\sin x = \sin(-\pi) = 0$. We substitute $x(z) = 2\arcsin[\tanh(q+z)] + \alpha(z)$ into $\sin x$. Expanding the trigonometric function to first order in ϵ generates the equation

$$2\text{sech}(q+z)\tanh(q+z) + \epsilon\left[-\cosh z\,\text{sech}q + \text{sech}(q+z)[1+z\tanh q]\right]\left[1-2\tanh^2(q+z)\right] = 0 \quad (1.89)$$

There is no hope of solving this analytically, unless a simplification strategy is introduced. We anticipate solutions $|z| \simeq \pi$ or much larger. Therefore, as above, we may set $\sinh z \to \text{sign}(z)\cosh z$; in which case $\tanh(q+z) \to \text{sign}(z)$. At the leftward extrema $z < 0$, so we substitute $\tanh(q+z) \to -1$. The simplified equation is

$$\epsilon\cosh z\,\text{sech}q = \text{sech}(q+z)[2\; + \epsilon\times(1+z\tanh q)]\;. \quad (1.90)$$

We contend, and shall show, that $\epsilon(1+z\tanh q)$ can be neglected in comparison to 2. The resulting equation can be solved using the steps outlined above. The leftward extrema of motion occur at the time:

$$\check{z} = \frac{1}{2}\log_e\left[\frac{4-\epsilon-\sqrt{16-8\epsilon+\epsilon^2\tanh^2 q}}{\epsilon(1+\tanh q)}\right] \approx \frac{1}{2}\ln\left[\frac{1}{8}\epsilon(1-\tanh q)\right] + \frac{\epsilon}{8} + \left[\frac{3}{128} - \frac{\tanh^2 q}{128}\right]\epsilon^2 + \ldots \quad (1.91)$$

The intercept with the position axis occurs when the momentum is zero, $p(z) = 2\omega_0 \,\text{sech}(q+z) \;+\alpha'(z) = 0$. We anticipate solutions $|z| \simeq \pi$ or much larger. At the rightward extrema $z > 0$, so following the simplification strategy we substitute $\tanh(q+z) \to +1$. The condition $p(z) = 0$ simplifies to

$$-\text{sech}q \sinh z = \text{sech}(q+z)[-2 \,+\epsilon\, +\epsilon(-1+z)\tanh q] \,. \tag{1.92}$$

We contend that the term in ϵ can be neglected in comparison to -2. The resulting equation can be solved using the substitution steps noted above. The rightward extrema of motion occurs at the time:

$$\begin{aligned} \hat{z} &= \frac{1}{2}\log_e\left[\frac{4+\epsilon\tanh q+\sqrt{16+8\epsilon\tanh q+\epsilon^2}}{\epsilon(1+\tanh q)}\right] \\ &\approx \frac{1}{2}\ln\left[\frac{8}{\epsilon(1+\tanh q)}\right] + \frac{1}{8}\epsilon\tanh q + \frac{1}{128}[1-3\tanh^2 q]\epsilon^2 + \ldots \end{aligned}$$

For brevity, we define

$$\xi(\epsilon, q) \equiv \left[\frac{4+\epsilon\tanh q+\sqrt{16+8\epsilon\tanh q+\epsilon^2}}{4-\epsilon-\sqrt{16-8\epsilon+\epsilon^2\tanh^2 q}}\right] \tag{1.93}$$

such that $(\hat{z}-\check{z}) = \frac{1}{2}\ln(\xi)$. ξ is positive for $\epsilon \leq 1$. The half-cycle, or separatrix crossing time, is $\omega_0\tau/2 = (\hat{z}-\check{z})$ given by:

$$\omega_0\frac{\tau}{2} = \frac{1}{2}\log_e \xi(\epsilon,q) \approx \ln\left[\frac{8\cosh q}{\epsilon}\right] + \frac{\epsilon}{8}[-1+\tanh q] - \frac{\epsilon^2}{64}\cosh 2q\,\text{sech}^2 q + \ldots \tag{1.94}$$

We now indicate the changes to $\check{z}$ and $\hat{z}$ if we had retained terms in ϵ that we anticipated to be small compared with ± 2. We insert the known first order values of $\check{z}$ and $\hat{z}$ into Eqs. (1.90) and (1.92), respectively, and then solve for their new values. At the left and right extrema:

$$\frac{\delta z}{\check{z}} = -\frac{\epsilon}{4}\left[\frac{1}{\check{z}}+\tanh q\right] \quad \text{and} \quad \frac{\delta z}{\hat{z}} = -\frac{\epsilon}{4}\left[\frac{1}{\hat{z}}+\tanh q - \frac{\tanh q}{\hat{z}}\right] \,. \tag{1.95}$$

Thus, the (modulus of) relative fractional changes are smaller than ϵ.

1.11.1 ΔH crossing separatrix

We are now in a position to compute the changes in Hamiltonian, and the time-averaged value $\langle \dot{H}\rangle = \Delta H/(\tau/2)$. From the integral in (1.69), it follows that $\Delta H(0 \to z) = 4\dot{\omega}_0[z+\tanh q - \tanh(q+z)]$. The simplification strategy implies $\tanh(q+z) \to \text{sign}(z)$. The individual increments entering and leaving the separatrix are

$$\begin{aligned} \Delta H(\check{z}\to 0) &= -\Delta H(0\to\check{z}) = -4\dot{\omega}[1+\check{z}+\tanh q] \\ &\approx -2\dot{\omega}_0\left[2+\ln\left[\frac{\epsilon}{8}(1-\tanh q)\right] + 2\tanh q + \frac{1}{4}\epsilon + \ldots\right] \\ \Delta H(0\to\hat{z}) &= 4\dot{\omega}[-1+\hat{z}+\tanh q] \\ &\approx -2\dot{\omega}_0\left[2+\ln\left[\frac{\epsilon}{8}(1+\tanh q)\right] - 2\tanh q - \frac{1}{4}\epsilon\tanh q + \ldots\right] \end{aligned}$$

There is an asymmetry introduced by the initial phase q of the oscillation. If q is positive then a larger fraction of the total change $\Delta H(\check{z}\to\hat{z})$ occurs before $t=0$; and for $q<0$ the larger part of ΔH occurs after $t=0$. The complete change of Hamiltonian is

$$\Delta H/\dot{\omega}_0 = -8 \;+4\ln\left[\frac{8\cosh(q)}{\epsilon}\right] \;+\frac{\epsilon}{2}\,(-1+\tanh q) \;+\ldots \tag{1.96}$$

We may form the ensemble average over the distribution of oscillation phases $\rho(q) = (1/\pi)\,\mathrm{sech}\,q$.

$$\langle \Delta H \rangle = \int_{-\infty}^{+\infty} \Delta H \times \rho(q) dq = \dot{\omega}_0 \left[-8 \ + 4\ln(16/\epsilon) - \frac{\epsilon}{2} \right] = \Delta H(q=0) + \dot{\omega}_0 \ln(16) \qquad (1.97)$$

Finally, we compute the time-averaged rate of change of Hamiltonian:

$$\begin{aligned} \overline{\dot{H}} &= \frac{\Delta H(\hat{z}) - \Delta H(\check{z})}{\tau/2} = 4\omega_0\dot{\omega}_0 \left[1 - \frac{2}{\hat{z} - \check{z}} \right] = 4\omega_0\dot{\omega}_0 \left[1 - \frac{4}{\ln[\xi(\epsilon, q)]} \right] \\ &\approx 4\omega_0\dot{\omega}_0 \left[1 - \frac{2}{\ln[(8\cosh q)/\epsilon]} + \frac{\epsilon}{4} \frac{(-1 + \tanh q)}{(\ln[(8\cosh q)/\epsilon])^2} + \ldots \right] \end{aligned}$$

1.12 Construction of the phase-space distribution

Our ultimate objective has been to construct the final distribution of Hamiltonian $\rho(H_T)$ when the voltage ramp is complete, from the initial distribution of Hamiltonian values $\rho(H_0)$ of the unbunched particle beam prior to initiating the ramp. Ideally, one value H_0 maps uniquely to a single H_T. However, there is a spread function such that each H_0 maps to a band $\tilde{H}_T + \Delta H(q)$. This implies that the density ρ at each value of H_T is contributed to by a range of H_0 values. Unfortunately, this complicates greatly the construction of ρ.

This mapping for the double-average $\tilde{H}_T$ is given in Sec. 1.8.2 to first and second order in $m = H/(2V)$. H_0^- is independent of x or q, and t; and so does not possess a time or ensemble average. Four steps must be cascaded: (1) $H_0^- \to H_0^+$; (2) $H_0 \to H_c$; (3) the increment at capture given in Sec. 1.11; and (4) $H_c \to H_T$. Remarkably, provided the voltage ramp is sufficiently slow, the result of the cascade is mostly independent of the details of the voltage ramp.

The spread function $\Delta H(q)$ represents phase mixing due to imperfect adiabaticity. The width of ΔH scales approximately as $1/T$. The final spread $\Delta H(q)$ is the sum of four sequential contributions: sudden voltage turn-on, the pre-capture rotation, the capture, and post-capture libration. The pre- and post-capture increments are given in Sec. 1.8.4, while the capture increment is given in Sec. 1.11.1.

Fig. 1.10 is a cartoon of the construction of $\rho(H_T)$ from the relationship between the average value $\tilde{H}(T)$ and $H(0)$ and from the knowledge of the distribution of the deviations $\rho(\Delta H)$ which is derived in Sec. 1.8.6. The figure depicts a case which is unrealistic, constant ΔH, but that is so mathematically simple that all steps in the construction can be achieved analytically. But it must be emphasised: numerical methods must be adopted for realistic examples.

There are some conceptual and notational challenges. We reiterate the notation.

- H_T is a general value of Hamiltonian at time $t = T$; it could correspond to any value of q.
- $\tilde{H}_T = \langle \overline{H(T, q)} \rangle$ is the double average.
- H_0 is a general value of Hamiltonian at time $t = 0$; it could correspond to any q value.
- $\Delta H \equiv a^2 + b^2$ where a, b are the coefficients of $\cos 2q$ and $\sin 2q$ respectively. ΔH is introduced in section 1.8.6. It is the function at $t = T$, and its value depends (in principle) on H_0.

To understand the need for two similar symbols, H_T, $\tilde{H}_T$, it may help to remember that the time evolution of a particle having Hamiltonian H_T may differ depending on its oscillation phase q, because its response to the perturbation $\dot{V}$ differs. We found the Hamiltonian at the end of the capture process, $\tilde{H}_T$, to depend on the value before the sudden voltage turn on (H_0^-):

$$\tilde{H}_T = (H_0^- + V_0)\sqrt{V_T/V_0} \quad \text{for } H_0^- < V_0; \text{ and } \quad \tilde{H}_T = 2\sqrt{H_0^- V_T} \quad \text{for } H_0^- \geq V_0. \qquad (1.98)$$

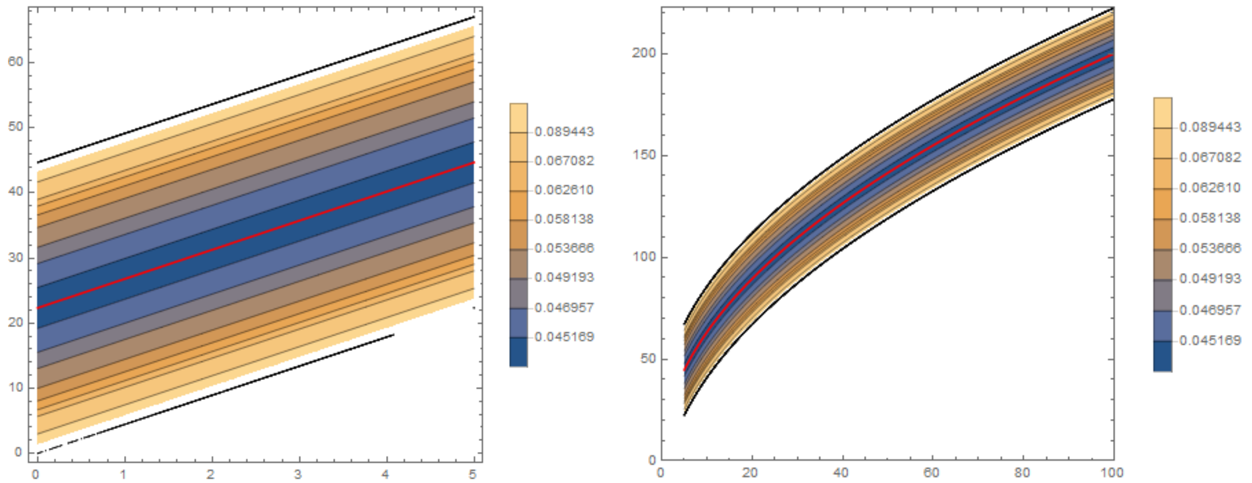

Figure 1.10: Construction the relationship between the final particle density $\rho(H_T)$ and the initial particle density $\rho(H_0)$. The plots show the dwell function $[\Delta H^2 - (H_T - \tilde{H}_T)^2]^{(-1/2)}$. The red curve is the double average $\tilde{H}$. Left Fig: $\tilde{H}_T = (H_0^- + V_0)\sqrt{V_T/V_0}$. Right Fig: $\tilde{H}_T = 2\sqrt{H_0^- V_T}$. The numerical values are $V_0 = 5$, $V_T = 100$, $\Delta H = \sqrt{V_0 V_T}$. The abscissa and ordinate are the initial and final Hamiltonian values H_T and H_0, respectively. $\rho(H_T)$ is found by summing over all values $\rho(H_0)$ that fall on the line $H_T =$ constant.

The lower range of H_0^- gives the effect of the sudden voltage turn on. We suppose that every H_0^- contributes a density to the final form of the total particle density. Each of these contributions is of the form

$$\rho(H_T, H_0) = \rho(H_0)/\left[\Delta H^2 - (H_T - \tilde{H}_T)^2\right]^{1/2}/\pi\ . \tag{1.99}$$

We substitute either of the expressions Eq. (1.98) into Eq. (1.99) to eliminate $\tilde{H}_T$ in favour of H_0. Here the active free variable is H_T, and H_0 is a parameter. We shall sum over all values H_0^- that can contribute density to $\rho(H_T)$. Let $\mathcal{I}(H_T, H_0)$ be the indefinite integral of $\rho(H_T, H_0)$ with respect to H_0. Let the lower and upper values of the range of H_0 be notated H_a and H_b, respectively. The density contributed by that range is

$$\rho(H_T) = \mathcal{I}(H_T, H_b) - \mathcal{I}(H_T, H_a)\ .$$

The ranges of integration $H_0 = [H_a, H_b]$ are complicated and must be formed systematically, as below. The ranges become further complicated if ΔH is a function of H_0; which is actually the case. But, for simplicity, we shall take ΔH as constant.

In principle, the variance ΔH may depend on H_0. It is simple to find ΔH corresponding to $H_0 = 0$. Using the condition $H_T > 0$ for H_0 implies that $\Delta H(H_0 = 0) \leq \sqrt{V_0 V_T}$. In fact, we know the equality to apply; because this is the spread from sudden turn-on. For compactness, we write $C = V_T/V_0$ be the ratio of final to initial voltage; and $L_T \equiv \sqrt{V_0 V_T} = V_0\sqrt{C}$ be the lift at $t = T$.

Lower integration range $H_0 \leq V_0$

We sum over all values $0 \leq H_0^- \leq V_0$ that can contribute density to $\rho(H_T)$. We substitute $\tilde{H}_T = (H_0^- + V_0)\sqrt{C}$. The density Eq. 1.99 is singular at the zeros of its denominator; the argument of the square root can be written

$$\Delta H^2 - (H_T - \tilde{H}_T)^2 \equiv (\Delta H + H_T - \tilde{H}_T)(\Delta H - H_T + \tilde{H}_T)\ .$$

From this it follows that the indefinite integral has two possible forms, depending on the range of H_T. We want the integral relevant[20] to $(\Delta H + H_T - \tilde{H}_T) \geq 0$ and $\rho(H_T)$ a real quanity (i.e. not complex). Suppose for example that the initial particle density $\rho(p)$ is constant. The corresponding density $\rho(H_0) \propto 1/\sqrt{H_0}$.
Let $B = \Delta H + H_T - L_T$. The indefinite integral is

$$\mathcal{I}(H_T, H_0) = \frac{1}{\pi\sqrt{B\sqrt{C}}}\mathcal{F}\left[-\arcsin\sqrt{\frac{B - H_0\sqrt{C}}{2\Delta H}}, \frac{2\Delta H}{B}\right]. \tag{1.100}$$

Here $\mathcal{F}(\phi, m)$ is the (incomplete) elliptic integral of the first kind; and $\mathcal{F}(\pi/2, m) = \mathcal{K}(m)$. The upper and lower limits of H_0 depend on the range of H_T; there are three such ranges.

$\underline{L_T - \Delta H \leq H_T \leq 2L_T - \Delta H}$
Substitute into Eq. 1.100 the lower and upper limits $H_0 = 0$ and $H_0 = B/\sqrt{C}$, respectively.

$\underline{2L_T - \Delta H \leq H_T \leq L_T + \Delta H}$
Substitute into Eq. 1.100 the lower and upper limits $H_0 = 0$ and $H_0 = V_0$, respectively.

$\underline{L_T + \Delta H \leq H_T \leq 2L_T + \Delta H}$
Substitute the lower and upper limits $H_0 = (-\Delta H + H_T - L_T)/\sqrt{C}$ and $H_0 = V_0$, respectively.

Figure 1.11 blue curve shows the function $\rho(T)$ across the lower range of H_T for the case that $\Delta H = L_T = \sqrt{V_0 V_T}$. In the case that $\hat{H}_T = (H_0^- + V_0)\sqrt{C}$ the dwell function is singular when H_0 equals 0 or V_0. And when ΔH is constant, the singular behaviour survives integration of H_0 to give a peak at $H_T = \Delta H + V_0\sqrt{C}$. However, in practice ΔH is a falling function of H_0 over the lower range, because other non-adiabatic processes have not yet contributed to the spread. Thus a bump rather than peak is seen in realistic cases. Depending on the details of the deviation ΔH there may be a small dimple for values $H_T \to 0$. The falling density for $H_T > \Delta H + V_0\sqrt{C}$ is compensated by rising density contributed by the integral for the upper range $H_0 \geq V_0$.

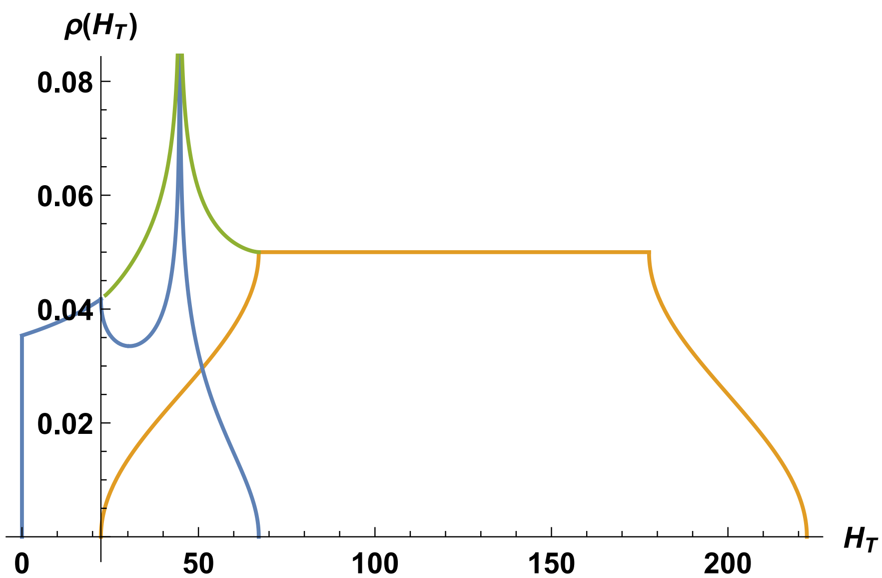

Figure 1.11: Final density $\rho(H_T)$ for $\rho(p)$ =constant and $\rho(H_0) = 1/\sqrt{H_0}$. Blue/gold curves: density contributed by lower and upper integration ranges of H_0. Olive-green curve: sum of contributions over the overlapping range of H_0.

[20] Wolfram *Mathematica*-11 finds the other form of the integral, relevant to $\Delta H - H_T + \tilde{H}_T > 0$.

Upper integration range $H_0 \geq V_0$

We sum over all values $V_0 \leq H_0^- \leq \hat{H}_0$ that can contribute density to $\rho(H_T)$. For compactness, let the maximum $\hat{H}_T \equiv 2\sqrt{\hat{H}_0 V_T}$. We substitute the average $\tilde{H}_T = 2\sqrt{H_0 V_T}$. The integration range is the solution of $\Delta H^2 - (H_T - 2\sqrt{H_0 V_T})^2 = 0$, namely $H_0 = (H_T \pm \Delta H)^2/(4V_T)$.

Suppose for example that the initial particle density $\rho(p)$ is constant. The corresponding density $\rho(H_0) \propto 1/\sqrt{H_0}$. For brevity, let $B = H_T - 2\sqrt{H_0 V_T}$. The indefinite integral is

$$\mathcal{I}(H_T, H_0) = \int^{H_0} \frac{1}{\sqrt{H_0}} \rho(H_T, H_0) dH_0 = \frac{-1}{2\pi\sqrt{V_T}} \arctan\left[\frac{B}{\sqrt{\Delta H^2 - B^2}}\right] . \tag{1.101}$$

The integral is simplified by a change of variable to $u = \sqrt{H_0}$. The upper and lower limits of H_0 depend on the range of H_T; there are three such ranges.

<u>$2L_T - \Delta H \leq H_T \leq 2L_T + \Delta H$</u>

Substitute the lower and upper limits $H_0 = V_0$ and $H_0 = (\Delta H + H_T)^2/(4V_T)$, respectively. At the upper limit, the arc tangent tends to $-\pi/2$.

<u>$2L_T + \Delta H \leq H_T \leq \hat{H}_T - \Delta H$</u>

Substitute the lower and upper limits $H_0 = (-\Delta H + H_T)^2/(4V_T)$ and $H_0 = (\Delta H + H_T)^2/(4V_T)$, respectively. At the integration limits, the argument of the arc-tangent becomes $\pm 1/0 \to \infty$, and therefore the angle becomes $\pm\pi/2$. Hence the integral is $\rho(H_T) = 1/(2\sqrt{V_T})$; and the final density is constant (over this range of H_T) when $\rho(p)$ is constant.

<u>$\hat{H}_T - \Delta H \leq H_T \leq \hat{H}_T + \Delta H$</u>

Substitute the lower and upper limits $H_0 = (-\Delta H + H_T)^2/(4V_T)$ and $H_0 = V_T$, respectively. At the lower limit, the arc tangent tends to $+\pi/2$.

Figure 1.11 gold-curve shows the function $\rho(T)$ across the upper range of H_T for the case that $\Delta H = L_T = \sqrt{V_0 V_T}$. The rising density at lower H_T compensates the falling density contributed by the integral for the lower range $H_0 \leq V_0$. At the upper end of H_T, the density $\rho(H_T, H_0)$ is truncated because the maximum value of H_0 is limited to $\hat{H}_0$. This adjustment has the effect of introducing a tail at the largest H_T values.

Note, we have assumed $2L_T + \Delta H < \hat{H}_T - \Delta H$. In extreme cases, V_0 or ΔH or both may be sufficiently large that the inequality is reversed. When this occurs, the three integration ranges are re-ordered; and the outcome $\rho(H_T)$ is markedly different.

Falling density $\rho(p)$

We can consider a more realistic form for the density prior to voltage turn-on: $\rho(p) = [1 - (p/\hat{p})^2]$. The corresponding density immediately before voltage turn on is $\rho(H_0) = [1 - H_0/\hat{H}_0]/\sqrt{H_0}$. We sum over all H_0^- values that can contribute density to $\rho(H_T)$. We suppose that V_0/V_T is vanishingly small, and that the lower integration range $H_0^- < V_0$ can be neglected. We suppose again that the spread ΔH is constant. The integration limits are $H_a = (-\Delta H + H_T)^2/(4V_T)$ and $H_b = (\Delta H + H_T)^2/(4V_T)$. The final density is

$$\rho(H_T) = \int_{H_a}^{H_b} \frac{(1 - H_0/\hat{H}_0)}{\sqrt{H_0}} \rho(H_T, H_0) dH_0 \;=\; \frac{8\hat{H}_0 V_T - 2H_T^2 - \Delta H^2}{16\hat{H}_0 V_T^{3/2}} . \tag{1.102}$$

Below we show the intermediate result before the integration limits are inserted.

$$\left\{ \sqrt{\Delta H^2 - B^2}[3H_T + 2\sqrt{H_0 V_T}] + [\Delta H^2 + 2H_T^2 - 8\hat{H}_0 V_T] \arctan\left[\frac{B}{\sqrt{\Delta H^2 - B^2}}\right] \right\} \Bigg|_{H_a}^{H_b}$$

At the limits H_a, H_b, $\Delta H^2 - B^2 = 0$ and the arc-tangent contributes $\pm\pi/2$. In the case that the spreads ΔH are negligible, the density tends to $\rho(H_T) = (4\hat{H}_0 V_T - H_T^2)/(8\hat{H}_0 V_T^{3/2})$; this is identical in form to the initial distribution as a function of momentum $\rho(p) = [1 - (p/\hat{p})^2]$. The spread ΔH is small if the adiabaticity condition is satisfied throughout the entire process, and the sudden turn-on voltage V_0 is very small compared to the final value V_T.

Varying spread ΔH

Above we took the spread to be constant, leading to fairly simple expressions for the density $\rho(H_T)$. The general case of variation $\Delta H(H_0)$ is intractable, except by numerical methods. However, the case where $\Delta H = \Delta H_n \sqrt{1 + a_n H_0}$ can be pursued to analytic results for $\rho(H_T)$. The constants ΔH_n and a_n can be chosen piecewise subject to the continuity of the pieces. The index n numbers the segments.

Complication

There is a complication in performing the sum of the three sequential spreads to find the nett effect: the ranges of phase q are different. The pre- and post-capture increments are given accurately for $m > 2$ and $m < 1/2$, respectively, and have range $q \approx [-\pi, +\pi]$; whereas the separatrix crossing increment is valid for $m = 1$, and has range $q \to \pm\infty$. However, if $\Delta H(q)$ is weighted by the distribution of phase on the separatrix $\rho(q) = (1/\pi)\,\mathrm{sech}\, q$ the range may be reduced to $\pm\pi$ because a very small fraction (5%) of particles lies beyond $q = \pi$. This complication is in fact an artefact, caused by using simple approximate functions in place of the real ones. We replaced the initial modulated distribution of phase q (which is depopulated at the extrema of q) by a uniform one. Later, when calculating form factors, we replaced Jacobi functions by trigonometric ones. We replaced the actual range of phase $Q = [-2K(m, +2K(m)]$ by the range $q = [-\pi, +\pi]$. There is a continuum of values m, and a corresponding continuum of ranges of Q. In that context, the distribution of oscillation phase on the separatrix is part of a natural progression, not an exception.

1.12.1 Transformation of particle density functions

The general transformation of density functions is $\rho(X) = \rho(Y)(dY/dX)$ evaluated at $Y(X)$. We shall make three conversions of this form, and use suffixes to distinguish between them.

- $\rho_0(p)$ = density as function of momentum (prior to voltage turn-on)
- $\rho_1(H_0)$ = density as function of initial hamiltonian value; where $H_0 \propto p^2$
- $\rho_2(H_T)$ = density as function of final hamiltonian value; where $H_T \propto \sqrt{H_0}$
- $\rho_1(H_0) = (dp/dH_0)\rho_0(p(H_0))$
- $\rho_2(H_T) = (dH_0/dH_T)\rho_1(H_0(H_T)) = (dH_0/dH_T)(dp/dH_0)\rho_0(p(H_0(H_T)))$
- $H_0 = \frac{1}{2}Ap^2$; so $dH_0/dp = Ap$ and $p = \sqrt{2H_0/A}$
- $H_T = 2\sqrt{H_0 V_T}$; so $dH_T/dH_0 = \sqrt{V_T/H_0}$ and $H_0 = H_T^2 V_T/4$.
- $p = H_T\sqrt{V_T}/\sqrt{2A}$
- $$\rho_2(H_T) = \frac{1}{Ap}\sqrt{\frac{H_0}{V_T}} \times \rho_0(p(H_0(H_T))) = \frac{1}{\sqrt{2AV_T}} \times \rho_0\left[\frac{H_T\sqrt{V_T}}{\sqrt{2A}}\right]$$

Thus the initial and final particle densities $\rho_0(p)$ and $\rho_2(H_T)$, respectively, of an adiabatic capture have the same functional form. For example if $\rho(p) \propto [1 - (p/\hat{p})^2]$ then $\rho(H_T) \propto [1 - (H_T/\hat{H}_T)^2]$. Here the circumflex $\hat{X}$ denotes maximum value of X.

1.13 Separatrix crossing points

We defined the separatrix family (abbreviated SF) at the instant t_0 to be the entirety of trajectories that touch the curve $(x_0, p_0 = \pm\omega_0\sqrt{1+\cos x_0})$ at $t_0 = 0$; where $\omega_0 = \sqrt{A \times V(t_0)}$. We call this curve (x_0, p_0) the instantaneous separatrix (abbreviated IS). The moment t_0 could be at any time during the voltage ramp. If $t_0 = T$ is the terminus of the ramp, such that the voltage becomes frozen at the value $V(T)$, then all these trajectories will eventually (at $t > T$) terminate on the fixed points $x = \pm\pi$. If $t_0 < T$, then all these trajectories will be swept into the interior of the growing IS. We may trace (in x, p space) the trajectories backward and forward in time from t_0, and superpose the instantaneous curve (x_0, p_0). Examples are shown in Fig. 1.8. It is observed that each member of the SF touches the IS twice. Cursory inspection of such a graph may lead to the identification of (x_0, p_0) as a grazing point, and $\approx (-x_0, p_0)$ as a crossing point. However, this is an illusion: (x_0, p_0) *is* the crossing point, and $\approx (-x_0, p_0)$ has little or no significance. Explaining that statement provides an opportunity to expound upon several points: (1) the nature of phase space, and the peculiarities of trajetcories when the Hamiltonian is time dependent; and (2) demonstrate an alternative way to genereate the separatrix family - by Taylor series.

But first a simple observation. What appears to be the grazing point, is actually the crossing point - because the IS is moving! It is easy to make the mistake of thinking the IS is frozen; but it is, in fact, expanding. If at $t = 0$ the SF of trajectories each touch the IS, then at $t < 0$ they move toward the IS; and at $t > 0$ they all move away from the IS. At the apparent left crossing point, when $x_0 > 0$, the time is earlier ($t < 0$) and the IS would have been smaller, and the family member definitely outside (so actually no crossing). At the apparent right crossing point, when $x_0 < 0$, the time is later ($t > 0$) and the IS will have been larger, and the family member definitely inside (so, actually, not crossing). On the apparent grazing point, $H(t_0) = 2V(t_0)$; shortly before $H(t_0 - \delta t) > 2V(t - \delta t_0)$; and shortly after $H(t_0 + \delta t) < 2V(t_0 + \delta t)$ where $\delta t > 0$ is a small quantity; and so it truly is a crossing point.

In the most restrictive sense, the phase space is the entire configuration of points (x, p) accessible to the dynamical system; if we stop at that interpretation there is no problem. But we may also think of the phase space as being occupied by trajectories, or flow lines. If that view is adopted, then we must not forget that different trajectories progress/advance (in phase space) at different rates. The interpretation of trajectories in the phase space becomes even more perilous when the Hamiltonian varies. If trajectories cross, there is no physical inference to be made about a shared or coincident time; because the traversal of a given graphical crossing point (x, p) occurs at different times for each the two trajectories. The apparent crossing points are illusory and irrelevant because they occur at times ($\neq t_0$) when the IS would have been or will be at a different location.

1.13.1 Bounding curves

For convenience we shall move the origin of time such that $t_0 = 0$. The separatrix family is the manifold of trajectories that are captured at $t = 0$. To visualize this family, locally in time, one may trace backward in time all the trajectories terminating on the IS at $t = 0$. By "locally" we mean the interval $t = [-\tau, 0]$ where τ is the libration period of small amplitude motion. The tracing may be accomplished either by numerical integration or by the methods introduced in Sec. 1.11. Fig 1.12-left shows the result for positive momenta moving rightward. Trajectories are bounded in the near past by a single composite curve: the IS and the SF member terminating on $x_0 \to +\pi$, whichever is the larger value of p. This boundary is slighly larger than the one introduced by Sayasov, who omits the contribution of the IS, and reflects that all trajectories on the IS at $t_0 = 0$ are captured. There are similar findings for the negative momenta travelling left toward $x_0 = -\pi$. The methods of Sec. 1.11 cannot be extrapolated beyond $\pm\tau$. To access earlier (or later) times we must rely on numerical integration of the motion equations. If the trajectories are traced into the distant past, we find that the SF trajectory terminating on $x_0 \to \pi$ is nolonger a bounding trajectory. This is because there are additional trajectory crossings (one by another) during the rotation era that modify the stratification of the flowlines.

One may also visualize the SF, locally, going forward in time for the interval $t = [0, +\tau]$. Fig 1.12-right shows the result for positive momenta moving rightward. All trajectories are bounded in the near future by a single composite curve: the IS and the SF member eminating from $x_0 \to -\pi$, whichever is the larger value of p. There are similar findings for negative momenta travelling left.

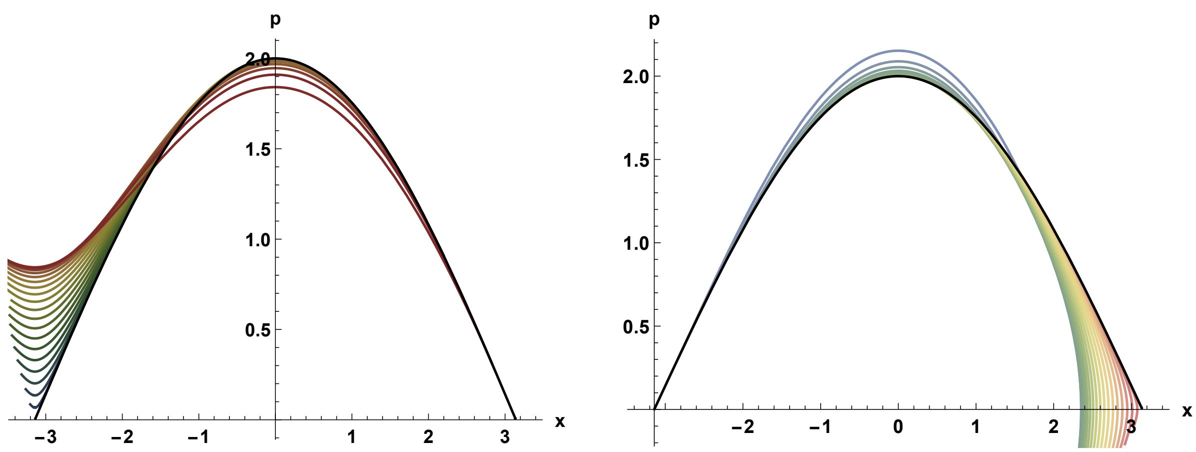

Figure 1.12: Left: trajectories terminating on instantaneous separatrix (IS) traced backward in the interval $t = [-\tau, 0]$. Right: trajectories eminating from the IS traced forward in the interval $t = [0, +\tau]$. Darker and lighter shades of the same colour denote the same trajectory at eralier and later times, respectively. Frequency slew rate $\beta = \epsilon \times \omega = 0.05$.

1.13.2 Vector methods

Vector methods simplify the task of determining the separation or angle between two arc segments (in terms of vector scalar or cross products). Accordingly, we introduce locus vectors (in phase space) for the unperturbed motion $\mathbf{X} = [x(z), x'(z)]$ and perturbation $\mathbf{A} = [\alpha(z), \alpha'(z)]$. These dimensionless quantities can be manipulated just like geometrical vectors. As time flows on, the vector $\mathbf{X}$ moves along the IS. The tangent vector is $\mathbf{X}' \equiv d\mathbf{X}/dz = [x'(z), x''(z)]$. The unit tangent is $\mathbf{X}'/|\mathbf{X}'|$. The unit normal is $\mathbf{n} = [-x''(z), x'(z)]/|\mathbf{X}'|$. Likewise the tangent vector to $\mathbf{A}$ is $\mathbf{A}' = d\mathbf{A}/dz = [\alpha'(z), \alpha''(z)]$. Specifically, the instantaneous separatrix is the locus generated by $\mathbf{X} = [2 \arcsin[\tanh(q + z)], 2\,\text{sech}(q + z)]$, where $z = \omega_0 t$ and q is the initial phase. The unit normal is $\mathbf{n} = [\sinh(q + z), \cosh(q + z)]/\sqrt{\cosh[2(q + z)]}$.

1.13.3 Crossing point

If $\omega(t)$ is held constant, the IS becomes frozen. A trajectory on the IS is of the form $\mathbf{X}$. A real trajectory is of the form $\mathbf{r} = \mathbf{X} + \mathbf{A}$ where $\mathbf{A}$ is the perturbative motion induced by $\dot{\omega}$. The tangent vectors to these locii are the derivatives $\mathbf{X}'$ and $\mathbf{r}'$. The cross product of the two tangent vectors is $\mathbf{X}' \wedge \mathbf{r}' = \mathbf{X}' \wedge \mathbf{A}'$. At the crossing point $\mathbf{A}' = [0, 0]$, and so the product is zero; and the vectors are locally parallel. This property follows from two causes: (1) the motion starts precisely at $t = 0$ when $\omega(t) = \omega_0$; and (2) the imposed initial conditions $\alpha(0) = \alpha'(0) = 0$. The former implies there has been no time for additional forces to develop and thus $\alpha''(0) = 0$. From these causes it follows that $\alpha'' \propto \epsilon z$ and $\alpha \propto \epsilon z^3$ locally in the vicinity of the touching point. If there had been a $\Delta\omega$ at $t = 0$ to drive the perturbation, things would have turned out differently. If $\omega(t)$ is allowed to vary, the IS expands. There is no real trajectory of that form, but it is a useful fiction. Under this condition, the tangent vector to the IS becomes $\mathbf{X}' + \epsilon\mathbf{X}$; the product of tangent vectors gains an additional term, and is no longer zero at the crossing point; thus a crossing of the locii emerges.

1.13.4 Taylor expansion

We define the separatrix family at the instant t_0 to be the entirety of trajectories that touch the curve $(x_0, p_0 = \pm\omega_0\sqrt{1+\cos x_0})$ at $t_0 = 0$. We now demonstrate how to find family members using Taylor series. Let $z = \omega_0 t$ be dimensionless time. We introduce the trial function $x(z) = \sum_{k=0}^{N} x_k z^k$. The initial conditions at $t = 0$ imply $x_0 = x(0)$ and $x_1 = x'(0) = \dot{x}(0)/\omega = p(0)/\omega$. The equation of motion is $z'' + (\omega(z)/\omega_0)^2 \sin[x(z)]$. The secret to simplicity is to avoid expanding the sine in a power series. Instead, take derivatives and insert $z = 0$.

$$(n+2)!\, x_{n+2} \;+\; \frac{1}{\omega_0^2}\left(\frac{d}{dz}\right)^n \left[\omega^2(z) \times \sin x(z)\right]\Big|_{z=0} = 0\,, \tag{1.103}$$

allows us to read off the coefficients x_k (in tems of lower coefficients in the series). We are invoking super-determinism: the entire future of $x(z)$ is predictable form the infinite set of derivatives at $t = 0$. It is a matter of choice at what order we truncate the Taylor series for $\omega(z)$. If we expect that ω changes little over the time scale appropriate to $x(z)$, then we may truncate to second order: ω''' and higher order terms are discarded. Let $\omega' = \epsilon\omega_0$ and $\omega'' = \delta\omega_0$ where ϵ, δ are dimensionless. For example, the first three coefficients are:

$$\begin{aligned} 2\,x_2 &= -\sin x_0\,, \qquad 6\,x_3 = -[2\epsilon \sin x_0 \;+ x_1 \cos x_0] \\ 24\,x_4 &= [-2(\delta+\epsilon^2) + \cos x_0]\sin x_0 \;- 4\epsilon\, x_1 \cos x_0 + x_1^2 \sin x_0 \end{aligned}$$

$x(z) \equiv x[z, \epsilon, \delta]$ is the complete solution including the effect of the perturbation $\omega(t)$. To find the perturbation, we form the difference $\alpha(z) = x[z, \epsilon, \delta] - x[z, 0, 0]$, and likewise for $\alpha'(z)$. For brevity and simplicity, we shall assume that the terms in ϵ^2 and δ may be discarded. Truncated to order z^5 the perturbation is:

$$\alpha = -\frac{1}{3}z^3 \epsilon \sin x_0 \;-\frac{1}{6}z^4 \epsilon\, x_1 \cos x_0 \;+\frac{1}{60}z^5 \epsilon[2\sin(2x_0) + 3x_1^2 \sin x_0]\,. \tag{1.104}$$

As $x_0 \to 0$, the coefficients of odd powers (z^3, z^5, etc) tend to zero, and we must take higher even powers such as z^6.

Tracing the trajectory $\mathbf{A} = [\alpha(z), \alpha'(z)]$ about $z = 0$ for the time interval $t = [-\tau, +\tau]$ leads to distinctive cusp and[21] lobed trajectories. If $x_0 < 0$ thus cusp appears for $\alpha' > 0$; and if $x_0 > 0$ the cusp appears at $\alpha' 0$. As $x_0 \to 0$, the lobes shrink to zero. Examples are given in Fig. 1.8 for the cases $x_0 = \pm\pi/2$. Additional examples are provided in Fig. 1.13.

1.13.5 Other crossing point

As noted above, the apparent crossing point $\approx (-x_0, p_0)$ is bogus and (in itself) of no interest. However, it does betray an interesting property of the perturbative motion: that it returns momentarily to zero. We look for solutions $z \neq 0$ of $\alpha(z) = 0$ and $\alpha'(z) = 0$; and denote these instances by z_x and z_p, respectively. The equation $\alpha(z) = 0$ has two distinct solutions (in addition to three repeated roots at $z = 0$). The relevant solution tends to zero as $x_0 \to 0$. The perturbation touches the separatrix, so we substitute $x_1 = \sqrt{2}\sqrt{1+\cos x_0} = 2\cos(x_0/2)$, yielding

$$z_x = \frac{10\cos x_0 - \sqrt{10}\sqrt{(9 + 16\cos x_0 - 15\cos 2x_0)}}{4(3 + 5\cos x_0)\sin(x_0/2)} \approx -x_0 + \frac{41x_0^3}{120} + \ldots \tag{1.105}$$

[21] Cusp (coloquial): a pointed end where two curves meet. Lobe (coloquial): a curved or rounded projection.

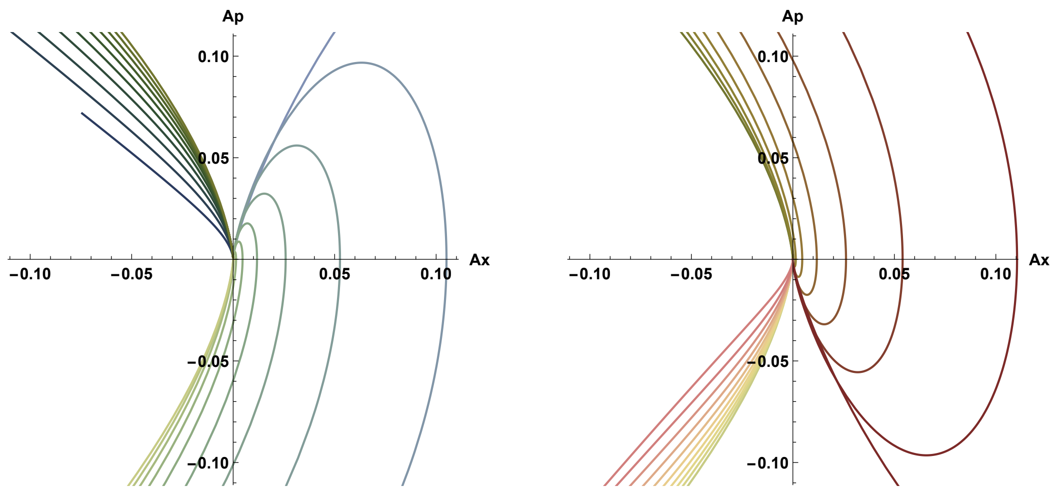

Figure 1.13: Perturbative trajectories $[A_p, A_x] \equiv [\alpha, \alpha']$ for the interval $t = [-\tau, +\tau]$. Left: trajectories such that $x_0 \leq 0$ at $t_0 = 0$. Right: trajectories such that $x_0 \geq 0$ at $t_0 = 0$. Within a single plot, darker and lighter versions of the same colour denote the same trajectory at earlier and later times, respectively. Frequency slew rate $\beta = \epsilon \times \omega = 0.05$.

The roots of $\alpha'(z) = 0$ have properties similar to those of α; the relevant solution tends to zero as $x_0 \to 0$, namely

$$z_p = \frac{4\cos x_0 - \sqrt{(14 + 24\cos x_0 - 22\cos 2x_0)}}{2(3 + 5\cos x_0)\sin(x_0/2)} \approx -\frac{3x_0}{4} + \frac{7x_0^3}{32} + \ldots \tag{1.106}$$

Because we are dealing with Taylor series expansions, the domain of validity is $|z_x| < 1$ and $|z_p| < 1$ and $|x_0| < 1$. The roots z_x, z_p are unequal (unless $x_0 = 0$) and they display different symmetries depending on the sign of the initial position coordinate x_0. If $x_0 = 0$ the apparent grazing point and bogus crossing point coalesce into a single crossing point.

It is not entirely clear what is the implication of $z_x \neq z_p$. Nevertheless, we may insert the solution z_x into the motion $\mathbf{X} + \mathbf{A}$ to find (an approximation for) the other crossing point. Now at $t = 0$, $x_0 = 2\arcsin[\tanh(q + z)]$ implies $x_0 \approx 2q$. Additionally $\mathbf{A}(z_x = -x_0) \approx [0, 0]$; and so the other crossing point is $\approx \mathbf{X}(z_x = -2q) = [2\arcsin[\tanh(-q)], 2\,\mathrm{sech}(-q)] = [-x_0, p_0]$.

1.13.6 Separation of locii

We make another estimate of the bogus crossing based on the separation between the perturbed and unperturbed trajectories. Because $\alpha(z)$ and $x[z, 0, 0]$ are in motion away from the common initial point x_0, p_0, the separation has to be measured in the co-moving frame, and is measured along the unit-normal vector, $\mathbf{n}$. The separation is $S = \mathbf{n} \cdot \mathbf{A} = \alpha \sin\theta + \alpha' \cos\theta$ where $\cos\theta$ is the direction cosine between $\mathbf{n}$ and $(0, 1)$. The sign of θ derives from the sign of x_0. Fig. 1.14 shows examples of $S(z)$. We develop an approximation for S for short timescale $|z| < 1$ and small angle $|x_0| < \pi/4$. To first order in z, the unit normal is

$$\mathbf{n}(z) \approx \frac{1}{\sqrt{1 + \sin^2(x_0/2)}} [\sin(x_0/2) + z\,\mathcal{C}, 1 - z\,\mathcal{C}\sin(x_0/2)] \ , \quad \mathcal{C} \equiv \frac{\cos^2(x_0/2)}{1 + \sin^2(x_0/2)} \ . \tag{1.107}$$

We truncate $\mathbf{A}$ to order z^4, that is

$$\mathbf{A} \approx -\epsilon\, z^2 \left[\frac{1}{3} z \sin x_0 + \frac{1}{6} z^2\, x_1 \cos x_0,\ \sin x_0 + \frac{2}{3} z\, x_1 \cos x_0\right] \ . \tag{1.108}$$

The perturbation emanates from the separatrix, so $x_1 = \sqrt{2}\sqrt{1+\cos x_0} = 2\cos(x_0/2)$. The equation $S = \mathbf{n}\cdot\mathbf{A} = 0$ is the product of a cubic and quadratic in z. There are three distinct roots (in addition to $z = 0$), one real and two complex. The real root is the location in time z_b of the other IS crossing:

$$z_b \approx -\frac{3x_0}{4} - \frac{91x_0^3}{256} \,. \tag{1.109}$$

The sign of z_b is opposite to that of x_0. So if $x_0 > 0$, the crossing occurs before $t = 0$; and if $x_0 < 0$, the crossing occurs after $t = 0$. We have demonstrated this property for small values of x_0, but it is a general property of the separatrix family of curves. The time ordering of z_b with respect to the zeros of α, α' is $z_x < z_b < z_p$ for $x_0 > 0$; and $z_p < z_b < z_x$ for $x_0 < 0$.

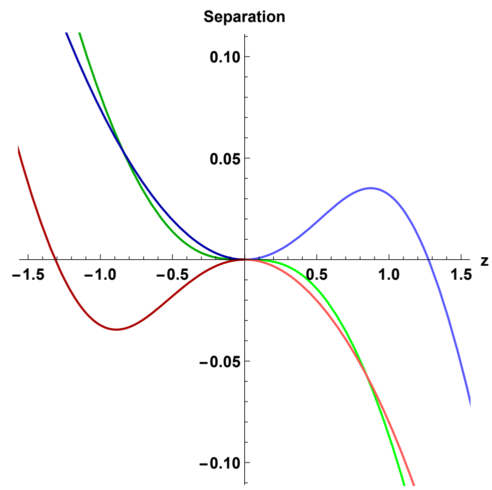

Figure 1.14: Three separation curves $S(z)$: emanating from $x_0 = 0$ (green), $x_0 = -\pi/2$ (blue) and $x_0 = +\pi/2$ (red). The tone, darker or lighter, indicates before or after $t = 0$.

1.13.7 Different form for the perturbation α

In this section we made a strict division between the induced perturbation α and pure unperturbed basis; such that the function α has the property $\alpha''(0) = 0$ at the moment when $\omega(t) = \omega_0$. In preceding articles, in Secs. 1.10 and 1.11, we found certain expressions for α, α' in terms of expansions in $y = e^z$ These particular functions do not have the precise property that $\alpha'' = 0$ at $z = 0$; and so $\mathbf{A}'$ is not precisely parallel to $\mathbf{X}'$ at $z = 0$. There is a subtle mathematical reason for this discrepancy: we took for our basis function Eq. (1.87) wherein the argument of the hyperbolic functions was not $z = \omega_0 t$ but $Z = \int_0^t \omega(u)du$. This gives the basis functions a flavour of the perturbation $\omega(t)$; the chosen basis was not pure un-perturbed functions. There are three consequences. First, that $\alpha''(0) = -2\epsilon\,\mathrm{sech}\,q$ at $z = 0$. Second, that the mathematical expressions for α, α' are much simpler, facilitating simple expressions for the extrema of motion. Third, provided that we retain those forms for α, α' it is legitimate to write $x(t) = 2\arcsin[\tanh(q + \omega_0 t + \dot{\omega}_0 t^2/2)]$ for $|\dot{\omega}_0 t| \ll \omega_0$.

1.14 Conclusion

We have started from the premise that dilution of the density of particles in phase-space is equivalent to and a surrogate for emittance increase. *Equivalent* because the areas contained within contours of constant density (knowns as percentage emittance) expand during a non-adiabatic process. In this article we presented all the ingredients for a calculation of the final ($t = T$) phase-space density

ρ, as a function of initial Hamiltonian values, resulting from a near-adiabatic transformation of the time-dependent Hamiltonian from $H(0) = \frac{1}{2}Ap^2$ to $H(T) = H(0) + V(T)[1 - \cos x]$. Analytic forms are given for all the ingredients, including the ΔH increments arising at the time of capture known as separatrix crossing. Unfortunately, the cascading of four processes leads to complexity; and so combination of the ingredients requires numerical computation. Consequently, the goal of a single formula that exactly describes $\rho[H(T)]$ in terms of the voltage ramp $V(t)$ has eluded us; and we cannot definitively point to one ramp as being better than another. However, in the limit that all processes are near-adiabatic and the initial voltage is vanishingly small we have found $\tilde{H}_T$ to be a universal function of H_0^- independent of the form of the ramp. In such cases, $\rho[H(T)]$ is an image of the original momentum distribution $\rho(p)$. Deviations from these forms, notably the spreads arising from poor adiabaticity, are shown to depend on the choice of the voltage ramp. Further, the analysis has generated detailed insight into the mechanisms at play and the factors that influence emittance increase. And the analysis provides a framework for numerical calculation that could be expected to require less computer resources than demanded by particle tracking simulations.

1.14.1 Acknowledgment

Author thanks Massimo Giovannozzi and Elias Metral (CERN-PS) for bringing to his attention the large body of literature (cited in [21]) concerned with change of adiabatic invariants. Author thanks Henry Lovelace III and Keith Zeno (BNL-CA) for their encouragement and support of this study, and for sharing the experimental data taken at the Brookhaven AGS reported in Ref.[39].

1.15 Jacobi elliptic functions

Some functions are defined by a differential equation or as the inverse of an indefinite integral; and such are the Jacobi elliptic functions. The Hamiltonian, Eqn. (1.36), does not change the nature of the stable $(x, p) = (0, 0)$ and unstable $(x, p) = (\pi, 0)$ Fixed Points (FP); and so the Jacobi elliptic functions can be used as approximate solutions to the dynamical equations provided the Jacobi parameter $m = H(t)/V(t)/2$ satisfies $|\delta m/m|$ per oscillation period τ is very small. This leads to the condition $|\dot{m}\tau/m| \ll 1$ or $|\dot{H}/H - \dot{V}/V|\tau \ll 1$. The m-definition means that m is calculated from known H, V. $H(t)$ is calculated by integrating Eqn. (1.37). These functions are defined both inside $(m < 1)$ and outside the bucket $(m > 1)$; and on the instantaneous separatrix $m = 1$. Let $K(m)$ and $E(m)$ be the complete elliptic integral of the first and second kinds, respectively.

Libration, $m < 1$

There are bounded motions inside the bucket with period $\tau = 4K(m)/\omega$. The two independent solutions are

$$\begin{aligned} x(t) &= 2\arcsin[\sqrt{m}\,\mathrm{sn}(\omega t, m)] \\ p(t) &= 2\sqrt{m}\,\omega\,\mathrm{cn}(\omega t, m) \\ x(t) &= 2\arcsin[\sqrt{m}\,\mathrm{cd}(\omega t, m)] \\ p(t) &= -2\sqrt{(1-m)m}\,\omega\,\mathrm{sd}(\omega t, m) \end{aligned}$$

Rotation, $m > 1$

There are unbounded motions outside the bucket with two periods equal $\tau = 4K(1/m)/(\omega\sqrt{m})$. The two independent solutions are

$$\begin{aligned} x(t) &= 2\arcsin[\mathrm{sn}(\sqrt{m}\omega t, 1/m)] \\ p(t) &= 2\sqrt{m}\,\omega\,\mathrm{dn}(\omega t, 1/m) \\ x(t) &= 2\arcsin[\mathrm{cd}(\sqrt{m}\omega t, 1/m)] \\ p(t) &= -2\sqrt{m-1}\,\omega\,\mathrm{nd}(\omega t, 1/m) \end{aligned}$$

Transition, $m = 1$

The two independent solutions are

$$\begin{aligned} x(t) &= 2\arcsin[\tanh(\omega t)] \\ p(t) &= 2\,\omega\,\mathrm{sech}(\omega t) \\ x(t) &= \pi \\ p(t) &= 0 \end{aligned}$$

Notice that the FPs are built into the Jacobi functions, and that for each sign ($\pm$) of momentum there is only one astable FP in the range $x = [-\pi, \pi]$. Although it might seem that each RF bucket has a share in two astable FPs, a perturbation of the synchronous phase reveals there is really only one per bucket.

1.16 Form Factor Integrals

Now we turn to the particular case of the periodic potential $U(x) = [1 - \cos x]$ and evaluation of the form-factor integrals Eqn. (1.40). In this section, a primed quantity X' does not denote a derivative. The prime denotes rotation, and its absence libration.

1.16.1 Integrals

The potential $[1-\cos x] = 2\sin^2(x/2)$ is traced out by a motion $x(s)$. Inside the bucket, substitute $x(s) = 2\arcsin[\sqrt{m}\,\mathrm{sn}(\omega s, m)]$, leading to $[1-\cos x(s)] = 2m\,\mathrm{sn}^2(\omega s, m)$. Let $K = K(m)$ and $\omega\tau = 4K$. Including the oscillation initial phase, the integrand of $F_k(q)$ is $2m\,\mathrm{sn}^2[\omega s + q(2K/\pi), m]s^k$. Let $u = \omega s$ and $Q = q(2K/\pi)$ and $F_k \equiv 2m\mathcal{I}_k$ where

$$\mathcal{I}_k(m,q) = \int_{-\tau/4}^{+\tau/4} \mathrm{sn}^2(\omega s + Q, m) s^k ds \Big/ \int_{-\tau/4}^{+\tau/4} ds = \int_{-K}^{+K} \left[\frac{u}{\omega}\right]^k \mathrm{sn}^2(u+Q,m)du \Big/ \int_{-K}^{+K} du$$

Outside the bucket, substitute $x = 2\arcsin[\mathrm{sn}(\sqrt{m}\omega s, 1/m)]$, leading to $[1-\cos x(s)] = 2\,\mathrm{sn}^2(\sqrt{m}\omega s, 1/m)$. Let $F'_k(q)$ be the form factor for motion outside the bucket. Let $K' = K(1/m)$ and $\tau = 4K'/(\omega\sqrt{m})$ be two periods. Including the initial phase, the integrand of $F'_k(q)$ is $2\,\mathrm{sn}^2[\sqrt{m}\omega s + Q, m]$. Let $u = \sqrt{m}\,\omega s$ and $Q = q(2K'/\pi)$ and $F'_k \equiv 2\mathcal{I}'_k$ where

$$\begin{aligned} \mathcal{I}'_k(m,q) &= \int_{-\tau/4}^{+\tau/4} \mathrm{sn}^2(\sqrt{m}\omega s + Q, 1/m) s^k ds \Big/ \int_{-\tau/4}^{+\tau/4} ds \\ &= \int_{-K'}^{+K'} \left[\frac{u}{\sqrt{m}\omega}\right]^k \mathrm{sn}^2(u+Q,1/m)du \Big/ \int_{-K'}^{+K'} du = \left[\frac{1}{\sqrt{m}}\right]^k \mathcal{I}_k(1/m, q) \end{aligned}$$

To summarise:

$$\text{inside} \quad F_k(q) = 2m\,\mathcal{I}_k(m,q) \quad \text{and outside} \quad F'_k(q) = 2\mathcal{I}_k(1/m,q)/\sqrt{m^k}\,.$$

When $k = 0$ the integral is unchanged when the origin is shifted, so $\mathcal{I}_0$ can be found in closed form:

$$\mathcal{I}_0(m,q) = \mathcal{I}_0(m,0) = \frac{1}{2K}\int_{-K}^{+K} \mathrm{sn}^2(u)du = C_0(m) \equiv \frac{1}{m}\left[1 - \frac{E(m)}{K(m)}\right] \quad \text{and} \quad \mathcal{I}'_0(m,0) = \mathcal{I}_0(1/m,0)$$

In order to compute $\mathcal{I}_k(m,q)$ for $k > 0$, we have to resort to replacing $\mathrm{sn}^2(u,m)$ by its Fourier series, which are given in Ref.[55] [Chap.10, Art. 22.735, Ex. 5, p.520]

$$\mathrm{sn}^2(u,m) = C_0(m) - \frac{2\pi^2}{mK^2}\sum_{n=1}^{\infty} C_n \cos[u \times (n\pi/K)] \quad \text{and} \quad C_n = \frac{nq^n}{1-q^{2n}} \tag{1.110}$$

Here $0 \le q < 1$ is the nome $q = \exp[-\pi K(1-m)/K(m)]$. Substituting the expression for $\mathrm{sn}^2(\ldots)$ into the integral $\mathcal{I}_k(m,q)$ we obtain

$$\mathcal{I}_k(m,q) = \left(\frac{2K}{\omega\pi}\right)^k \int_{-\pi/2}^{+\pi/2} v^k \left\{C_0 - \sum_{n=1}^{\infty} C_n \cos 2n(v+q)\right\} dv \Big/ \int_{-\pi/2}^{+\pi/2} dv \tag{1.111}$$

1.16.2 Approximate forms

In principle, the integrals may be computed exactly from Eqn. (1.111). However, strong progress can be made with much simpler approximate expressions that are valid for $m \ll 1$ inside (and $m \gg 1$ outside) the RF bucket. The starting point is the first-order Taylor expansion:

$$\mathrm{sn}^2(u) \approx (m/8)\sin 2u(\sin 2u - 2u)\,.$$

This is accurate to a few percent or less for $m \le \frac{1}{2}$.

Inside the bucket

$$\begin{aligned}
\mathcal{I}_0(m,q) &\approx (1/2) + m(1-2\cos 2q)/16 \qquad \mathcal{I}_0(m,0) \approx 1/2 - m/16 \\
\mathcal{I}_1(m,q) &\approx [8(2+m)\sin 2q - 2m\cos 2q(4q + \sin 2q)]/64 \qquad \mathcal{I}_1(m,0) = 0 \\
\mathcal{I}_2(m,q) &\approx [2(8+m)\pi^2 + 12(8 - m(\pi^2-6))\cos 2q - 3m\cos 4q + 48mq\sin 2q]/384 \\
\mathcal{I}_2(m,0) &\approx [m(69 - 10\pi^2) + 16(6+\pi^2)]/384
\end{aligned}$$

$$\begin{aligned}
F_0(m,q) &= m[1 + m(1-2\cos 2q)/8] \\
F_1(m,q) &= m/(32\omega)[8(2+m)\sin 2q - 2m\cos 2q(4q + \sin 2q)] \\
F_2(m,q) &= m/(192\omega^2)[2\pi^2(8+m) + 12(8 - m(\pi^2-6))\cos 2q - 3m\cos 4q + 48mq\sin 2q] \\
F_0(m) &= \langle F_0(m,q)\rangle = m(1 + m/8) \\
F_1(m) &= \langle F_1(m,q)\rangle = 0 \\
F_2(m) &= \langle F_2(m,q)\rangle = m/(96\omega^2)[8\pi^2 - m(12 - \pi^2)]
\end{aligned}$$

Outside the bucket

$$(\sqrt{m})^k\,\mathcal{I}'_k(m,q) = \mathcal{I}_k(1/m,q)\,. \tag{1.112}$$

$$\begin{aligned}
F_0'(m,q) &= 1 + (1 - 2\cos 2q)/(8m) \\
F_1'(m,q) &= \frac{1}{32m\sqrt{m}\,\omega}\left[8(2m+1)\sin 2q - 2\cos 2q\,(4q + \sin 2q)\right] \\
F_2'(m,q) &= \frac{1}{192m^2\omega^2}\left[2\pi^2(1+8m) + 12(8m+6-\pi^2)\cos 2q - 3\cos 4q + 48q\sin 2q\right] \\
F_0'(m) &= \langle F_0'(m,q)\rangle = 1 + 1/(8m) \\
F_1'(m) &= \langle F_1'(m,q)\rangle = 0 \\
F_2'(m) &= \langle F_2'(m,q)\rangle = 1/(96m^2\omega^2)[-12 + (1+8m)\pi^2]
\end{aligned}$$

1.17 Whence kinetic energy T and potential energy U?

Gottfried Liebniz (circa 1680) was the first to introduce the idea of Kinetic Energy, calling it *vis viva* (living force). Thomas Young (1807) just called it energy. William Thomson, Lord Kelvin (1824–1907) added the adjective “kinetic” to separate it from “potential energy”. The symbol T was introduced by Lagrange (1788) in Mécanique Analytique and subsequently adopted W.R. Hamilton (1834). It is sometimes suggested that T originates from “travail mécanique” (mechanical work) or “quantitié de travail” (quantity of work). However, there is no mention of the words “vis viva” or “travail” in Mécanique Analytique. The terms mechanical work and quantity of work were introduced by Poncelet and Coriolis (circa 1829) to distinguish between two formulae with different calibration factors, namely $T = m \times v^2$ and $T = \frac{1}{2}m \times v^2$.

The origin of the symbol U for potential energy is equally unclear. Lagrange (1788) used the symbol V. William Rowan Hamilton (1834) called the potential energy the “force function” but introduced the symbol U in “On a General Method in Dynamics”[22]; and this was adopted by William Rankine (1858). It has been suggested that U derives from, in Hamilton’s words, its “great utility in theoretical mechanics”. However, the root may be more pedestrian. Mathematicians often group symbols alphabetically, such as x, y, z. And, likewise, Hamilton introduced the triplet T, U, V: $m\ddot{x} = dU/dx$ and $m\dot{x} = dV/dx$. The quantity V is the path integral of the kinetic energy T.

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