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Alpha magnet

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Abstract: Introduction of the beam dynamics of an alpha magnet, with its Hamiltonian and F matrix are discussed in this work.

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1 Introduction

The Alpha magnet is a special kind of achromatic bending magnet. The name is given due to the particle trajectory that resembles the Greek letter α (as shown in Fig. 1). In principle, particles enter the magnet at an angle of 40.7° and leave the magnet with zero dispersion.

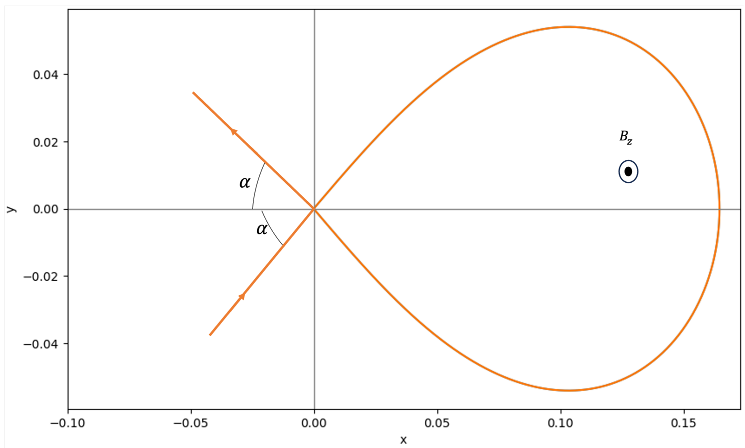

Figure 1: Alpha magnet with the α-shaped orbit.

The most common type of alpha magnet has a quasi-quadrupolar field dependence of $B_z = Gx$, where $G = \frac{\mathrm{d}B_z}{\mathrm{d}x}$ is the quadrupolar field gradient. One of the possible vector potential of the alpha magnet in the Cartesian system can therefore be given by

$$
\begin{aligned}
A_x &= 0 \\
A_y &= \frac{G}{2}(x^2 - z^2) \\
A_z &= 0
\end{aligned} \tag{1}
$$

Eqn. 1 obeys Maxwell equation of $\nabla \times \mathbf{B} = 0$. The corresponding magnetic field in Cartesian Frame (CS) is given by

$$
\begin{aligned}
B_x &= Gz \\
B_y &= 0 \\
B_z &= Gx
\end{aligned} \tag{2}
$$

The position of the reference particle with $\vec{r_0} = x\hat{i} + y\hat{j} + z\hat{k}$ as it travels through the alpha magnet is given as [1]

$$
\begin{aligned}
x_c &= \cos\phi \left[\frac{2\gamma m v(1+a)}{Gq}\right]^{1/2} \\
y_c &= \left(\frac{\gamma m v}{Gq}\right)^{1/2} [2E(\phi,k) - F(\phi,k)] \\
&= \left(\frac{\gamma m v}{Gq}\right)^{1/2} \left[\phi - \frac{k^2\phi^3}{2} + \frac{k^2}{40}(4-5k^2)\phi^5 + \mathcal{O}(\phi^6)\right] \\
z_c &= 0
\end{aligned} \tag{3}
$$

where α is the entrance angle, $k = \sqrt{\frac{1+a}{2}}$ and $a = \sin\alpha$. $E(\phi,k)$ and $F(\phi,k)$ are elliptic integral of the second and first kind respectively. The derivative of the reference trajectory in the alpha magnet has the following expression:

$$
\begin{aligned}
\dot{x} &= v\sqrt{1-Q(s)^2} = vL(s) \\
\dot{y} &= vQ(s) \\
\dot{z} &= 0
\end{aligned} \tag{4}
$$

with $C = \cos\phi$, $S = \sin\phi$, $L(s) = \sqrt{1-(aS^2-C^2)^2}$ and $Q(s) = aS^2 - C^2$ (For simplification, we simply denote $\phi(s)$ as ϕ, Q as $Q(s)$ and L as $L(s)$ from here onwards).

2 Frenet-Serret Frame (FS)

According to the Fresnet-Serret (FS) Coordinate system, the unit vectors are given as :

$$
\begin{aligned}
\hat{s} &= \frac{\mathrm{d}\vec{r_o}}{\mathrm{d}s} = \frac{\mathrm{d}\vec{r_o}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} \\
\hat{n} &= \frac{\frac{\mathrm{d}\hat{s}}{\mathrm{d}s}}{\left\|\frac{\mathrm{d}\hat{s}}{\mathrm{d}s}\right\|} = \frac{\frac{\mathrm{d}\hat{s}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s}}{\left\|\frac{\mathrm{d}\hat{s}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s}\right\|} = \frac{\frac{\mathrm{d}\hat{s}}{\mathrm{d}t}}{\left\|\frac{\mathrm{d}\hat{s}}{\mathrm{d}t}\right\|} \\
\hat{b} &= \hat{s} \times \hat{n}
\end{aligned} \tag{5}
$$

where $\left\|\frac{\mathrm{d}\hat{s}}{\mathrm{d}s}\right\| = \Omega(s) = \frac{qGx_c(s)}{\gamma m v}$ is the numerical value of the curvature of the trajectory.

Using eqn. 4 and 5 , the unit vector of the FS system is given as

$$
\begin{aligned}
\hat{s} &= L\ \hat{i} + Q\ \hat{j} \\
\hat{n} &= \frac{\dot{y}}{v}\hat{i} - \frac{\dot{x}}{v}\hat{j} \\
&= Q\ \hat{i} - L\ \hat{j} \\
\hat{b} &= -\hat{k}
\end{aligned} \tag{6}
$$

The position vector of a particle that is not too far from the reference particle in the FS frame is now given in eqn. 7 (dropping z_c as $z_c = 0$).

$$\begin{aligned}\vec{r} &= \vec{r_0} + n\hat{n} + b\hat{b} \\ &= x_c\hat{i} + y_c\hat{j} + n(Q\ \hat{i} - L\ \hat{j}) - b\hat{k}\end{aligned} \tag{7}$$

The transformation from Cartesian to local coordinate (s,n,b) is given as

$$\begin{aligned}x &= x_c + nQ \\ y &= y_c - nL \\ z &= -b\end{aligned} \tag{8}$$

3 Median Plane Symmetry

As mentioned in [2], the magnetic field with median plane symmetry can, in principle, be described by

$$\begin{aligned}A_s &= (1 + \Omega n)\vec{A} \cdot \hat{s} = a(s)n + c(s)n^2 + e(s)b^2 \\ A_n &= \vec{A} \cdot \hat{n} = 0 \\ A_b &= \vec{A} \cdot \hat{b} = 0\end{aligned}$$

Note that $A_s \neq \vec{A} \cdot \hat{s}$. Instead, $\vec{A} \cdot \hat{s} = \dfrac{A_s}{1 + \Omega n}$. Therefore, if we perform the curl using 23, the B_b we get is not $B \cdot \hat{b}$ as the length scale of $A_s \neq 1$. This complicates the comparison of B field at median plane later. An easy trick to do this is by dividing A_s by $(1 + \Omega n)$. Then the length scale became 1, a normal curl can be used to compare B.

According to the Maxwell equation, $\nabla \times \nabla \times A = \nabla \times B = 0$. However, if we perform the curl for the field described above,

$$\nabla \times B = \begin{bmatrix} -2c(s) - 2e(s) + a(s)\Omega(s) \\ a'(s) \\ 0 \end{bmatrix}_{n=0,b=0}$$

To satisfy the Maxwell equation,

$$a(s) = \rho(s)\left[2c(s) + 2e(s)\right] \tag{9}$$

for $\Omega(s) = 1/\rho(s)$,

$$a'(s) = 0$$

This is fine for a constant dipole field with $a(s) = B_0$. However, for a field with a gradient such as $a(s) = B(s)$, $a'(s) \neq 0$, and this violates the Maxwell equation. To conserve the Maxwell equation again, we can add another term (that is not contributing to the second order Hamiltonian needed for F matrix),

$$\begin{aligned} A_s &= a(s)n + c(s)n^2 + e(s)b^2 \\ A_n &= f(s)b^2 \\ A_b &= g(s)nb \end{aligned} \tag{10}$$

$$\therefore \boldsymbol{\nabla} \times B = \begin{bmatrix} -2c(s) - 2e(s) + a(s)\Omega(s) \\ -2f(s) + g(s) + a'(s) \\ 0 \end{bmatrix}_{n=0,b=0}$$

so that $a(s) = B(s)$, $a'(s) = 2f(s)$. To simplify the formulation for B_n, we can choose $g(s)$ as a constant so that $g'(s) = 0$.

4 Vector Potential in FS frame

As shown in eqn 2 [1], we assume that the ideal magnetic field for a particle that is slightly away from the reference orbit to be

$$\vec{B}(s) = Gz\hat{i} + Gx\hat{k}$$

, where x and z are the Cartesian coordinates. In order to determine the field in FS frame, we can take the dot product of $\vec{B}$ and $\vec{r}$ in 6:

$$\begin{aligned} B_s &= \vec{B} \cdot \hat{s} = GzL = -GbL \\ B_n &= \vec{B} \cdot \hat{n} = GzQ = -GbQ \\ B_b &= \vec{B} \cdot \hat{b} = -Gx = -G(x_c + nQ) \end{aligned} \tag{11}$$

Comparing 11 with the curl by using 10, we can determine $e(s)$ as

$$\begin{aligned} -GbQ &= \frac{2be(s)}{1 + n\Omega(s)} \\ \therefore e(s) &= \frac{-GQ(s)}{2}(1 + n\Omega(s)) \end{aligned}$$

Assuming the reference particle, $n = 0, b = 0$, all the first order terms from the Hamiltonian has to be zero (no net force onto the reference particle along the reference trajectory),

$$\begin{aligned} p_n' &= -\frac{qa(s)}{P_0} - \Omega(s) = 0 \\ \therefore -\frac{qa(s)}{P_0} &= \Omega(s) = \frac{qGx_c(s)}{P_0} \\ a(s) &= -Gx_c(s) \end{aligned}$$

Using 9, we can determine $c(s)$ as

$$\begin{aligned} a(s) &= \rho(s)\,[2c(s) + 2e(s)] \\ \therefore c(s) &= -\frac{Gx_c(s)}{2}\Omega(s) + \frac{GQ(s)}{2}(1 + n\Omega(s)) \end{aligned}$$

Thus, the vector potential in FS frame can now be written as

$$\begin{aligned} A_s &= -Gx_c(s)n + \left[-\frac{Gx_c(s)}{2}\Omega(s) + \frac{GQ(s)}{2}(1 + n\Omega(s))\right] n^2 - \frac{GQ(s)}{2} b^2 \\ A_n &= f(s)b^2 \\ A_b &= gnb \end{aligned} \tag{12}$$

5 Hamltonian in FS

The Hamiltonian in FS frame without the presence of the scalar potential ($qV = 0$) is given as

$$H_t(s, P_s, n, P_n, b, P_b; t) = c\sqrt{m^2c^2 + (1 + n\Omega)(P_s - qA_s)^2 + (P_n - qA_n)^2 + (P_b - qA_b)^2} \tag{13}$$

Re-arranging eqn 13 to depend on s instead of t,

$$H_s(t, E, n, P_n, b, P_b; s) = -(1 + \Omega n)\sqrt{E^2/c^2 - m^2c^2 - (P_n - qA_n)^2 - (P_b - qA_b)^2} - qA_s \tag{14}$$

Substituting 12 into 14

$$\begin{aligned} H_s = &-(1 + n\Omega(s))\sqrt{\frac{E^2}{c^2} - m^2c^2 - (P_n - qb^2 f(s))^2 - (P_b - gqnb)^2} \\ &- q\left[-Gx_c(s)n + \left[-\frac{Gx_c(s)}{2}\Omega(s) + \frac{GQ(s)}{2}(1 + n\Omega(s))\right] n^2 - \frac{GQ(s)}{2} b^2\right] \end{aligned} \tag{15}$$

In order to change the canonical conjugate from $(t, -E)$ to $(-z, \Delta P)$, we can perform canonical transformation by using the generating function

$$G(t, \Delta P) = \beta c \left(\int \frac{1}{\beta c} ds - t \right) \left(\frac{E_0}{\beta c} + \Delta P \right) \tag{16}$$

Perform substitution and add $\frac{\partial G}{\partial s} = \frac{m^2c^2}{P_0} + P_0 + \Delta P$ to 15. Expanding the Hamiltonian to second order

$$H_s \approx \frac{m^2c^2}{P_0^2} + \frac{p_n^2}{2} + \frac{p_b^2}{2} + \frac{\Delta p^2}{2\gamma^2} - \frac{Gqn^2Q(s)}{2P_0} + \frac{Gqb^2Q(s)}{2P_0} - \Delta pn\Omega(s) + \frac{Gqn^2x_c(s)\Omega(s)}{2P_0} \tag{17}$$

6 Transfer Matrices

The corresponding F matrix can now be obtained by taking the product between the Hessian Matrix (second order derivative of H) and S

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{18}$$

$$F = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{Gq(Q(s)-x_c(s)\Omega(s))}{\mathrm{P0}} & 0 & 0 & 0 & 0 & \Omega(s) \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{GqQ(s)}{\mathrm{P0}} & 0 & 0 & 0 \\ -\Omega(s) & 0 & 0 & 0 & 0 & \frac{1}{\gamma^2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{19}$$

The transfer matrix can then obtained by solving the integral of $M = I + Fds$ (I is the identity matrix).

A sample calculation was performed using the `odeint` solver in python. The particle was taken to be an electron with energy of 2 MeV. The field gradient was assumed to be

-1.0 T/m (negative G for negative charge). The result is given below:

$$M = \begin{pmatrix} -1.02 & -0.213 & 0 & 0 & 0 & -0.00175 \\ 0.0252 & -1.01 & 0 & 0 & 0 & -0.0169 \\ 0 & 0 & -0.747 & 0.69 & 0 & 0 \\ 0 & 0 & -0.686 & -0.705 & 0 & 0 \\ 0.017 & 0.00176 & 0 & 0 & 1 & -0.193 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{20}$$

As a comparison, it is compared with the results generated by ELEGANT in [3]

$$M_{ele} = \begin{pmatrix} -1.0 & -0.21 & 0 & 0 & 0 & 0 \\ 0 & -1.0 & 0 & 0 & 0 & -0 \\ 0 & 0 & -0.74 & 0.69 & 0 & 0 \\ 0 & 0 & -0.66 & -0.74 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.21 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{21}$$

and the transfer matrix generated using T. Planche's FFA alpha (electron with positive G, transverse coordinate in cm)

$$M_{TP} = \begin{pmatrix} -1.000005 & -21.02646 & 0.000000 & 0.000000 & 0.000000 & 0.1904447e-02 \\ -0.6224379e-04 & -1.001302 & 0.000000 & 0.000000 & 0.000000 & -0.2411418e-05 \\ 0.000000 & 0.000000 & -0.7354586 & 68.99606 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & -0.6609298e-02 & -0.7396578 & 0.000000 & 0.000000 \\ -0.2580737e-05 & -0.1958119e-02 & 0.000000 & 0.000000 & 1.000000 & -19.27754 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 1.000000 \end{pmatrix}$$

The symplecticity is also checked as

$$M^T S M - S = \begin{pmatrix} 0. & 0.0355676 & 0. & 0. & 0. & 0.0342821 \\ -0.0355676 & 0. & 0. & 0. & 0. & 0.0035922 \\ 0. & 0. & 0. & -0.000025 & 0. & 0. \\ 0. & 0. & 0.000025 & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. \\ -0.0342821 & -0.0035922 & 0. & 0. & 0. & 0. \end{pmatrix}$$

References

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[3] J. Lewellen, F. Krawczyk, et al., Simulation of alpha magnet elements in dipole-only tracking codes, Proceeding of FEL2014, Basel, Switzerland, THP023 (2014) 735–738.

[4] E. Courant, H. Snyder, Theory of the alternating-gradient synchrotron, Annals of Physics 3 (1) (1958) 1–48. doi:https://doi.org/10.1016/0003-4916(58)90012-5.
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A Curl in FS frame

Curl in any curvilinear frame:

$$\nabla \times v = \frac{1}{h_u h_v h_w} \begin{bmatrix} h_u \hat{u} & h_v \hat{v} & h_w \hat{w} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u v_u & h_v v_v & h_w v_w \end{bmatrix}$$

In the case of Fresnet-Serret system,

$$\begin{aligned} \vec{r} &= \vec{r_0} + n\hat{n} + b\hat{b} \\ \mathrm{d}\vec{r} &= \frac{\partial \vec{r}}{\partial s}\mathrm{d}s + \frac{\partial \vec{r}}{\partial n}\mathrm{d}n + \frac{\partial \vec{r}}{\partial b}\mathrm{d}b \end{aligned}$$

$$h_s \equiv \left|\frac{\partial \vec{r}}{\partial s}\right| \ , \ h_n \equiv \left|\frac{\partial \vec{r}}{\partial n}\right| \ , \ h_b \equiv \left|\frac{\partial \vec{r}}{\partial b}\right| \tag{22}$$

$h_n = h_b = 1$. For a standard FS system with $\hat{b} = \hat{s} \times \hat{n}$,

$$\begin{aligned} \frac{\mathrm{d}\vec{r_0}}{\mathrm{d}s} &= \hat{s} \\ \frac{\mathrm{d}\hat{n}}{\mathrm{d}s} &= -\Omega(s)\hat{s} + \omega(s)\hat{b} \\ \frac{\mathrm{d}\hat{b}}{\mathrm{d}s} &= -\omega(s)\hat{n} \\ \frac{\partial \vec{r}}{\partial s} &= \frac{\partial \vec{r_0}}{\partial s} + n\frac{\partial \hat{n}}{\partial s} + b\frac{\partial \hat{b}}{\partial s} \\ &= \hat{s} - n\Omega(s)\hat{s} \text{ , for } \omega(s) = 0 \\ \therefore h_s &\equiv 1 - n\Omega(s) \end{aligned}$$

Note the sign of $n\Omega(s)$ is different as different convention is used for the convenience of canonical momenta. The curl of the vector potential (or the magnetic field) is thus given as

$$\begin{aligned} \nabla \times A &= B_s\hat{s} + B_n\hat{n} + B_b\hat{b} \\ B_s &= \frac{\partial A_b}{\partial n} - \frac{\partial A_n}{\partial b} \\ B_n &= \frac{\partial A_s}{\partial b} - \frac{1}{h_s}\frac{\partial A_b}{\partial s} \\ B_b &= \frac{1}{h_s}\left[\frac{\partial A_n}{\partial s} - \frac{\partial (h_s A_s)}{\partial n}\right] \end{aligned} \tag{23}$$

B Canonical Momenta in FS

Assuming the generating function $F(p, \bar{q})$ is given as

$$F(p_x, p_y, p_z; s, n, b) = \vec{p} \cdot (\vec{r_0}(s) + n\hat{n}(s) + b\hat{b}(s)) \tag{24}$$

where $\vec{p} = \gamma m\vec{v}$ is the linear momentum in Cartesian system. Using the property of FS (Courant Snyder system with $\frac{\mathrm{d}\hat{n}}{\mathrm{d}s} = +\Omega(s)\hat{s}$),

$$\begin{aligned}\frac{\mathrm{d}\vec{r_0}}{\mathrm{d}s} &= \hat{s} \\ \frac{\mathrm{d}\hat{n}}{\mathrm{d}s} &= \Omega(s)\hat{s} + \omega(s)\hat{b} \\ \frac{\mathrm{d}\hat{b}}{\mathrm{d}s} &= -\omega(s)\hat{n}\end{aligned}$$

$\omega(s) = 0$ (no torsion) in this case, the new canonical momenta are now

$$\begin{aligned}P_s = \frac{\partial F}{\partial s} &= \vec{p}\cdot(\frac{\mathrm{d}\vec{r_0}}{\mathrm{d}s} + n\frac{\mathrm{d}\hat{n}}{\mathrm{d}s} + b\frac{\mathrm{d}\hat{b}}{\mathrm{d}s}) \\ &= \vec{p}\cdot\left[\hat{s} + n(\Omega(s)\hat{s} + \omega(s)\hat{b}) - b\omega(s)\hat{n}\right] \\ &= \vec{p}\cdot\left[(1 + n\Omega(s))\,\hat{s} + \omega(s)\left(n\hat{b} - b\hat{n}\right)\right] \\ &= \vec{p}\cdot[(1 + n\Omega(s))\hat{s}] \\ P_n = \frac{\partial F}{\partial n} &= \vec{p}\cdot\hat{n} \\ P_b = \frac{\partial F}{\partial b} &= \vec{p}\cdot\hat{b}\end{aligned} \tag{25}$$

According to [4], the vector potential $\vec{A}$ in FS frame can be obtained by replacing $\vec{p}$ in eqn 25 $\omega(s) = 0$

$$\begin{aligned}A_s &= \vec{A}\cdot[(1 + n\Omega(s))\hat{s}] \\ A_n &= \vec{A}\cdot\hat{n} \\ A_b &= \vec{A}\cdot\hat{b}\end{aligned} \tag{26}$$

In principle, it is possible to determine the vector potential in the FS frame by using eqn 26. However, the vector potential defined in the Cartesian is not the full expansion. Higher order terms are dropped for convenience. If we perform directly the transformation using eqn 26, it leads to a very complicated gauge that is hard to resolve and simplify to the second order Hamiltonian. Trials were done using eqn 26, but the mathematics was messy and that's why, the vector potential was found using a 'cheated' way of working it backward.