Needle Model for Simulation of Space-Charge Dominated Beam Dynamics In an Isochronous AVF Cyclotron

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This note comprises two documents: the original research paper written in 1992, and a historical commentary (the Prologue and Epilogue) written in 2023.

1 Prologue

The history of the computer program PICN is one of evolution. Cyclotrons and the particle orbits within them have, approximately, circular symmetry; leading to the adoption of cylindrical coordinates (radius r, azimuth θ, vertical displacement) for their description. Often, the azimuthal direction is also called longitudinal. In 1969, Mort Gordon[1] predicted that space-charge forces in a isochronous cyclotron would lead to vortex structures in the radial-azimuthal plane; because the vertical magnetic field deflects azimuthal motion into the radial direction and visa versa. In a reference frame rotating synchronously with the particle beam, the equilibrium motion (in the radial-azimuthal plane) satisfies the vector cross product:

$$m_0\,\boldsymbol{\omega} \wedge \mathbf{v} = q\mathbf{E} = q(\mathbf{E}_{\rm sc} + \Delta\mathbf{E}_{\rm rf})\ . \tag{1}$$

Here m_0 is the particles mass, $\boldsymbol{\omega}$ is the angular velocity vector, parallel to the magnetic field; and $\mathbf{v}$ is the velocity vector. The electric field vector, $\mathbf{E}$, has two contributions: the space-charge field $\mathbf{E}_{\rm sc}$ from all sources inside the cyclotron vacuum chamber, and the residual field $\Delta\mathbf{E}_{\rm rf}$ being the difference between the general RF field and the value at the charge centre responsible for acceleration. In a state of equilibrium, individual particles circulate along equipotential surfaces within the cloud charge.

Eq. 1, with $\Delta\mathbf{E}_{\rm rf} = \mathbf{0}$, is the exact analogue in atmospheric physics of the internal motion within a cyclone. For a cyclone there is a radial pressure gradient, but the winds flow in the azimuthal direction due to the Coriolis force that arises from the spinning frame of the earth. As is well known, cyclones (having reached a steady state) are approximately circular; and likewise for isolated charge clouds in an isochronous cyclotron. The cyclone spans the diameter of the coordinate system which is centred at the eye of the storm. Contrastingly, the charge cloud lies along a radius, and is offset from the centre of the cyclotron coordinate system. For that reason, the words radial and azimuthal have different interpretations in these two different contexts. Nevertheless, the coordinate-independent Eq. 1 places the radial and azimuthal components (v_r, v_θ) and (E_r, E_θ) on a footing of equal importance.

1.1 Equilibrium charge distribution

For a charge cloud well separated from other clouds, the space-charge electrical force points outward from the charge centre, and so individual particles circulate within the cloud charge. If $\Delta E_{\rm rf} = \mathbf{0}$, the equilibrium shape arising from Eq. 1 is a disc or "flying saucer". Of course, the cloud is not isolated; there are charges in other RF-packets distributed in azimuth, and there are charges in all the accelerated turns along the diameter of the cyclotron. There is a small literature devoted to the estimation of the nett field from all those sources. Many factors are at play: whether the cyclotron receives pulsed or continuous-wave beam; the harmonic number and whether all RF-packets are filled; and whether the turn structure is separated or not; and the RF-compression of bunches as the beam accelerates; and whether beam is injected into the centre-region or at an intermediate radius as in a ring cyclotron; and the shielding effect of image charge distributions on the inner surface of the vacuum chamber (which may be a pill-box or annulus with rectangular cross-section).

Nevertheless, we can invoke symmetry arguments to infer a few properties of the space-charge field without resorting to detailed calculation. Figure 1 is a cartoon of the first few separated turns of charge density with harmonic number $h = 4$ and all RF packets filled; the effect of RF-compression is not included. Consider the field at one packet due to all others. There will be cancellations between fields emanating from different clouds. By inspection, the azimuthal force

is zero at all of the charge centres. The deviation of longitudinal force at head and tail falls as the inverse of harmonic number. At the centre $r \to 0$, the cancellation is perfect. At other radii, there is a net outward radial force that is proportional to radius. These observations imply that the isolated cloud model will work well toward the central region; and the effect of other turns will become important at large radii.

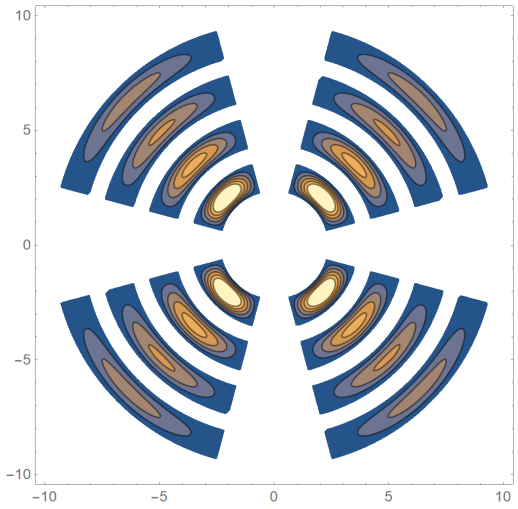

Figure 1: Cartoon of charge density in the mid plane of a cyclotron with separated turns and harmonic number $h = 4$. In the laboratory frame the spokes rotate at angular frequency ω.

Gordon[1] considers both individual bunches and the collective motion within a radial sector of bunches (which is called a spoke in Fig. 1) and suggests that both execute vortex motion. Gordon proposes that for the collective cloud of neighbour turns, and non-zero deviations of the RF-acceleration $\Delta\mathbf{E}_{\rm rf} \neq \mathbf{0}$, the equilibrium shape is the sector of a circle (or pie-wedge); and that the radial forces are insignificant except at $r \to 0$ and $r \to R$. The explanation is brief, and it is unclear precisely what conditions and assumptions are made. Despite having discovered the individual vortices, Gordon discards them as a special case and states: "the local vortices are so small and feeble that their presence can be neglected entirely." Of course, it would have been difficult for Gordon to imagine the hundred-fold increase of current (from $\simeq 10\mu$A at the MSU cyclotron to the milli-Ampere level at the PSI cyclotrons) enabled by development of ion sources[4] and charge-exchange injection[5, 6].

Gordon's interest lies in the MSU 50 MeV cyclotron[2, 3], and he identifies the longitudinal field of the collective cloud as degrading the turn separation, and focuses on exploring that effect. Contrastingly, in the PSI Injector-II cyclotron[7], constructed two decades after the MSU cyclotron, it is the central region and radial-azimuthal space-charge that generates intriguing effects; and it will be the discus of charge that concerns us herein (the work of 1992-93).

1.2 Contributions of Stefan Adam

Steffan Adam presented[8] a method for calculating Accelerated Equilibrium Orbits (AEOs) and a Hamiltonian formulation of compression of the bunch length in RF-phase in 1981. The calculations are performed by a program called MATADOR (MAtrix Techniques for Acceleration studies and Display of ORbits). At the same conference, Adam et al presented[9] preliminary methods for the calculation of longitudinal space-charge effects. (Radial space-charge forces are entirely omitted. This is only legitimate if cylinders all lie along the same straight line.)

In the first variant, the beam is represented by a series of (some 50 to 100) thin discs (circular surfaces with no volume) of charge. The discs have the diameter of the beam transverse cross sec-

tion; and the discs are arranged along the orbit with their axes directed along the orbit. The discs have no internal structure, and do not change their shape or size. What is important is the longitudinal force law between discs. The DISKS program was patched into MATADOR. Subsequently, and also reported in 1981, the disks were soon replaced by rigid uniformly filled cylinders (but, confusingly, the program retained the name DISKS). The rotation axis of the cylinders is placed along the AEO. Each cylinder may take an individual value of charge, to give the longitudinal distribution. What is important is the longitudinal force law between cylinder-ends. Values for the transverse and longitudinal shape are taken from measurements of the the respective beam profiles, both parameterised in terms of binomial distributions.

Later, 1983-84, phenomena emerged (in simulations of low energy orbits near the central region) which pointed to inadequacy of the disc and cylinder models when the orbit is strongly curved, because the radial component of space-charge forces was not considered. Adam[10, 11] replaced the DISKS cylinder model with Particle-In-Cell (PIC) methods[12] that facilitate the addition of radial forces. The beam is represented by several thousand super-particles, each of which is a sphere. The spheres are rigid and have no internal dynamics; but they are like clouds and may be interpenetrating. The sphere diameter is equal to the transverse beam radius. The charge distribution within is homogeneous but not uniform; it is taken to be binomial in radius $[1 - (r/a)^2]^5$ with power law index inferred from experimentally measured beam profiles. The space-charge force computation is sophisticated. The charges are distributed onto a grid; the forces are calculated on the grid from the net charge density; and the forces are then propagated back to the charge centres. The force calculation depends on convolution of the electric-field Green's function for a spherical binomial charge density with the net charge density of the ensemble of spheres. The convolution is performed by Fourier methods. This PIC formulation is reported in Adam's doctoral thesis[10], 1985, along with an account of AEOs and MATADOR. During the period 1985 to 1992, it appears that the PIC program became somewhat "stand-alone", less reliant on MATADOR. In 1986, Joho[13] describes plans to raise the current in Injector-II by combating the longitudinal spiral instability with a high-harmonic flat-top RF system dedicated to the first few turns.

An important observation in Ref. [10] is that the "super-particles" representing the beam should not have the interaction force law of point charges or of filaments. Due to the low number of super-particles, there is a disproportionate number of close encounters and scattering events; leading to unrealistic effects. This problem is eliminated by endowing the super-particles with a finite (non-zero) size, and adopting a force law that rises from zero at the super-particle-centre; and then falls outside the particle bounding radius. These remarks were made in the context of the sphere model, but also reflect experience[9] with the disc and cylinder models (and a precursor filament model). Baartman[14], 1986, describing his buncher simulation program, SPUNCH, makes similar remarks:

> "The chief advantage of using discs rather than point particles is that the electric field from a disc is non-singular; collision effects are thereby avoided. A similar technique is employed in the program PARMILA[Ken Crandall 1967]"

One aspect of the spherical PICS model is very unsatisfactory: the rigidity of the sphere, and the absence of internal dynamics of the cloud of charge. The increase of computational power between 1985 and 1992 made it possible to remedy those defects. Adam proposed to replace each sphere by an ensemble of "needles", or more accurately rods: thin vertical cylinders. Adam and Joho invited Koscielniak to execute that proposal at PSI in 1992. The computer model with rigid spheres was renamed PICS, and the new model with "needles" was christened PICN. The few thousand spheres were replaced by one hundred thousand needles.

1.3 Contribution of Koscielniak

Koscielniak inherited many excellent features from Adam's PICS: good data-structures; clean division of sub-routines; and gilded graphical facilities. Much of the PIC functionality could be used

with only one change: replacing the Green's function for the sphere with that of a rod. But there were also significant changes. The basic equations of motion were updated, following Cazoll[15] and Kleeven[16], to give a better representation of dynamics in an Azimuthally Varying Field. Although the basic beam-bunch description was still the layout of spheres, each was the repetition of a cluster (around a thousand) of needles. The needles may each be assigned an individual charge value, but they all have the same height. Thus technically, the ensemble is not a sphere; rather it is a pill-box shape (a short vertical cylinder). The ensemble of needles must have the same statistical properties as the sphere based on a binomial distribution; an ensemble generator was written to do that.

It was realized by Koscielniak that the so-called "spiral-nebula instability"[1] was in fact just the ensuing dynamics following a mis-match between initial coordinates and space-charge forces. (He rediscovered the equilibrium distribution implied by Eq. 1.) Moreover, now that the spheres had become composite particles (containing many needles) the internal dynamics had to be accounted for. This would facilitate making a correspondence and comparison between the pure-sphere and needle models. Subroutines were written to equilibriate (by choosing and adjusting initial coordinate values) the needles within a sphere, with or without space-charge forces. Further, treating the entire bunch as a single sphere, gave matched conditions that eliminate the "instability". (Because the matched beam radial width is much greater than its vertical height, these equilibriated beam packets are referred to as "flying saucers" in the 1993 paper.) After they have equilibriated, there is no growth of the azimuthal RF-phase extent of the bunches. Finally, the basic time-integration method was revised: the leap-frog replaced by a low-memory variant of Runge-Kutta. This had two benefits: longer time steps, and the ability to stop (and restart) the calculation at precisely any time, rather than at the grid of leap-frog time steps. The properties of and results from the new simulation program PICN[17] were reported in 1993.

2 Epilogue

There was an immediate response to the 1993 paper[17]. Martin Reiser informed us (Adam & Koscielniak) of his 1966 paper[18] on space-charge effects. Later, in 1996, Mort Gordon[19] and Felix Marti wrote to us of "space-charge effects reconsidered" for ensembles of needles. Both these papers deal with analytic calculation of space-charge forces, but neither discuss the resulting beam dynamics.

Adam contributed an oral presentation[20] on space-charge effects to the 1995 Cyclotron Conference. He describes three types of simulation model: the cylinder model[2]; the sphere model; and the needle model. He also explains how the combination of radial and vertical betatron oscillations can be thought of as giving rise to a charge-sphere. Adam is less clear about the individual contributions to the decade of progress made on space-charge simulations. Adam programmed the cylinder and sphere models, and orchestrated the needle model. Koscielniak created the detailed physics model of the needles, in which an ensemble of vertical rods approximates a sphere, and programmed and tested that model. Adam neglects to acknowledge Koscielniak's insight that the "spiral instability" is in fact a mismatch; and that Koscielniak gave conditions (in the 1992 internal report and 1993 Conf. Proc) for the matched beam distribution in the radial-azimuthal plane: it is a disk with symmetry axis vertical. Adam goes on to describe extension of the spheres model to include interaction between neighbouring turns. Drawing on and considering the work of Gordon[1], this was a natural development for the PICS simulations. Adam concludes with a discussion of halos generated as an initially mismatched bunch transforms to a matched "round" distribution (i.e. disk).

[1] The term spiral-nebula was coined by Edwin Hubble in his book "Realm of the Nebulae". The modern term is spiral-galaxy.

[2] Called the cylinder model in his 1985 Thesis, but confusingly labelled DISK in the 1995 paper

PICS and PICN were still in use in 2001, as reported by Adam[20] and Andreas Adelmann at the 16th Int. Cyclotron Conf. This report confirms experimentally that RF-phase packets of beam (i.e. bunches) wind up into circular distributions in the mid-plane of radius-azimuth. The report confirms again the prediction of matched circular distributions by Koscielniak in 1993, although this work is not referenced in 2001.

In 2004 Adelmann[22] and colleagues ushered in a new era of beam simulations for cyclotrons and for space-charge and other calculations made possible by sophisticated programming (such as heavily parallel instruction execution on multiplre CPUs) and the enormous advances in computational power made since the early 1990s.

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PICN SIMULATION OF SPACE-CHARGE-DOMINATED BEAM DYNAMICS IN AN ISOCHRONOUS AVF INJECTOR CYCLOTRON

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Koscielniak was a visiting scientist hosted by Stefan Adam and Werner Joho – under a collaboration agreement between the two laboratories (TRIUMF and PSI) for the exchange of highly qualified personnel.

This manuscript was not published in 1992, as there was no obvious mechanism for a report authored by a TRIUMF scientist to be released as a PSI laboratory report, nor the converse.

THE WORK IS DIVIDED IN TWO MAIN PARTS

I. THEORY AND COMPUTAIONAL DETAILS
II. EXAMPLE SIMULATIONS USING THE “PICN” COMPUTER PROGRAM

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1 Introduction

As has been noted by several authors[1][2][3], space-charge is important in isochronous cyclotrons for the following reason: there is no longitudinal focusing and there is strong radial-azimuthal coupling. Because the space-charge electric fields are severely non-linear, practical investigation must proceed by numerical methods[4][5]. This note describes the extension of the original cyclotron beam simulation code (PICS), devised by Adam[3], to include betatron motion in the median plane. We shall assume the machine operates in a regime where the vertical focusing is adequate.

The original code PICS (also named PIC in earlier publications) considered the beam to be formed of charged spheres, and completely neglected the internal motions within these spheres. The main argument to justify this simplification was that the betatron oscillations are much faster movements than the deformations of a beam bunch due to space charge forces. With this reasoning, the common motion of all particles situated on a vertically oriented cylinder can be based on the averaging of the space charge forces over several vertical and horizontal-longitudinal betatron oscillation periods. The assumption that a group of cylinders can be packed into a sphere, which will then move in a way representative of all its cylinders, stays on weaker grounds. It only holds if the space charge force has small second and higher radial and azimuthal derivatives. From theoretical investigations on effects of transversal space charge forces, it was quite clear that the effect inside a ball was merely to reduce the tune and increase the betatron amplitude.
These assumptions allowed us to represent the 3-dimensional charge distribution of a beam bunch by a 2-dimensional distribution of centres of charged spheres. The PICS model had then only to integrate the first order equations for the motion of the centres of charged spheres, and in the absence of electric fields (due either to space-charge or R.F. acceleration) there is no motion at all. The electric force law between spheres was devised so that for wide separations the Coulomb form applies Q^2/R^2, and for small separations (i.e. interpenetrating spheres) the mutual force goes to zero as the separation R. This corresponds with our expectation that two exactly coincident spheres have no force between them, but rather there is a tendency for them to expand in radius. PICS cannot model the increase in radius, but it is reasoned that the expansion would be slight since the magnetic betatron focusing predominates.

The new simulation code, to be called PICN, assumes that the radial and the vertical betatron motions are decoupled, just as in the code PICS. However, the median plane internal motions within a charged sphere are now to be included explicitly. The sphere is decomposed into cylinders and the cylinders are divided into vertical rods or needles. There are now new freedoms in the motion: rods within the same initial sphere need not have the same centre nor the same oscillation frequency, and time-dependent modulations of tune and betatron amplitude (because of forces between the spheres) are now allowed. The new model assumes that all needles have the same, fixed height; i.e. the spheres are replaced by full cylinders [1]. It is expected that this simplification introduces only minor errors into the PICN-model. Because of these modifications, the artificial force law used in PICS is replaced by the force between two uniformly charged vertical rods, and has a singularity at zero separation between the rods.

We have described the basic features of the extended cyclotron beam simulation code PICN. It is now appropriate to state the questions that PICN should help us to answer.

[1] To let the height depend on the radial betatron amplitude, would be difficult, since this quantity can now vary during the motion. It would also considerably complicate the task of finding the space-charge force.

- Does the new model, PICN, lead to qualitatively different migration and clustering of the particles as compared with PICS?

- Are the spheres really stationary in the statistical sense, that is do cylinders move with a common centre, and do needles stay on a cylinder?

To summarize, PICS treats the charged spheres as a statistically stationary entity such that the cylinders of which it is composed have a common centre, and the vertical rods of which a cylinder is composed have a common tune. PICN, however, allows these assumptions to be dropped, and offers a possibility to observe new collective phenomena.

2 Toward an equation of motion

While attempting to formulate an equation for beam motion, there is an implicit expectation that the forces on a particle are divisible into two parts: $\mathbf{F}^{ext} + \mathbf{F}^{int}$.
$\mathbf{F}^{ext}$ are applied electromagnetic (EM) fields to guide motion.
$\mathbf{F}^{int}$ are forces resulting from the mutual Coulombic repulsion of the particles.

Our intention is to give a relativistically correct description of the motion. Quantities in the beam frame are denoted by a superfix prime. Now, $\mathbf{F}^{ext}$ is more simply found in the laboratory frame, while $\mathbf{F}^{int}$ is more easily found in the beam frame. Hence we have two alternatives:

(i) Equations in the laboratory frame; $\mathbf{E}_{ext}$, $\mathbf{B}_{ext}$ are known, but we must transform $\mathbf{x} \rightarrow \mathbf{x}'$ to solve for $\mathbf{E}'_{int}$ and then back-transform to give $\mathbf{E}_{int}$, $\mathbf{B}_{int}$.

(ii) Equations in the beam co-moving frame; $\mathbf{E}'_{int}$ is known, but we must transform $\mathbf{x}' \rightarrow \mathbf{x}$ to solve for $\mathbf{E}_{ext}$, $\mathbf{B}_{ext}$ and then back-transform to give $\mathbf{E}'_{ext}$, $\mathbf{B}'_{ext}$.

Though we shall sketched how (ii) might be achieved, ultimately we shall use alternative (i).

2.1 Reference particle – definition

Since the sum over all particle-particle interactions is zero, the centre of mass experiences no net Coulombic force (either in its rest-frame or in the laboratory frame). So, provided that we are careful to place the reference particle at the centre of charge,

$$\text{the rate of change of reference momentum is} \quad \frac{d}{dt}\mathbf{p}_{ref} = \mathbf{F}^{ext}(\mathbf{x}_{ref}) \ . \tag{1}$$

Here $\mathbf{F}^{ext}$ is the force field evaluated at the reference particle. When the the centre of charge lies outside the distribution, as for example occurs for a charged annulus, then we shall continue to use (1), but must reinterpret the reference as being a *notional* particle used purely for convenience. Note, if all particles are of a single specie and in the same charge state, then the centre of mass and of charge are coincident.

2.2 Thoughts on the Hamiltonian approach

In most accounts of cyclotrons, the equations of motion are formulated in the laboratory frame and are derivable from the laboratory relativistic Hamiltonian. There is rarely (if ever) an attempt to transform the motions into a referential which is co-moving with the beam. Let us examine this observation, in the light of possibilities open to us.

The simple relativistic Hamiltonian is:

$$H = c[(\mathbf{P} - q\mathbf{A})^2 + (m_o c)^2]^{\frac{1}{2}} + q\Phi(\mathbf{x}) \ .$$

Here $\mathbf{P}$ is the momentum conjugate to position $\mathbf{x}$ such that $\mathbf{P} = \mathbf{p} + q\mathbf{A}$ where $\mathbf{p} = \gamma_u m_o \mathbf{u}$ is the ordinary kinetic momentum and $\mathbf{A}(\mathbf{x})$ is the magnetic vector potential. The first order equations,

$$\frac{d}{dt}\mathbf{x} = +\frac{\partial H}{\partial \mathbf{P}} = \mathbf{u} \qquad \text{and} \qquad \frac{d}{dt}\mathbf{P} = -\frac{\partial H}{\partial \mathbf{x}} \ ,$$

$$\text{lead to} \qquad \frac{d}{dt}\mathbf{p} \ = \ q\mathbf{E} \ + \ q\mathbf{u} \wedge \mathbf{B}$$

because $-\nabla\Phi - \frac{d}{dt}\mathbf{A} = \mathbf{E}$ and $\mathbf{B} = \nabla \wedge \mathbf{A}$. Here $\mathbf{B}$ and $\mathbf{E}$ are the magnetic and electric fields, respectively.

Suppose now that we wish to write the equations in a new referential (i.e. another inertial frame); then we encounter the problem (see reference[7] section 12.1) that the Hamiltonian above does not transform in a manifestly covariant manner, because it is the time-like component of a 4-vector and the space-like components are not specified.

Why might we wish to make a change of reference frame? In an accelerator there are several possibilities.

(1.) Suppose that in the laboratory frame, there are applied static magnetic fields, an applied electric field wave, and no internal forces (i.e. no space-charge). Then one can find a time-independent Hamiltonian by transforming to a frame where the $\mathbf{E}$-wave is (approximately) stationary.

(2.) Suppose that in the laboratory frame, there are static magnetic fields, no applied electric fields, but significant internal E.M. fields. Then in a frame which moves with the beam, the problem of finding the internal fields is considerably simplified, because there are usually no sources of $\mathbf{B}$ (i.e. no net currents).

There are other good reasons for choosing <u>not</u> not approach the problem by means of the Hamiltonian. Though, in certain circumstances, the Hamiltonian formalism may be used to demonstrate 'nice' symmetry or periodicity properties of the motion (particularly for single particle motion), it does not lend itself to a treatment of non-adiabatic acceleration nor to the description of self-consistent self-forces.
We choose **not** approach the problem by means of the Hamiltonian.

2.3 Working with differences

Accelerator physicists love to work with differences, for example $\mathbf{x}_{gen} - \mathbf{x}_{ref}$ and $\mathbf{v}_{gen} - \mathbf{v}_{ref}$; where $\mathbf{x}$ and $\mathbf{v}$ are position and velocity vectors, and the subscripts 'gen' and 'ref' denote the general

and reference particles, respectively. We may form these differences either in the laboratory frame or in a frame co-moving with the beam. The choice is ours, and both possibilities are explored in this document.

Usually, a principle goal of the investigation is to find equations for small motions (often oscillatory) about a reference point. Also, it can be useful to transform away some overall translation of the system; and so excise large cumulative distance integrals. Both these aims can be achieved in either of two ways:

(1a) Write down the equation of motion for a reference particle, $d\mathbf{p}_{ref}/dt = \mathbf{F}_{ref}$.

(1b) Write down the equation for a general particle, $d\mathbf{p}_{gen}/dt = \mathbf{F}_{gen}$.

(1c) Make the replacements $\mathbf{x}_{gen} = \mathbf{x}_{ref} + \Delta\mathbf{x}$ for positions and $\mathbf{v}_{gen} = \mathbf{u}_{ref} + \Delta\mathbf{v}$ for velocities.

(1d) Finally, subtract these equations to give the evolution of laboratory frame differences. In the non-relativistic case we should find:

$$\Delta\dot{\mathbf{x}} = \Delta\mathbf{v} \qquad \text{and} \qquad m_0\Delta\dot{\mathbf{v}} = (\mathbf{F}_{gen} - \mathbf{F}_{ref}) \equiv \Delta\mathbf{F} \ .$$

(2a) Perform as (1a) and solve for the motion $\mathbf{x}_{ref}(t)$, $\mathbf{u}_{ref}(t)$ as functions of time t.

(2b) Make a (relativistic) change of referential upon the motion of the general particle into a new frame which co-moves with $\mathbf{x}_{ref}(t)$.

(2c) The resulting equations, for $\mathbf{x}'_{gen}(\tau)$ and $\mathbf{v}'_{gen}(\tau)$, will be for the evolution of differences in the co-moving frame.

Here the dot notation for time derivatives is employed, and the superfix prime indicates the new referential with proper time τ.

Route (2) has the advantage that the self-fields are easy to find, but the external forces (confining electric and magnetic fields in the laboratory) must be transformed into the co-moving frame.

Route (1) has the advantage that we can immediately report the laboratory positions of the particles; but the self-fields may be difficult (and/or lengthy) to calculate.

Route (1) is the most widely adopted, and the one we shall follow. However, we stress that the equations are not (as sometimes erroneously stated) those as seen in a frame rotating with the bunch; rather they are laboratory frame equations for the difference between a general and a reference particle.

3 Differential equations in the beam frame

Here we start to investigate route (2). The idea is to solve for the reference particle (which does not see the self-fields) and then transform the equation for a general particle into an inertial frame

in which the reference particle is at rest.

First, let us find how the velocities transform when we go to the co-moving referential. Let the beam frame translate toward the right, with speed v. Motion parallel to v will be denoted $\|$, and perpendicular indicated by $\perp$. We shall use primed variables to denote quantities in the beam frame, and unprimed to indicate laboratory frame quantities. Hence, the velocity component $u = dx/dt$ in the laboratory becomes $u' = dx'/dt'$ in the co-moving frame. Let $\gamma'_u = \gamma_{u'}$ be the gamma factor in the beam frame. The velocity 4-vector for the general particle in both frames is

$$U_4 = \gamma_u(u_\perp,\, u_\|,\, ic) \qquad \text{and} \qquad U'_4 = \gamma'_u(u'_\perp,\, u'_\|,\, ic)$$

$$\text{with} \qquad \gamma_u = 1/\sqrt{1-(u_\perp^2+u_\|^2)/c^2} \quad ,\text{etc..}$$

When the beam frame moves at speed v, as measured in the laboratory frame, then the general transformations relating U_4 and U'_4 are

$$\begin{aligned} \gamma'_u\, u'_\| &= \gamma_v\gamma_u\,[u_\| - v] \\ \gamma'_u\, u'_\perp &= \gamma_u u_\perp \\ \gamma'_u &= \gamma_v\gamma_u\,[1 - v u_\|/c^2]\,. \end{aligned}$$

There are 2 special cases to consider.

(i) First, $u_\| = v$ and $u_\perp = 0$ would imply that $u'_\| = 0$, $u'_\perp = 0$ and $\gamma'_u = 1$, and of course $\gamma_u = \gamma_v$. This is the reference particle.

(ii) Second, $u_\| = v + \Delta u_\|$ and $u_\perp \neq 0$ with conditions $\Delta u_\| \ll v$ and $u_\perp \ll v$, would imply that to first order

$$\left.\begin{aligned} \gamma'_u &\simeq 1 \\ u'_\perp &\simeq \gamma_v u_\perp \\ u'_\| &\simeq \gamma_v^2 \Delta u_\| \end{aligned}\right\} \quad \begin{aligned} &\text{that is, the} \\ &\text{overall translation} \\ &\text{has been removed.} \end{aligned}$$

Next we shall find the equations relating forces to acceleration in the new referential. In the laboratory frame:

$$\begin{aligned} \tfrac{d}{dt}\mathbf{p} &= F \qquad\qquad & \textit{where} \quad \mathbf{p} &= m_0\gamma_u u \\ \tfrac{d}{dt}W &= u\cdot F \qquad\qquad & \textit{where} \quad W &= m_0\gamma_u c^2\,. \end{aligned}$$

The appropriate 4-vector transformation to the co-moving frame is:

$$\gamma_u \frac{d}{dt}\left(p_\perp,\, p_\|,\, \frac{W}{c}\right) \quad \Rightarrow \quad \gamma'_u \frac{d}{dt'}\left(p'_\perp,\, p'_\|,\, \frac{W'}{c}\right)\,.$$

In general

$$\begin{aligned} \gamma'_u \frac{d}{dt'}\, p'_\perp &= \gamma_u \frac{d}{dt}\, p_\perp \\ \gamma'_u \frac{d}{dt'}\, p'_\| &= \gamma_v\gamma_u \left[\frac{d}{dt}\, p_\| - \frac{v}{c^2}\frac{d}{dt}W\right] \\ \gamma'_u \frac{d}{dt'} W' &= \gamma_v\gamma_u \left[\frac{d}{dt} W - v\frac{d}{dt}\, p_\|\right]\,. \end{aligned}$$

Note, if the reference particle is accelerating in the laboratory frame, there is a continuous change of referential. The next step is to substitute the laboratory frame expressions for $\dot{\mathbf{p}}$ and $\dot{W}$ into the right hand side.

3.1 Reference particle

For the special case (i) of $u_\parallel = v$ and $u_\perp = 0$, instantaneously, we find:

$$\begin{aligned}
\frac{d}{d\tau} p'_\perp &= \gamma_v \frac{d}{dt} p_\perp = \gamma_v F_\perp \\
\frac{d}{d\tau} p'_\parallel &= F_\parallel \\
\frac{d}{d\tau} W' &= 0
\end{aligned}$$

where τ is the "proper time" of the frame co-moving with the beam.

3.2 General particle

For the special case (ii) of $u_\parallel = v + \Delta u_\parallel$ and $\Delta u_\perp \neq 0$, we obtain:

$$\begin{aligned}
\frac{d}{d\tau} p'_\perp &\simeq \gamma_v F_\perp \\
\frac{d}{d\tau} p'_\parallel &\simeq F_\parallel - \frac{\gamma_v^2}{c^2} v \left[\Delta u_\parallel F_\parallel + \Delta u_\perp F_\perp\right] \\
\frac{d}{d\tau} W' &\simeq \gamma_v^2 \left[\Delta u_\parallel F_\parallel + \Delta u_\perp F_\perp\right] .
\end{aligned}$$

Since in the co-moving referential, the proper time is used, and the γ-factors are almost (or identically) unity, the Newtonian Kinematic relations apply; and so

$$\mathbf{u}' = \frac{d}{d\tau}\mathbf{x}' \qquad \textit{and} \qquad \mathbf{p}' = m_0 \mathbf{u}' .$$

3.3 Evolution of differences

It is a straight forward matter to compare general and reference particle, and to form the differences

$$\begin{aligned}
m_0 \frac{d}{d\tau} \Delta u'_\perp &\simeq \gamma_v \Delta F_\perp \\
m_0 \frac{d}{d\tau} \Delta u'_\parallel &\simeq \Delta F_\parallel - \frac{\gamma_v^2}{c^2} v \left[\Delta u_\parallel F_\parallel + \Delta u_\perp F_\perp\right] \\
m_0 \frac{d}{d\tau} \gamma'_u &\simeq \frac{\gamma_v^2}{c^2} \left[\Delta u_\parallel F_\parallel + \Delta u_\perp F_\perp\right]
\end{aligned}$$

$$\textit{with} \qquad \mathbf{x}' = \mathbf{x}'_0 + \Delta\mathbf{x}' \qquad \text{and} \qquad \mathbf{u}' = \mathbf{v} + \Delta\mathbf{u}' \qquad \text{and} \qquad \Delta\mathbf{F} = \mathbf{F}(\mathbf{x}') - \mathbf{F}(\mathbf{x}'_0) .$$

Here $\mathbf{x}'_0$ and $\mathbf{x}'$ are the respective positions of the reference and general particles. Now, because the speed of light "c" is so very large, we shall often find

$$\left.\begin{array}{rcl} m_0\frac{d}{d\tau}\Delta u'_\perp &\simeq& \gamma_v \Delta F_\perp \\ m_0\frac{d}{d\tau}\Delta u'_\parallel &\simeq& \Delta F_\parallel \\ m_0\frac{d}{d\tau}\gamma'_u &\simeq& 0 \end{array}\right\} \quad \begin{array}{l}\text{are sufficiently}\\ \text{accurate equations of}\\ \text{motion for differences}\\ \text{in the co-moving frame.}\end{array}$$

3.4 Motion under external and internal forces

The forces acting may be divided into externally applied, and internally active:

$$\mathbf{F}^{tot} = \mathbf{F}^{int} + \mathbf{F}^{ext} \ .$$

The internal forces in the co-moving frame shall be written

$$\mathbf{F}'_{int} \ .$$

The reference particle is only acted upon by the external fields; these fields may be magnetic or electrical or both. Let $\mathbf{v}$ and $\mathbf{p}_{ref} = m_0\gamma_v\mathbf{v}$ be the reference velocity and momentum, found by solving

$$\frac{d}{dt}\mathbf{p}_{ref} = \mathbf{F}^{ext} \qquad \text{and} \qquad \frac{d}{dt}W_{ref} = \mathbf{v}\cdot\mathbf{F}^{ext} \ .$$

The general particle momentum evolves according to

$$\begin{array}{rcl} m_0\frac{d}{d\tau}\Delta u'_\perp &\simeq& [\gamma_v\Delta\mathbf{F}^{ext} + \mathbf{F}'_{int}]\cdot\mathbf{e}_\perp \\ m_0\frac{d}{d\tau}\Delta u'_\parallel &\simeq& [\Delta\mathbf{F}^{ext} + \mathbf{F}'_{int}]\cdot\mathbf{e}_\parallel \\ \gamma(\Delta u') &=& 1 \end{array}$$

where $\mathbf{e}_\perp$ and $\mathbf{e}_\parallel$ are unit vectors perpendicular and parallel to $\mathbf{v}$, respectively. Because the meanings of $\perp$ and $\parallel$ refer to $\mathbf{v}$, so it follows that if $\mathbf{v}(t)$ varies direction so will the unit vectors $\mathbf{e}_\perp$ and $\mathbf{e}_\parallel$.

Note, we *do not* require the $\mathbf{F}^{ext}$ to be transformed (!) but we do have to transform the dependence on arguments

$$\mathbf{F} = \mathbf{F}[\mathbf{x}(\mathbf{x}',\tau),\, t(\mathbf{x}',\tau)] \ ,$$

and finding a convenient explicit form might be difficult. Supposing this can be done, then

$$\mathbf{F}^{ext}_{ref} = q[\mathbf{E}_{ref} + \mathbf{v}\wedge\mathbf{B}_{ref}] \qquad \text{and} \qquad \mathbf{F}^{ext}_{gen} = q[\mathbf{E}_{gen} + \mathbf{u}\wedge\mathbf{B}_{gen}] \ .$$

Let $\mathbf{E}_{gen} - \mathbf{E}_{ref} = \Delta\mathbf{E}[\mathbf{x}(\mathbf{x}')]$ and

$$(\mathbf{u}\wedge\mathbf{B}_{gen}) - (\mathbf{v}\wedge\mathbf{B}_{ref}) = (\mathbf{v}+\Delta\mathbf{u})\wedge(\mathbf{B}_{ref}+\Delta\mathbf{B}) - (\mathbf{v}\wedge\mathbf{B}_{ref}) = \Delta\mathbf{u}\wedge\mathbf{B}_{ref} + \mathbf{v}\wedge\Delta\mathbf{B} \ .$$

Thus for the reference motion

$$\frac{d}{dt}\mathbf{p}_{ref} = q[\mathbf{E}_{ref} + \mathbf{v}\wedge\mathbf{B}_{ref}] \ , \qquad \frac{d}{dt}W_{ref} = q\mathbf{v}\cdot\mathbf{E}_{ref} \ ,$$

and for the general particle

$$m_0\frac{d}{d\tau}\Delta u'_\perp = \gamma_v q\left[\Delta\mathbf{E}+\left(\frac{\Delta u'_\perp}{\gamma_v},\frac{\Delta u'_\parallel}{\gamma_v^2}\right)\wedge(\mathbf{B}^{ref}_\perp,\mathbf{B}^{ref}_\parallel)+\mathbf{v}\wedge\Delta\mathbf{B}\right]_\perp + F'_{int,\perp}$$

$$m_0\frac{d}{d\tau}\Delta u'_\parallel = q\left[\Delta\mathbf{E}+\left(\frac{\Delta u'_\perp}{\gamma_v},\frac{\Delta u'_\parallel}{\gamma_v^2}\right)\wedge(\mathbf{B}^{ref}_\perp,\mathbf{B}^{ref}_\parallel)+\mathbf{v}\wedge\Delta\mathbf{B}\right]_\parallel + F'_{int,\parallel}\,.$$

These are a workable set of equations, but it is perhaps a too radical departure from previous formulations.

Before pursuing the laboratory frame equations, we make the following observation. Obtaining simple equations can depend as much (or more) on choosing a felicitous coordinate basis as upon choosing the 'right' referential. For instance, the use of polar coordinates can transform away, completely, the acceleration of the reference particle when the reference orbit is circular.

4 Differential equations in the laboratory frame

Previously, v was the reference speed and u the general speed. Unfortunately, due to an oversight, we have chosen from now onward to reverse the symbols for reference and general velocities, that is the reference velocity is now to be $\mathbf{u}$ and the general is to be $\mathbf{v}$.

4.1 Formal equations and definition of quantities

The centre of mass (and charge) moves with velocity $\mathbf{u}$. For the reference particle, the rate of change of momentum and energy are given by:

$$m_0\frac{d}{dt}(\mathbf{u}\,\gamma_u)=\mathbf{F}_0^{ex} \qquad \text{and} \qquad m_0c^2\frac{d}{dt}\gamma_u=\mathbf{u}\cdot\mathbf{F}_0^{ex}\,.$$

Here $\mathbf{F}_0^{ex}$ is the externally applied force at the position of the reference particle. Consider an arbitrary particle in the bunch which has some motion with respect to the distribution; its velocity is $\mathbf{v}$. For the general particle the rate of change of momentum and energy in response to forces is

$$m_0\frac{d}{dt}(\mathbf{v}\,\gamma_v)=\mathbf{F}^{ex}+\mathbf{F}^{sc} \qquad \text{and} \qquad m_0c^2\frac{d}{dt}\gamma_v=\mathbf{v}\cdot(\mathbf{F}^{ex}+\mathbf{F}^{sc})\,. \tag{2}$$

$\mathbf{F}_{ex}$ is an externally applied force due to magnets and cavities, say. $\mathbf{F}_{sc}$ is the force due to space-charge; that is arising from the whole assembly of which the 'test' particle is a member.

We may use the energy equations to eliminate the time derivatives of the γ-factors from the momentum equations. Let $\mathbf{F}$ be the sum of all forces in the laboratory frame, and let us *define* the vector $\mathbf{F}_2$ by the expression:

$$\mathbf{F}_2\equiv\mathbf{F}-m_0\mathbf{v}\frac{d}{dt}\gamma_v\;; \tag{3}$$

in which case

$$m_0\gamma_v\frac{d}{dt}\mathbf{v}=\mathbf{F}_2\,.$$

One again, $\mathbf{F}_2$ is divisible into applied $\mathbf{F}_2^{ex}$ and space-charge parts $\mathbf{F}_2^{sc}$. For the reference particle:

$$m_0\gamma_u\frac{d}{dt}\mathbf{u} = \mathbf{F}_2^{ex} \equiv \mathbf{F}_0^{ex} - \mathbf{u}(\mathbf{u}\cdot\mathbf{F}_0^{ex})/c^2 .$$

For the general particle:

$$m_0\gamma_v\frac{d}{dt}\mathbf{v} = \mathbf{F}_2^{ex} + \mathbf{F}_2^{sc} \equiv \mathbf{F}^{ex} + \mathbf{F}^{sc} - \mathbf{v}[\mathbf{v}\cdot(\mathbf{F}^{ex} + \mathbf{F}^{sc})]/c^2 .$$

The next step is to find expressions for $\mathbf{F}^{ex}$; we shall deal with $\mathbf{F}^{sc}$ below in section 5. The reference particle at location $\mathbf{x}_0$ experiences an external magnetic field $\mathbf{B}_0$ and an applied electric field $\mathbf{E}_0$. Hence the reference value of $\mathbf{F}_2$ is

$$\frac{1}{q}\mathbf{F}_2^{ex}(\mathbf{x}_0) = \mathbf{u}\wedge\mathbf{B}_0 + \mathbf{E}_0 - \mathbf{u}(\mathbf{u}\cdot\mathbf{E}_0)/c^2 .$$

The general particle at location $\mathbf{x}$ sees a magnetic field $\mathbf{B}^{ex} = \mathbf{B}_0 + \Delta\mathbf{B}$ and electric field $\mathbf{E}^{ex} = \mathbf{E}_0 + \Delta\mathbf{E}$. Thus the value of $\mathbf{F}_2^{ex}$ at the general particle is

$$\frac{1}{q}\mathbf{F}_2^{ex}(\mathbf{x}) = \mathbf{v}\wedge\mathbf{B}^{ex} + \mathbf{E}^{ex} - \mathbf{v}(\mathbf{v}\cdot\mathbf{E}^{ex})/c^2 .$$

4.2 Evolution of differences

We subtract the reference motion from the general motion, giving:

$$m_0[\gamma_v\dot{\mathbf{v}} - \gamma_u\dot{\mathbf{u}}] = \Delta\mathbf{F}_2^{ex} + \mathbf{F}_2^{sc}(\mathbf{x}) \qquad \text{where} \quad \Delta\mathbf{F}_2^{ex} = [\mathbf{F}_2^{ex}(\mathbf{x}) - \mathbf{F}_2^{ex}(x_0)] .$$

We substitute $\mathbf{v} = \mathbf{u} + \Delta\mathbf{v}$ where appropriate. The magnetic part of $\Delta\mathbf{F}_2^{ex}$ is

$$\mathbf{u}\wedge\Delta\mathbf{B} + \Delta\mathbf{v}\wedge\mathbf{B}_0 + \Delta\mathbf{v}\wedge\Delta\mathbf{B} .$$

Note, to get the correct invariant of motion, it is essential to retain the 2^{nd} order term. The electric part of $\Delta\mathbf{F}_2^{ex}$ is

$$\Delta\mathbf{E} - \mathbf{u}(\Delta\mathbf{v}\cdot\mathbf{E}_0 + \mathbf{v}\cdot\Delta\mathbf{E})/c^2 - \Delta\mathbf{v}(\mathbf{v}\cdot\mathbf{E}^{ex})/c^2$$

If we expand to first order[2] and evaluate the dot-products, we find

$$\Delta\mathbf{E} - \mathbf{u}(\Delta\mathbf{v}\cdot\mathbf{E}_0 + u\Delta E_{\|})/c^2 - \Delta\mathbf{v}(uE_{0,\|})/c^2$$

which can be written as

$$(1/{\gamma_u}^2)\Delta\mathbf{E}_{\|} + \Delta\mathbf{E}_{\perp} - \mathbf{u}(\Delta\mathbf{v}\cdot\mathbf{E}_0)/c^2 - \Delta\mathbf{v}(uE_{0,\|})/c^2$$

As a final step, we shall assume that $\gamma_v \approx \gamma_u$ to a high degree of approximation; then we find the incremental velocity $\Delta\mathbf{v}$ evolves according to:

$$\begin{aligned} m_0\gamma_u\frac{d}{dt}\Delta\mathbf{v} &= F_2^{sc} + q[\mathbf{u}\wedge\Delta\mathbf{B} + \Delta\mathbf{v}\wedge\mathbf{B}_0 + \Delta\mathbf{v}\wedge\Delta\mathbf{B}] \\ &\quad + q[(1/{\gamma_u}^2)\Delta\mathbf{E}_{\|} + \Delta\mathbf{E}_{\perp} - \mathbf{u}(\Delta\mathbf{v}\cdot\mathbf{E}_0)/c^2 - \Delta\mathbf{v}(uE_{0,\|})/c^2] . \end{aligned} \tag{4}$$

[2] I am not sure if this is completely valid...

5 The space-charge force

The purpose of this section is to demonstrate that the space-charge force $\mathbf{F}^{sc}$ is electromagnetic, rather than simply electric. The space-charge term can be expressed in terms of the Coulombic electric fields, due to the assembly of particles (i.e. bunch), as measured (or calculated) in the frame co-moving with the centre of mass of the group. That is $\mathbf{F}^{sc}$ is written in terms of $\mathbf{E}'$, and as resolved into components parallel and perpendicular to velocity $\mathbf{u}$. Now define $\|$ and $\perp$ to mean parallel and perpendicular to $\mathbf{u}$. If there is a fixed vector $\mathbf{A}$, then the values of $\mathbf{A}_\|$ and $\mathbf{A}_\perp$ will change if $\mathbf{u}$ alters direction.

5.1 An expression for space-charge force

Let the space-charge electric field in a frame which is co-moving with $\mathbf{u}$ be $\mathbf{E}'_{sc}$. It is assumed that in the rest frame of the bunch there is an electric field $\mathbf{E}'_{sc}$ and no magnetic field $\mathbf{B}'_{sc}$. The electric field is resolved into components transverse and longitudinal to the beam axis: $E' = E'_\perp + E'_\|$ in the frame moving at velocity $\mathbf{u}$. The laboratory frame is the stationary frame of the principal observer. A Lorentz transformation gives the laboratory frame electric field:

$$E_\perp = \gamma_u E'_\perp \quad ; \quad E_\| = E'_\| \, .$$

In the laboratory frame, there is also a magnetic field:

$$B_\perp = \gamma_u (\mathbf{u} \wedge E'_\perp)/c^2 \quad ; \quad B_\| = \mathbf{0} \, .$$

Hence the space-charge force on a test particle is:

$$\mathbf{F}^{sc} = q \left[E'_\| + \gamma_u E'_\perp + \mathbf{v} \wedge (\mathbf{u} \wedge E'_\perp)\gamma_u/c^2 \right] \, . \tag{5}$$

The 'test-particle' velocity $\mathbf{v}$ is resolved into components perpendicular and parallel to the beam axis: $\mathbf{v} = v_\perp + v_\|$. The velocity $\mathbf{u}$ is defined to be parallel to the beam axis and so $\mathbf{u} = u_\|$; $u_\perp = \mathbf{0}$. A more useful expression for the space-charge force (5) is obtained by using the general vector identity

$$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

to replace the triple vector product, so:

$$\mathbf{v} \wedge (\mathbf{u} \wedge E'_\perp) = (\mathbf{v} \cdot E'_\perp)\mathbf{u} - (\mathbf{v} \cdot \mathbf{u})E'_\perp$$

$$= \left[(v_\perp + v_\|) \cdot E'_\perp\right] \mathbf{u} - \left[(v_\perp + v_\|) \cdot \mathbf{u}\right] E'_\perp = (v_\perp \cdot E'_\perp)\mathbf{u} - (v_\| \cdot \mathbf{u})E'_\perp \, .$$

Substituting the last expression gives

$$\mathbf{F}^{sc} = q \left\{ E'_\| + \gamma_u E'_\perp + \frac{\gamma_u}{c^2}[(v_\perp \cdot E'_\perp)\mathbf{u} - (v_\| \cdot \mathbf{u})E'_\perp] \right\}^{sc} \, .$$

This is resolved into longitudinal and transverse components:

$$\mathbf{F}^{sc}_\| = q \left\{ E'_\| + \frac{\gamma_u}{c^2}(v_\perp \cdot E'_\perp)\mathbf{u} \right\}^{sc}$$

$$\mathbf{F}^{sc}_\perp = q \left\{ \gamma_u E'_\perp - \frac{\gamma_u}{c^2}(v_\| \cdot \mathbf{u})E'_\perp \right\}^{sc}$$

which serves to show that the magnetic force acts to 'stretch' a bunch longitudinally and 'pinch' it transversely.

5.1.1 Energy equation for space-charge

$$m_0 c^2 \frac{d}{dt}\gamma_v = \mathbf{v}\cdot\mathbf{F}^{sc}$$

The vector $\mathbf{v}\wedge(\mathbf{u}\wedge E'_\perp)$ occurring in the space-charge force term, $\mathbf{F}^{sc}$ equation (5), is perpendicular to $\mathbf{v}$ and so their dot product is zero. Thus the magnetic field due to the beam cannot alter the beam energy; and so the equations are self-consistent.

$$m_0 c^2 \frac{d}{dt}\gamma_v = q\left[(v_\parallel \cdot E'_\parallel) + \gamma_u (v_\perp \cdot E'_\perp)\right]^{sc}$$

5.2 $\mathbf{F}_2$ for space-charge

With no approximation, the space-charge contribution to $\mathbf{F}_2$ is given by:

$$\frac{1}{q}\mathbf{F}_2^{sc} \equiv \; E'_\parallel \; - \; \frac{1}{c^2}\mathbf{v}(\mathbf{v}\cdot E'_\parallel) \; + \; \gamma_u E'_\perp\left(1 - \frac{\mathbf{v}\cdot\mathbf{u}}{c^2}\right) \; + \; \frac{\gamma_u}{c^2}(\mathbf{v}\cdot E'_\perp)(\mathbf{u}-\mathbf{v}) \; .$$

We now substitute $\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp = \mathbf{u} + \Delta\mathbf{v}$ and observe that

$$\frac{1}{q}\mathbf{F}_2^{sc} = \; (E'_\parallel + \gamma_u E'_\perp)\left(1 - \frac{u v_\parallel}{c^2}\right) \; - \; \frac{\Delta\mathbf{v}}{c^2}(v_\parallel E'_\parallel + v_\perp \gamma_u E'_\perp) \tag{6}$$

is an exact expression. Now, to first order

$$\left(1 - \frac{u v_\parallel}{c^2}\right) \; = \; \frac{1}{\gamma_u} \; .$$

In fact, $\gamma_v = \gamma_u$ which is a good approximation since the energy spread in a beam $\Delta\gamma/\gamma$ is typically 10^{-3} or less. Also note that the final term of (6) in $\Delta\mathbf{v}/c^2$ is negligible. Hence, now approximately,

$$\mathbf{F}_2^{sc} \; = \; \frac{q}{\gamma_u}\left[E'_\perp \; + \; E'_\parallel \frac{1}{\gamma_u}\right] \; .$$

Note, this could potentially be a poor approximation if there is a coherent oscillation such that the ensemble has, at any instant, a velocity substantially different from the notional reference particle on the equilibrium orbit.

5.3 Comparison of γ_v with γ_u

In the last section (**5.2**) and in the following we approximate the relativistic kinematic factor of a general particle γ_v by that of the reference particle γ_u, in order to get simple expressions for the relative acceleration. The reference velocity is $\mathbf{u}$, and the general velocity is $\mathbf{v} = \mathbf{u} + \Delta\mathbf{v}$ with $\Delta\mathbf{v} = \Delta\mathbf{v}_\parallel + \Delta\mathbf{v}_b ot$.

$$\frac{1}{\gamma_v^2} \; = \; \frac{1}{\gamma_u^2} \; - \; \frac{(2\mathbf{u}\cdot\Delta\mathbf{v} + \Delta\mathbf{v}^2)}{c^2} \; \approx \; \frac{1}{\gamma_u^2} \; - \; \frac{2u\Delta v_\parallel}{c^2} \; .$$

Thus, to first order, it is only the relative motion in the direction of $\mathbf{u}$ that contributes to the difference between the γ-factors, and the perpendicular motion $\Delta v_\perp$ is of no account unless $\Delta v_\parallel < |\Delta v_\perp|/u$.

Let us write $\gamma_v = \gamma_u(1+\epsilon)$ where $|\epsilon| \ll 1$. Then to first order $\epsilon = \gamma_u^2(u\Delta v_{\|}/c^2)$. For example, if $u/c = 1/\sqrt{2}$ then $\epsilon < 10^{-5}$ for $\Delta v_{\|} \leq 1$km/sec. Hence we conclude $\gamma_v = \gamma_u$ is a good approximation for this example.

5.4 On the beam frame magnetic field

It has been assumed that $\mathbf{B}_{sc} = \mathbf{0}$ in the beam frame. This needs immediate criticism and explanation. Strictly, the charges are in motion relative to the bunch centre, so acting as currents they are sources of $\mathbf{B}$. However, in the absence of special preparation and the establishment of ordering over macroscopic distances, the fluctuating microscopic fields are uncorrelated and so (the expectation value of the sum of the micro-fields) macroscopic $\mathbf{B} = \mathbf{0}$. Neither are there any microscopic consequences, for in the vicinity of any one charge there will be an (almost) equal number of currents flowing parallel and anti-parallel, so that there is no net magnetic force upon it.

6 Reference particle motion

The vectors are coordinate independent quantities. A particular representation is projected out by forming the 'dot product' with a complete set of basis vectors. As usual, a careful choice of basis can substantially facilitate the solution of the motion equations.

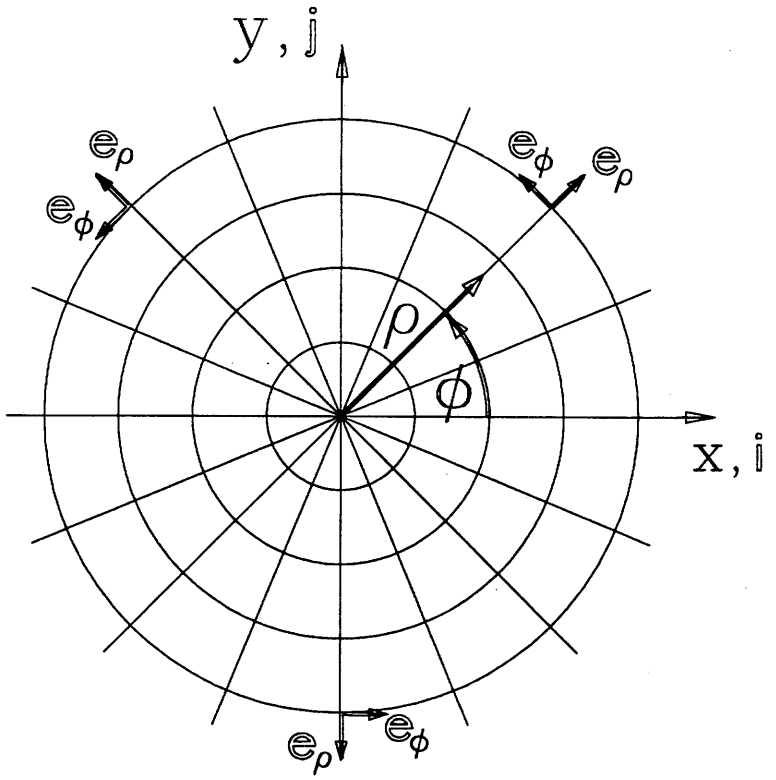

Figure 1: Variation of unit vector directions in a polar coordinate system.

6.1 Coordinate dependent representation

We appeal to the symmetry of a cyclotron, which is basically cylindrical and adopt cylindrical polar coordinates with origin at the centre of the cyclotron. This is a fixed set (in the laboratory frame) of locally orthogonal coordinates but which has the property that the directions in which the basis vectors $\mathbf{e}_\rho$, $\mathbf{e}_\phi$ point, varies from place to place according to the polar angle ϕ. This variation is sketched above, figure 1. $\mathbf{i}$ and $\mathbf{j}$ are unit vectors for the Cartesian coordinates x, y respectively.

The familiar expressions for velocity and acceleration when cartesian coordinates are used, assume a new form when expressed in cylindrical polars. The transformation of coordinates is not trivial. The problem is this, as the particle moves from place to place, so the cylindrical polar basis vectors change their orientation. Note, the basis vectors do not move, it is just that the particle may move to a place where the local basis vectors point in different directions to the basis vectors at the particles previous location.

The reference particle travels anti-clockwise in the median plane, and has time-dependent coordinates $(\rho, z = 0, \phi)$ and position vector $\mathbf{x}_0 = \rho\,\mathbf{e}_\rho$.

$$\text{The derivative of } \mathbf{x}_0 \text{ is} \qquad \mathbf{u} = \mathbf{e}_\rho(\phi)\frac{d}{dt}\rho + \rho\frac{d}{dt}\mathbf{e}_\rho(\phi)\ .$$

Evidently, we shall need expressions for $\frac{d}{dt}\mathbf{e}_\rho$ and $\frac{d}{dt}\mathbf{e}_\phi$, and these can be found after noting that

$$\begin{bmatrix} \mathbf{e}_\rho \\ \mathbf{e}_\phi \end{bmatrix} = \mathbf{M}(\phi)\begin{bmatrix} \mathbf{i} \\ \mathbf{j} \end{bmatrix} \qquad \text{where} \quad \mathbf{M}(\phi) = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \qquad \text{and} \quad \phi = \phi(t)\ .$$

Here $\mathbf{M}$ is the rotation matrix. Taking the time derivative of the unit vectors, and observing that $\frac{d}{dt}\mathbf{i} = 0$ and $\frac{d}{dt}\mathbf{j} = 0$ we find

$$\boxed{\dot{\mathbf{e}}_\rho = +\dot\phi\,\mathbf{e}_\phi \qquad \text{and} \qquad \dot{\mathbf{e}}_\phi = -\dot\phi\,\mathbf{e}_\rho\ .}$$

$$\text{Hence, the derivative of } \mathbf{x}_0 \text{ is} \qquad \mathbf{u} = \dot\rho\,\mathbf{e}_\rho + \rho\,\dot\phi\,\mathbf{e}_\phi\ ,$$

$$\text{and the derivative of } \mathbf{u} \text{ is} \quad \frac{d}{dt}\mathbf{u} = [\ddot\rho - \rho(\dot\phi)^2]\,\mathbf{e}_\rho + [2\dot\rho\,\dot\phi + \rho\,\ddot\phi]\,\mathbf{e}_\phi\ .$$

As soon as we have expressions for $\mathbf{B}_0^{ext}$ and $\mathbf{E}_0^{ext}$, we shall be in a position to find the reference motion.

6.1.1 The reference magnetic field

We shall take the magnetic field $\mathbf{B}_0^{ext}$ components $(B_\rho, B_z, 0)$ to be the isochronous field for a relativistic particle, and now give the expression for $B_z^0(\rho)$ in the median plane. We take equation (13–72) from Livingood[8] in our own rotation.

Let B^0_{cent} be the field at the centre of the cyclotron, as appropriate to $\gamma_u = 1$. Let $\omega_c = \dot\phi_c = (q/m_0)B^0_{cent}$ be the cyclotron frequency for a non-relativistic particle. Define the azimuthal

kinematic $\beta_\phi \equiv (\rho\dot{\phi}_c)/c$ and $\gamma(\rho) \equiv 1/\sqrt{1-\beta_\phi^2}$.

Then the reference field is $\quad B_z^0(\rho) = \gamma(\rho) B_{cent}^0 \quad$ in the median plane.

By symmetry, B_ρ and all its derivatives are equal to zero in the median plane (see Livingood[8] section 2.5).

6.1.2 The external electric field

We shall take the electric field $\mathbf{E}^{ex} = (E_\rho, 0, E_\phi)$ to be the field synchronous with a particle having azimuthal speed $\beta_\phi c$, that is a travelling wave:

$$E_\phi(\phi, t) = \cos[h(\phi - \omega_c t)] E_\phi(\rho)$$

with h the harmonic number and ϕ is the polar angle at which the particle encounters the electric field. For the reference particle, $\phi(t) = \omega_c t$. We can also imagine adding higher harmonics.

6.2 Reference motion in polar coordinates

From section 4, the reference particle equation of motion is:

$$\gamma_u \frac{d}{dt}\mathbf{u} \;=\; \left(\frac{q}{m_0}\right)\left[\mathbf{u} \wedge \mathbf{B}_0^{ext} \;+\; \mathbf{E}_0^{ext} - \mathbf{u}(\mathbf{u}\cdot\mathbf{E}_0^{ext})\frac{1}{c^2}\right] \;. \tag{7}$$

Let us set (q/m_0) = unity for convenience. In cylindrical polars, the vector product $\mathbf{u} \wedge \mathbf{B}_z^0$ can be written

$$\begin{vmatrix} \mathbf{e}_\rho & \mathbf{e}_z & \mathbf{e}_\phi \\ \dot{\rho} & 0 & \rho\dot{\phi} \\ 0 & B_z^0 & 0 \end{vmatrix} \;=\; -\mathbf{e}_\rho[\,\rho\dot{\phi}B_z^0\,] \;+\; \mathbf{e}_\phi[\,\dot{\rho}B_z^0\,] \;.$$

We multiply the motion equation by $\mathbf{e}_\rho$ and $\mathbf{e}_\phi$ to find:

$$\text{Radial equation} \qquad \gamma_u[\ddot{\rho} \;-\; \rho \times (\dot{\phi})^2] \;=\; -\rho\dot{\phi}B_z^0 \;-\; \dot{\rho}(\rho\dot{\phi}E_\phi)/c^2$$

$$\text{Azimuthal equation} \qquad \gamma_u[\,2\dot{\rho}\dot{\phi} + \rho\ddot{\phi}] \;=\; \dot{\rho}B_z^0 \;+\; E_\phi \;-\; \rho\dot{\phi}(\rho\dot{\phi}E_\phi)/c^2 \;.$$

There is no vertical equation, since the reference particle is supposed to move in the median plane.

Note that $B_z^0 = \gamma(\rho)\omega_c$ when $q/m_0 = 1$, but also note that $\gamma_u \neq \gamma(\rho)$ unless the radial speed $\dot{\rho} = 0$ for the reference particle. Here $\gamma_u = 1/\sqrt{1-[\beta_\phi^2 + (\dot{\rho}/c)^2]}$.

Artificially imposing the isochronous constraints that $\ddot{\phi} = 0$ and $\dot{\phi} = \omega_c$ into the radial and azimuthal equations yields

$$\gamma_u\ddot{\rho} = \rho\omega_c^2[\gamma_u - \gamma(\rho)] \;-\; \dot{\rho}\rho\omega_c E_\phi/c^2 \text{ and } 2\gamma_u\dot{\rho}\,\omega_c = \dot{\rho}\omega_c\gamma(\rho) \;+\; E_\phi/\gamma^2(\rho) \;,$$

which are *not mutually consistent* unless $E_\phi(\rho) \equiv 0$, in which case the only self-consistent solution is $\gamma_u = \gamma(\rho)$ and $\dot{\rho} = 0$, both constants; that is $\mathbf{u} = \rho\,\omega_c\mathbf{e}_\phi$. Consequently, if there is to be acceleration, then there is inevitably some loss of instantaneous isochronism. However, this does not exclude the possibility of turn-average isochronism.

Let us allow $E_\phi(\rho) \neq 0$, and insert the turn average quantities $\langle\dot{\phi}\rangle = \omega_c$ and $\langle\ddot{\phi}\rangle = 0$, and find turn averages for $\langle\dot{\rho}\rangle$ and $\langle\ddot{\rho}\rangle$. We rearrange the azimuthal equation into the form:

$$\langle\dot{\rho}\rangle\,\omega_c\,[2\gamma_u - \gamma(\rho)] \;=\; E_\phi/\gamma^2(\rho)\;.$$

On the time average, $\gamma_u = \gamma(\rho)$ because the magnetic field is shaped for isochronism, and so

$$\langle\dot{\rho}\rangle \;=\; E_\phi(\rho)/\omega_c\gamma^3(\rho) \tag{8}$$

which is in agreement with the alternative derivation below; (q/m_0) must be inserted on the right hand side to obtain the correct physical dimensions.

We rearrange the radial equation (turn-averaged) into the form:

$$\gamma_u\frac{\langle\ddot{\rho}\rangle}{\langle\rho\rangle} \;=\; \omega_c^2[\gamma_u - \gamma(\rho)] \;-\; \frac{\langle\dot{\rho}\rangle\omega_c}{c^2}E_\phi(\rho)\;.$$

Inserting the turn-average condition $\gamma_u = \gamma(\rho)$ and substituting for $\langle\dot{\rho}\rangle$ we obtain:

$$\frac{\langle\ddot{\rho}\rangle}{\langle\rho\rangle} \;=\; -\left[\frac{E_\phi(\rho)}{\gamma(\rho)^2\,c}\right]^2\;. \tag{9}$$

Thus, on the average, the beam decelerates in the radial direction. Note, for correct dimensionality $(q/m_0)^2$ must be inserted on the right hand side.

6.2.1 Alternative version of turn averaging

A few thought experiments show the motion $\dot{\rho}_{ref}(t)$ to be very complicated. On one reference orbit $\dot{\rho} = 0$, and one turn later on the next reference orbit $\dot{\rho}$ is once again zero. But, for the reference particle to get from one turn-and-radius to the next turn-and-radius, there must be some motion with $\dot{\rho} > 0$. Hence the motion must be approximately periodic, as sketched below.

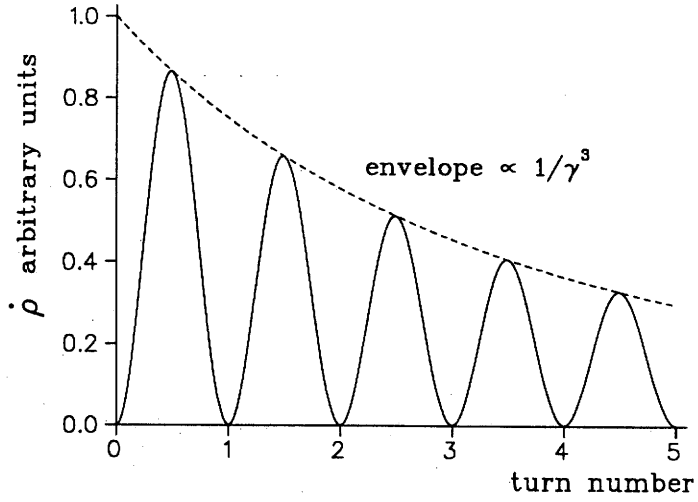

Figure 2: Oscillatory behaviour of $\dot{\rho}(t)$.

Given the previous observations, expecting to find an exact solution of the radial equation is, probably, demanding too much. However, we can hope to obtain turn average values $\langle\dot{\rho}\rangle$ and $\langle\ddot{\rho}\rangle$ which are slowly changing functions and can be found from general properties and by reasoning.

We can find the change of reference orbit radius due to acquiring extra kinetic energy qV where

$$V = \int_0^{2\pi} E_\phi(\rho)\,\rho(\phi)\,d\phi\,. \qquad \text{Now} \quad \dot{\rho} = \frac{d\rho}{d\gamma}\cdot\frac{d\gamma}{dt} \quad \text{but}$$

$$\frac{d\rho}{d\gamma} = \frac{1}{d\gamma/d\rho} = \frac{1}{\gamma_\rho^3}\frac{1}{\rho}\left(\frac{c}{\omega_c}\right)^2 \qquad \text{and} \qquad \frac{d\gamma}{dt} = \frac{1}{m_0c^2}\,qV\,\frac{\omega_c}{2\pi}$$

because time taken for a turn is constant. Let $\gamma_\rho \equiv \gamma(\rho)$. Hence, rewriting, we have

$$\langle\dot{\rho}\rangle = \frac{1}{\gamma_\rho^3}\,\frac{V}{2\pi\rho}\,\frac{q}{m_0}\,\frac{1}{\omega_c} = \left(\frac{q}{m_0}\,\frac{V}{2\pi\omega_c}\right)\cdot\frac{1}{\gamma_\rho^3\,\rho}$$

and if V is a constant, then the turn spacing diminishes because $1/(\rho\,\gamma_\rho^3)$ is monotonically decreasing. Let us replace the average electric gradient $V/(2\pi\rho)$ by $\langle E_\phi\rangle$ and write the cyclotron frequency in terms of the central magnetic field. The average radial speed becomes

$$\langle\dot{\rho}\rangle = \frac{1}{\gamma_\rho^3}\,\frac{\langle E_\phi\rangle}{\omega_c}\left(\frac{q}{m_0}\right) = \frac{1}{\gamma_\rho^3}\cdot\frac{E_\phi}{B_c^0}$$

The turn average radial acceleration $\langle\ddot{\rho}\rangle$ is simply the time derivative of $\langle\dot{\rho}\rangle$ and is a negative, monotonically decreasing function.

7 General particle motion

7.1 On the choice of coordinates

There are several choices for the coordinate system, none of them perfectly suited to completely simplify the problem.

(i) Acknowledge the cylindrical symmetry of the cyclotron applied $\mathbf{E}^{ex}$ and $\mathbf{B}^{ex}$ fields, and adopt polar coordinates. However for the space-charge calculation, this has the awkwardness that the position dependent $\mathbf{e}_\phi$ and $\mathbf{e}_\rho$ must be resolved with respect to $\mathbf{u}_\parallel$ and $\mathbf{u}_\perp$. There is also the fact that we intend calculating the $\mathbf{E}'_{sc}$ on a rectangular grid.

(ii) Acknowledge that with space-charge there are preferential directions parallel and perpendicular to $\mathbf{u}$, and adopt rectangular coordinates. In the absence of acceleration, these preferred directions are, respectively, along $\mathbf{e}_\phi$ and $\mathbf{e}_\rho$ of the reference particle. However the external $\mathbf{B}^{ex}$ and $\mathbf{E}^{ex}$ fields which are most naturally expressed in polar coordinates, may be awkward and must be carefully treated.

From the foregoing it is seen that there is a modicum of difficulty which can be moved around (that is re-partitioned between $\mathbf{F}^{ex}$ and $\mathbf{F}^{sc}$) but not eradicated. Accordingly, we compromise and take a local cartesian (rectangular) coordinate system (r, s, z) centred on the reference particle position expressed in polar coordinates (ρ, z, ϕ).

We have not yet considered acceleration; the awkwardness is extended, because $\mathbf{u}$ is no longer parallel to $\mathbf{e}_\phi$. Assuming we have a local cartesian system centred on the reference particle, there are two choices for the orientation.

(a) Unit vectors (and principal axes) of the local cartesian system to follow $\mathbf{e}_\rho$ and $\mathbf{e}_\phi$.
The forms for $\mathbf{v} = \mathbf{u} + \Delta\mathbf{v}$ and $\frac{d}{dt}\mathbf{v}$, etc. are quite simple; as are the forms for $\mathbf{B}^{ex}$ and $\mathbf{E}^{ex}$.
However, if $u_\rho = \mathbf{u} \cdot \mathbf{e}_\rho$ is non-zero (i.e. $\mathbf{u}$ not parallel to $\mathbf{e}_\phi$), then we encounter a problem with the space-charge calculation.
The Lorentz transformation of $\Delta\mathbf{x}$ to $\Delta\mathbf{x}'$ is non-trivial because $\mathbf{u}$ is not aligned with the coordinate axis parallel to $\mathbf{e}_\phi$.
Also, because $\parallel$ and $\perp$ refer to $\mathbf{u}$, we must take linear combinations (corresponding to a rotation) of the fields $\mathbf{E}'_{sc,\perp}$ and $\mathbf{E}'_{sc,\parallel}$ to give the fields along $\mathbf{e}_\phi$ and $\mathbf{e}_\rho$.

(b) Unit vectors (and principal axes) of the local cartesian system to follow $\mathbf{u}$, $\mathbf{e}_z$ and $\mathbf{u} \wedge \mathbf{e}_z$.
The forms for $\mathbf{x}, \mathbf{v}, \frac{d}{dt}\mathbf{v}$, etc. become really quite complicated, particularly if $\mathbf{u}$ is changing its angular orientation with respect to $\mathbf{e}_\rho$ and $\mathbf{e}_\phi$ at some rate $\dot{\alpha}$.
The space-charge remains fairly easy to calculate; the coordinates do not need to be rotated, and the fields $\mathbf{E}'_\perp$, $\mathbf{E}'_\parallel$ are automatically parallel and perpendicular to $\mathbf{u}$ as are the relative velocities $\frac{d}{dt}(s\,\mathbf{e}_\perp + r\,\mathbf{e}_\parallel)$.
However, the external applied forces $\mathbf{B}^{ex}, \mathbf{E}^{ex}$ become hard to find since they must be resolved along $\mathbf{e}_\perp$ and $\mathbf{e}_\parallel$ whereas the fields take the simplest form when resolved along $\mathbf{e}_\rho$ and $\mathbf{e}_\phi$.

In the sight of all this complexity, we shall consider the case of no acceleration, $\mathbf{E}_\phi = 0$; in which case both alternatives become identical and equivalent.

We take the coordinate system, senses of rotation, and principal directions adopted in the Cazoll Thesis[5]. We shall take s directed along the direction of $\mathbf{u}$, and take r directed radially outward from the cyclotron centre and perpendicular. We shall take (r, s, z) to form a right-handed orthogonal set. The system is sketched below, figure 3. Recall that $\mathbf{x}_0$ is the reference particle position. Hence, the general position vector is: $\mathbf{x} = \mathbf{x}_0 + \mathbf{e}_\rho r + \mathbf{e}_\phi s + \mathbf{e}_z z$ where $\mathbf{x}_0 = \rho\,\mathbf{e}_\rho(\phi)$. Note $\mathbf{e}_\rho(\phi)$ and $\mathbf{e}_\phi(\phi)$ are the polar unit vectors at the reference point, and not the ones at the general particle position. As indicated earlier, we prefer to work with the differences $\Delta\mathbf{x} = \mathbf{x} - \mathbf{x}_0$ and $\Delta\mathbf{v} = \mathbf{v} - \mathbf{u}$.

$$\begin{aligned} \text{Thus} \quad \Delta\mathbf{x} &= \mathbf{e}_\rho r + \mathbf{e}_\phi s + \mathbf{e}_z z \\ \text{and} \quad \Delta\mathbf{v} &= (\dot{r}\mathbf{e}_\rho + r\dot{\mathbf{e}}_\rho) + (\dot{s}\mathbf{e}_\phi + s\dot{\mathbf{e}}_\phi) + \mathbf{e}_z\dot{z} \\ &= (\dot{r} - s\dot{\phi})\mathbf{e}_\rho + (\dot{s} + r\dot{\phi})\mathbf{e}_\phi + \mathbf{e}_z\dot{z} \ . \end{aligned}$$

We must recall that ϕ is the coordinate of the reference particle, and we have specified the reference to be isochronous so that it rotates at constant angular velocity $\dot{\phi} = \omega_c$.

$$\text{Thus} \qquad \Delta\mathbf{v} = (\dot{r} - s\omega_c)\mathbf{e}_\rho + (\dot{s} + r\omega_c)\mathbf{e}_\phi + \dot{z}\mathbf{e}_z \ .$$

The time derivative of $\Delta\mathbf{v}$ will include terms in $\ddot{\phi}$. However, for enforced isochronism, $\ddot{\phi} = 0$. In this case,

$$\begin{aligned} \frac{d}{dt}\Delta\mathbf{v} &= (\ddot{r} - \dot{s}\dot{\phi})\mathbf{e}_\rho + (\dot{r} - s\dot{\phi})\dot{\mathbf{e}}_\rho + (\ddot{s} + \dot{r}\dot{\phi})\mathbf{e}_\phi + (\dot{s} + r\dot{\phi})\dot{\mathbf{e}}_\phi + \mathbf{e}_z\ddot{z} \\ &= \mathbf{e}_\rho[\ddot{r} - 2\dot{s}\omega_c - \omega_c^2 r] + \mathbf{e}_\phi[\ddot{s} + 2\dot{r}\omega_c - \omega_c^2 s] + \mathbf{e}_z\ddot{z} \ . \end{aligned}$$

The general equation of motion is given by expression (4). We write the form appropriate to $\mathbf{E}^{ex} = \mathbf{0}$.

$$\left(\frac{m_0}{q}\right)\gamma_u \frac{d}{dt}\Delta\mathbf{v} = [\Delta\mathbf{v} \wedge \mathbf{B}_0 + (\mathbf{u} + \Delta\mathbf{v}) \wedge \Delta\mathbf{B}]^{ex} + \frac{1}{\gamma_u}\left[\mathbf{E}'_{\parallel}\frac{1}{\gamma_u} + \mathbf{E}'_{\perp}\right]^{sc} .$$

Now $\mathbf{B}_0^{ex} = (B_\rho = 0, B_z^0, B_\phi = 0)$ is the reference field in the medium plane, and $\Delta\mathbf{B} = (\Delta B_\rho, \Delta B_z, 0)$ is the field at point (r, s, z).

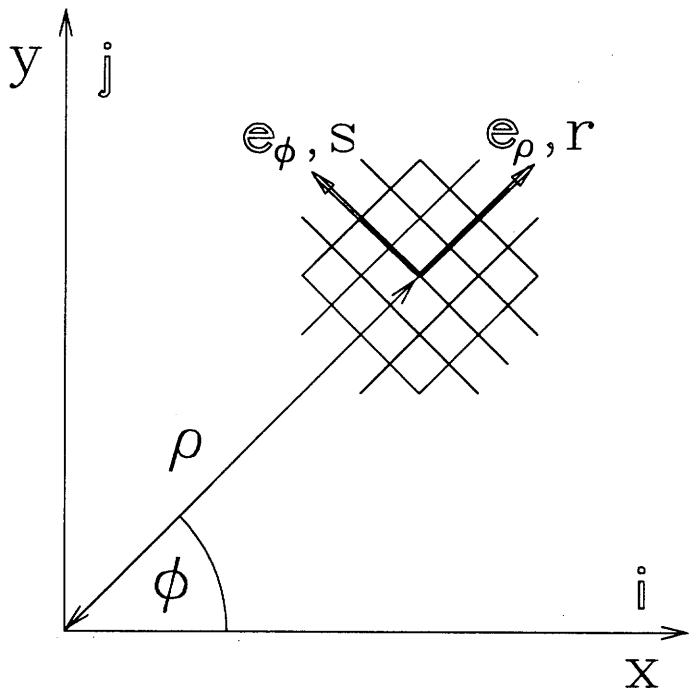

Figure 3: Sketch of relation between global polar and local cartesian coordinate systems.

7.2 Flat field, $\Delta\mathbf{B} = \mathbf{0}$

As noted before, the polar unit vectors vary in direction from place to place. Because, for the reference particle, these vectors are coincident with the r-axis and s-axis of the local cartesian system, we could (for the sake of clarity) re-label the unit vectors as follows: $\mathbf{e}_r \equiv \mathbf{e}_\rho^{ref} \equiv \mathbf{e}_\rho(\phi)$, and $\mathbf{e}_s \equiv \mathbf{e}_\phi^{ref} \equiv \mathbf{e}_\phi(\phi)$. For the flat magnetic field, there is a single Lorentz force term expressible as the determinant:

$$\Delta\mathbf{v} \wedge \mathbf{B}_0^{ex} = \begin{vmatrix} \mathbf{e}_\rho & \mathbf{e}_z & \mathbf{e}_\phi \\ (\dot{r} - s\omega_c) & \dot{z} & (\dot{s} + r\omega_c) \\ 0 & B_z^0 & 0 \end{vmatrix} = -(\dot{s} + r\omega_c)B_z^0\mathbf{e}_\rho + (\dot{r} - s\omega_c)B_z^0\mathbf{e}_\phi .$$

We compare the identities for magnetic and space-charge forces with that for acceleration

$$\frac{d}{dt}\Delta\mathbf{v} = \mathbf{e}_\rho[\ddot{r} - 2\dot{s}\omega_c - \omega_c^2 r] + \mathbf{e}_\phi[\ddot{s} + 2\dot{r}\omega_c - \omega_c^2 s] + \mathbf{e}_z\ddot{z}$$

and form the 'dot products' with $\mathbf{e}_r$, $\mathbf{e}_s$, $\mathbf{e}_z$ to find the respective equations of motion.

Note we have the following equivalent definitions for the electric field components: $\mathbf{E}'_{\parallel} \cdot \mathbf{e}_{\phi}^{ref} = E'_{\parallel} = \mathbf{E}'_{\parallel} \cdot \mathbf{e}_s = E'_s$ and $\mathbf{E}'_{\parallel} \cdot \mathbf{e}_{\rho}^{ref} = 0$, etc;
also we have: $\mathbf{E}'_{\perp} \cdot \mathbf{e}_{\rho} = E'_{\perp} = \mathbf{E}'_{\perp} \cdot \mathbf{e}_r = E'_r$ and and $\mathbf{E}'_{\perp} \cdot \mathbf{e}_{\phi}^{ref} = 0$, etc; and finally $E'_z = \mathbf{e}_z \cdot \mathbf{E}'_{\perp}$.
Hence we shall find motion with respect to the reference particle as follows:

$$\begin{aligned}
\gamma_u(\ddot{r} - 2\dot{s}\omega_c - \omega_c^2 r) &= -\left[\frac{q}{m_0}\right](\dot{s} + r\omega_c)B_z^0 + \left[\frac{q}{m_0}\right]\frac{1}{\gamma_u}E'_r \\
\gamma_u(\ddot{s} + 2\dot{r}\omega_c - \omega_c^2 s) &= +\left[\frac{q}{m_0}\right](\dot{r} - s\omega_c)B_z^0 + \left[\frac{q}{m_0}\right]\frac{1}{\gamma_u^2}E'_s
\end{aligned}$$

$$\text{Note} \quad \left(\frac{q}{m_0}\right)B_z^0 = \gamma(\rho)\omega_c \qquad \text{given} \quad \omega_c = \left(\frac{q}{m_0}\right)B_{cent}^0 \ .$$

Here B_{cent}^0 is the magnetic field at the cyclotron centre. For slow accelerations, $\gamma_u = \gamma(\rho)$ to good approximation, which allows the following expressions.

7.2.1 Summary of the laboratory frame equations

Radial Equation

$$\ddot{r} - \dot{s}\omega_c = \left[\frac{q}{m_0}\right]\frac{1}{\gamma_u^2}E'_r(r,s,z)$$

Azimuthal Equation

$$\ddot{s} + \dot{r}\omega_c = \left[\frac{q}{m_0}\right]\frac{1}{\gamma_u^3}E'_s(r,s,z)$$

Vertical Equation

$$\ddot{z} = \left[\frac{q}{m_0}\right]\frac{1}{\gamma_u^2}E'_z(r,s,z) \ .$$

We may wish to state the derivatives in turns rather than time. Since we have explicitly constrained $\dot{\phi}_{ref} = \omega_c$ to be constant, time and turns accumulate equally. Hence, there is a simple transformation between derivatives. Let $\theta = \omega_c t$, so one turn corresponds to $\theta = 2\pi$. Then

$$\frac{d}{dt} = \omega_c\frac{d}{d\theta} \quad \text{and} \quad \frac{d^2}{dt^2} = \omega_c^2\frac{d^2}{d\theta^2} \ .$$

The motion equations become:

$$\begin{aligned}
(r' - s)' &= (1/\omega_c^2)(q/m_0)\,E'_r(r,s,z)/\gamma_u^2 \\
(s' + r)' &= (1/\omega_c^2)(q/m_0)\,E'_s(r,s,z)/\gamma_u^3 \\
(z')' &= (1/\omega_c^2)(q/m_0)\,E'_z(r,s,z)/\gamma_u^2 \ ,
\end{aligned}$$

where the superfix prime denotes the derivative with respect to radian-turns.

Note, we have incorporated a slight inconsistency; the reference particle 'sees' a gradient magnetic field whereas the general particle sees a field which is flat about the reference particle – the problem disappears if there is no acceleration because we can take a flat, constant field with value γB_{cent}^0 everywhere.

7.2.2 Flat field $\nu = 1$ focusing

Some readers may be perturbed by the absence of explicit $\nu = 1$ focusing in the radial direction. Before giving a more detailed explanation, one may quickly argue that this apparent loss is due to the fact that a particle in our coordinates (r, s) with a positive radial displacement (i.e. $r > 0$) has automatically a greater azimuthal momentum (even when $\dot{s} = 0$), whereas in the 'standard' derivation (see, for instance, reference[8] section 2-6) such a particle has the same momentum as the reference particle but a smaller angular rotation $\dot{\phi}$. Let us also quickly note, that when we solve the cartesian equations of motion we shall, of course, obtain simple harmonic motion with $\nu = 1$.

Here we explicitly show how the assumption of unequal angular velocities enters the usual derivation (for example, Livingood[8]) of the equation of motion. Livingood adopts a polar coordinate system with origin at the cyclotron centre, and two separate radially directed vectors to the reference and general particles. The vectors of length R_{ref} and R_{gen} subtended angles θ and ϕ, respectively, with cartesian unit vector $\mathbf{j}$.
For the reference particle with coordinates $(R_{ref}=\rho,\, \theta)$ the equation of motion is

$$\ddot{\rho} \; - \; \rho(\dot{\theta})^2 = -(q/m)\rho\,\dot{\theta} B_z \; .$$

Imposing the condition $\ddot{\rho} = 0$ implies $\dot{\theta} = (q/m)B_z$.
For the general particle at $(R_{gen}=\rho+r,\, \phi)$

$$\ddot{r} \; - \; (\rho + r)(\dot{\phi})^2 \; = \; -(q/m)(\rho + r)\,\dot{\phi}\, B_z \; .$$

If both particles have the same azimuthal momentum (or velocity) then $\rho\dot{\theta} = (\rho + r)\dot{\phi}$. Substituting for $\dot{\phi}$ we find:

$$\ddot{r} \; - \; (\rho\dot{\theta})^2/(\rho + r) \; = \; -\rho(\dot{\theta})^2$$

and hence, by Taylor expansion, to first order $\ddot{r} + (\dot{\theta})^2 r \approx 0$. Replacing the time derivative with the turn derivative gives $r'' + r = 0$, because $\dot{\theta} = \omega_c$ is equal to the cyclotron angular frequency.

7.3 Toward an approximate form for smooth focusing

Thus far, our equations of motion have only the $\nu = 1$ focusing of a uniform vertical magnetic field. In practise, this is supplemented by stronger vertical and radial focusing due to carefully shaped magnetic fields which vary in the radial and azimuthal directions. This deficiency is now to be partially remedied.

Let us declare from the outset, that the equation of motion we shall devise is not strictly justifiable; and might benefit from a more careful study. It is no simple matter to find equations of motion in the median plane so as to describe a smooth focusing due to the combination of radial field gradients and sectored azimuthal variations of the magnetic field. One key fact remains: whatever coordinates and equations are used, the particle speed must be conserved when only magnetic forces are present; and this rule shall be our guide.

For the reference particle $\mathbf{u}^2 =$ constant, and for the general particle $(\mathbf{u}+\Delta\mathbf{v})^2 =$ constant (in the absence of electric fields). Hence, by subtraction, $\Delta\mathbf{v}^2 + 2\mathbf{u}\cdot\Delta\mathbf{v} =$ constant. In the coordinate system (r, s) previously introduced, and assuming no motion out of the median plane, this means

$$(r' - s)^2 + (s' + r)^2 \; + \; 2\rho(s' + r) = \textit{constant} \; . \qquad (10)$$

The first two terms of (10) are $\Delta\mathbf{v}^2$, and the last term is $2\mathbf{u}\cdot\Delta\mathbf{v}$. Taking the derivative of (10) with respect to time along the reference path gives

$$(r'-s)(r'-s)' \;+\; (s'+r+\rho)(s'+r)' = 0\,. \tag{11}$$

For the case of a uniform vertically directed magnetic field $(0, B_z, 0)$, the first integrals $(r'-s)$ and $(s'+r)$ are both constants and so (10) is also constant, as required.

7.4 Radial field gradient

Let us now investigate the case of a local radial field gradient. We take a local cartesian system $(r,\ s)$ centred at the reference particle; and radius vectors from the cyclotron centre to the reference and general particles of length ρ and $\rho+L$, respectively. The angle formed between the two radius vectors is θ.

We have already evaluated the flat field term $\Delta\mathbf{v}\wedge\mathbf{B}_0$; we must now evaluate $(\mathbf{u}+\Delta\mathbf{v})\wedge\Delta\mathbf{B}$. The additional field at point $(\mathbf{r}, s, z)$ is $\Delta\mathbf{B} = \mathbf{e}_z\Delta B_z + \mathbf{e}_\rho^{gen}\Delta B_\rho$, where $\mathbf{e}_\rho^{gen} = \mathbf{e}_\rho(\phi+\theta)$ is the polar unit vector at the general particle and is not necessarily parallel to $\mathbf{e}_r = \mathbf{e}_\rho(\phi)$. We assume zero azimuthal magnetic field $(B_\phi = 0)$ throughout the volume of the cyclotron, and so the field term in $\mathbf{e}_\phi^{gen}$ is zero.

$$\text{Now} \qquad \Delta B_z = \frac{\partial B_z}{\partial\rho}\delta\rho \;+\; \frac{\partial B_z}{\partial z}\delta z$$

with derivatives evaluated at the reference orbit in median plane, and $\delta\rho$ in the direction of $\mathbf{e}_\rho^{gen}$. Because there is no azimuthal dependence of the magnetic field, the possible term $\frac{\partial}{\partial\phi}B_z\times\delta\phi$ is everywhere zero. Due to the median plane symmetry, $\frac{\partial}{\partial z}B_z = 0$ [see Livingood[8], page 32, section 2.5]. It is conventional to write the first term on the right hand side in terms of the field index $n = \left(\frac{\partial B_z}{\partial\rho}/\frac{B_z}{\rho}\right)$.

$$\text{The radial component is given similarly:} \quad \Delta B_\rho = \frac{\partial B_\rho}{\partial\rho}\delta\rho \;+\; \frac{\partial B_\rho}{\partial z}\delta z\,.$$

Now $\frac{\partial}{\partial\rho}B_\rho = 0$ in the median plane, since $B_\rho = 0$ at all points in the median. We appeal to $\nabla\wedge\mathbf{B} = \mathbf{0}$ to find

$$\frac{\partial B_\rho}{\partial z} = \frac{\partial B_z}{\partial\rho} = n\frac{B_z^0}{\rho}\,.$$

We must also resolve $\mathbf{e}_\rho^{gen}\Delta B_\rho$ into components along $\mathbf{e}_r$ and $\mathbf{e}_s$. We shall write $\delta\rho$ as L and write δz simply as z. We shall use ρ to indicate the reference orbit radius.

$$\text{Thus} \quad (\mathbf{u}+\Delta\mathbf{v})\wedge\Delta\mathbf{B} \;=\; \begin{vmatrix} \mathbf{e}_r & \mathbf{e}_z & \mathbf{e}_s \\ (\dot r - s\omega_c) & \dot z & [\dot s + (\rho+r)\omega_c] \\ \Delta B_\rho\cos\theta & \Delta B_z & \Delta B_\rho\sin\theta \end{vmatrix}$$

where $\Delta B_z(r,s) = n(B_z^0/\rho)\times L$ and $\Delta B_\rho(z) = n(B_z^0/\rho)\times z$. If we consider the median plane motion with $z=0$ and $\dot z = 0$, the Lorentz force expression simplifies to:

$$-\mathbf{e}_r[\dot s + (\rho+r)\omega_c]nB_z^0L/\rho \;+\; \mathbf{e}_s[\dot r - \omega_c s]nB_z^0 z/\rho\,.$$

Substitution of the additional forces, $(\mathbf{u}+\Delta\mathbf{v})\wedge\Delta\mathbf{B}$, into the equations of motion gives

$$(s'+r)' = n(r'-s)\,L/\rho \quad , \qquad (r'-s)' = -n(s'+r+\rho)\,L/\rho \tag{12}$$

which automatically satisfies (11) and so the particle speed is conserved. To close the equations, we must express L in terms of r, s: $L = r\cos\theta + s\sin\theta + \rho(\cos\theta - 1)$ where θ is the angle subtended between radial vectors from the centre of the cyclotron to the general particle and to the reference particle.

The model of motion (12) does not allow for FFAG type focusing, and is somewhat cumbersome; also it has the property that $s \neq 0$, $r \neq 0$, $s' = 0$, $r' = 0$ is not an allowed solution. We shall look for an alternative simpler model of betatron motion, but must sacrifice the constant of motion (10). The new motion will have some constants of motion, but not the correct one (10).

One might be tempted to drop the second order terms which are the product of small quantities, that is neglect $\Delta\mathbf{v} \wedge \Delta\mathbf{B}$, and so consider

$$(s' + r)' = 0 \ , \qquad (r' - s)' = -n(r\cos\theta + s\sin\theta) \approx -nr \ .$$

The choice $-nr$ for the right hand side has two unfortunate consequences: (i) we cannot have static offsets for $s(0)$ and $r(0)$ for the initial conditions; and (ii) the particle speed is not correctly conserved, because $(r' - s) = ns(\theta)$. These undesirable properties imply that we should look for an alternative form for the focusing forces.

7.4.1 Isochronous radial field index

For consistency with the isochronous magnetic field assumed for the reference particle, in section 6.1.1, one should take

$$B_z^0(\rho) = \gamma(\rho)B_c^0 \ , \quad \frac{\partial B_z^0}{\partial \rho} = \gamma^3(\rho)B_c^0\left(\frac{\omega_c}{c}\right)^2\rho$$

and the local value of the field index as

$$n(\rho) = \gamma^2(\rho)\left(\rho\frac{\omega_c}{c}\right)^2 = \gamma^2(\rho)\beta^2(\rho) \qquad \left[\beta(\rho) = \beta_\phi = \frac{\rho\dot{\phi}_c}{c}\right] .$$

7.4.2 Vertical motion

Had we not constrained $\dot{z} = 0$, we should have stated an equation for vertical motion. To first order, the additional vertical force arising from the radial field gradient is $\rho\omega_c\Delta B_\rho$. Replacing $\Delta B_\rho = nB_z^0 z/\rho$ and replacing $(q/m_0)B_z^0 = \gamma(\rho)\omega_c$ we find the approximate Vertical Equation

$$(\ddot{z} - \omega_c^2 nz) = \left(\frac{q}{m_0}\right)\frac{1}{\gamma_u^2}E_z'(r, s, z)$$

The term in $-\omega_c^2 nz$ is *defocusing*, and it is for that reason that gradient focusing must be supplemented by "azimuthally varying field" (AVF) focusing.

7.5 Cazoll's approximation for the focusing

The Cazoll Thesis[5], section I.2.2, gives the form:

$$\mathbf{F}_{focus} = -\mathbf{e}_\rho(r - \rho_0\delta W/2W_0)(\nu^2 - 1)$$

where $2W_0 = m_0(\omega_c\rho_0)^2$ is the (non-relativistic) kinetic energy (K.E.) of the reference particle, δW is the additional K.E. of the general particle, and ν is the tune or number of betatron oscillations per turn. In general $\nu^2 \neq 1 + n$. In the absence of electrical forces, W_0 and δW are constants of motion. No explanation is given for the Cazoll choice of $\mathbf{F}_{focus}$, and we shall argue that it is incorrect.

Firstly, magnetic Lorentz forces are proportional to velocity and not energy. The focusing should aim to give no force for an off-momentum particle which is on the equilibrium orbit for that momentum. Hence the force should look like:

$$\mathbf{F}_{focus} = -\mathbf{e}_\rho(r - v_\theta/\omega_c)(\nu^2 - 1) \tag{13}$$

where v_θ is the azimuthal speed of the general particle relative to the reference particle. Along the line $s = 0$, v_θ is given by $r\omega_c + \dot{s}$. Hence $\mathbf{F}_{focus} = +\mathbf{e}_\rho\, s'(\nu^2 - 1)$ might seem a good choice. The equations of motion would become:

$$\text{azimuthal} \quad (s' + r)' = 0 \ , \quad \text{and radial} \quad (r' - \nu^2 s)' = 0 \ .$$

The equations have the 'nice' property of allowing non-zero static solutions for s and r. However, despite there being two first integrals of motion the particle speed (10) is not correctly conserved.

Secondly, the force is incomplete. To demonstrate this, let us extend the coordinate system to include one full quadrant of the cyclotron, as shown below in figure 4.

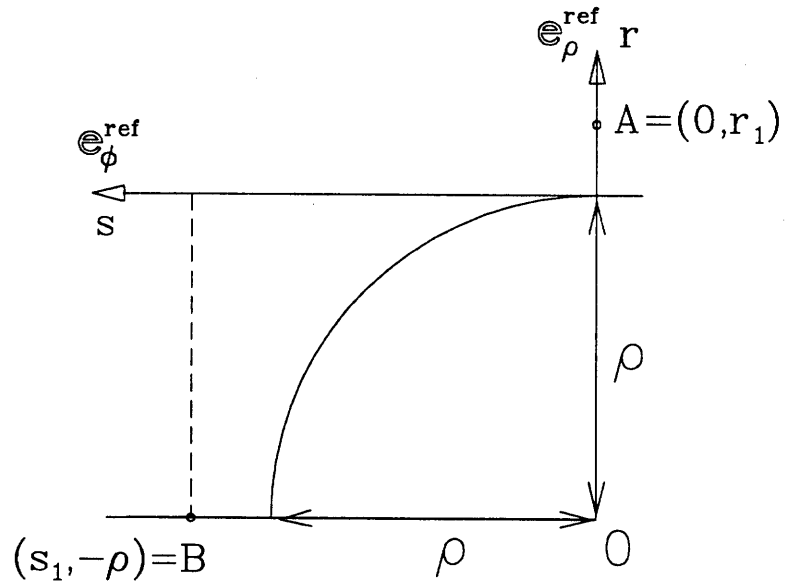

Figure 4: Full quadrant in terms of the local coordinate system r, s .

At point A, the radial displacement is r directed along OA, and the relative azimuthal speed is $v_\phi \equiv \Delta\mathbf{v}\cdot\mathbf{e}_\phi^{ref} = (\omega_c r + \dot{s})$ directed parallel to OB. The focusing force is $-n(r - v_\phi/\omega_c) = n\dot{s}/\omega_c$ directed along OA. The force at A can be found from (13), after putting $\nu^2 \equiv 1 + n$.

At point B, the radial displacement is $(s-\rho)$ directed along OB, and the relative azimuthal speed is $-v_\rho \equiv \Delta\mathbf{v}\cdot\mathbf{e}_\rho^{ref} = [\omega_c(s-\rho) - \dot{r}]$ directed parallel to OA. The focusing force is $-n[(s-\rho)+v_\rho/\omega_c] = -n\dot{r}/\omega_c$ directed along BO. Note well, the force at B cannot be found from (13), and so the expression for $\mathbf{F}_{focus}$ must be incomplete.

Let us try a focusing force which is proportional to the radial displacement (approximately l) from the equilibrium orbit of the off-momentum particle, and which is directed along the radius vector linking the general particle position to the centre of the cyclotron. However, we must also allow for the fact that such a particle has possibly greater momentum and so the focusing is weakened in proportion to the 'extra' azimuthal speed

$$v_\phi = \mathbf{v}\cdot\mathbf{e}_\phi^{gen} - \mathbf{u}\cdot\mathbf{e}_\phi^{ref} \ .$$

Let $\mathbf{e}_r = \mathbf{e}_\rho^{ref}$ and $\mathbf{e}_s = \mathbf{e}_\phi^{ref}$ be unit vectors along the rectangular coordinate directions r, s respectively, and let $\mathbf{e}_\rho^{gen}, \mathbf{e}_\phi^{gen}$ be unit vectors parallel and perpendicular to the general radius vector. Let θ be the angle subtended between radius vectors to the general and reference particles.

Let us suppose a focusing force of the form

$$\mathbf{F}_{focus} = -m[l - v_\phi/\omega_c]\mathbf{e}_\rho^{gen}$$

where $l = r\cos\theta + s\sin\theta$ and $m = \nu^2 - 1$. If we define $v_s \equiv \Delta\mathbf{v}\cdot\mathbf{e}_s = (\dot{s} + \omega_c r)$ and $v_r \equiv \Delta\mathbf{v}\cdot\mathbf{e}_r = (\dot{r} - \omega_c s)$, then we may write $v_\phi = [v_s\cos\theta - v_r\sin\theta]$. Using $\mathbf{e}_\rho^{gen} = \mathbf{e}_s\sin\theta + \mathbf{e}_r\cos\theta$ allows us to resolve the force components, and so find the equations of motion:

$$(s' + r)' = m\cdot\Delta v_\phi\cdot\sin\theta \ , \qquad (r' - s)' = m\cdot\Delta v_\phi\cdot\cos\theta$$

where $\Delta v_\phi = (s'\cos\theta - r'\sin\theta)$. It can be shown that these equations conserve neither $2\mathbf{u}\cdot\Delta\mathbf{v} + \Delta\mathbf{v}^2$ nor $\Delta\mathbf{v}^2$ where $\mathbf{u}$ and $\mathbf{v} = (\mathbf{u} + \Delta\mathbf{v})$ are the laboratory frame velocities of reference and general particle, respectively. Thus we are led, once more, to try another form for the equation of motion.

7.6 Kleeven's approximation for the focusing

Let us restore symmetry to the focusing, and make the force proportional and parallel to the displacements l radial and k azimuthal from the reference particle orbit; and also, to account for differing momenta, weaken the focusing force proportional and perpendicular to the 'extra' azimuthal v_ϕ and radial v_ρ speed of the general particle. Here, as before, $v_\phi = \mathbf{v}\cdot\mathbf{e}_\phi^{gen} - \mathbf{u}\cdot\mathbf{e}_\phi^{ref}$ and

$$v_\rho = \mathbf{v}\cdot\mathbf{e}_\rho^{gen} - \mathbf{u}\cdot\mathbf{e}_\rho^{ref} \ .$$

$$\text{Let} \quad \mathbf{F}_{focus} = -m\mathbf{e}_\rho^{gen}[l - v_\phi/\omega_c] \ + \ m\mathbf{e}_\phi^{gen}[k - v_\rho/\omega_c] \tag{14}$$

$$\text{where} \qquad \begin{bmatrix} l \\ k \end{bmatrix} = M(\theta)\begin{bmatrix} r \\ s \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} v_\rho \\ v_\phi \end{bmatrix} = M(\theta)\begin{bmatrix} v_r \\ v_s \end{bmatrix}$$

$$\text{and} \begin{bmatrix} \mathbf{e}_\rho \\ \mathbf{e}_\phi \end{bmatrix} = M(\theta) \begin{bmatrix} \mathbf{e}_r \\ \mathbf{e}_s \end{bmatrix} \quad \text{with} \quad M(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} .$$

$M(\theta)$ is the rotation matrix. Once more θ is the angle between radius vectors from the cyclotron centre to general and reference particles; the polar unit vectors $\mathbf{e}_\rho^{gen}$,$\mathbf{e}_\phi^{gen}$ and $\mathbf{e}_\rho^{ref}$,$\mathbf{e}_\phi^{ref}$ only become coincident when $\theta \equiv 0$. The coordinate system is sketched below, figure 5.

With the above substitutions, we find $\mathbf{F}_{focus} = m(\mathbf{e}_r s' - \mathbf{e}_s r')$. Hence the equations of motion:

$$\text{azimuthal} \quad (s' + r)' = -m \cdot r' \ , \quad \text{and radial} \quad (r' - s)' = +m \cdot s' \ . \tag{15}$$

There are two constants of motion:

$$s' + \nu \cdot r = const \quad \text{and} \quad r' - \nu \cdot s = const$$

with[3] $\nu = (m + 1)$. However, the fundamental constant (10) is not conserved because there is no term proportional to the the reference particle motion in our equation of motion. Nevertheless, these equations of motion appeal on the grounds of symmetry. They yield betatron motion in the

[3]Note that the definition of m in terms of the tune ν, differs from our earlier definition of $n + 1 = \nu^2$.

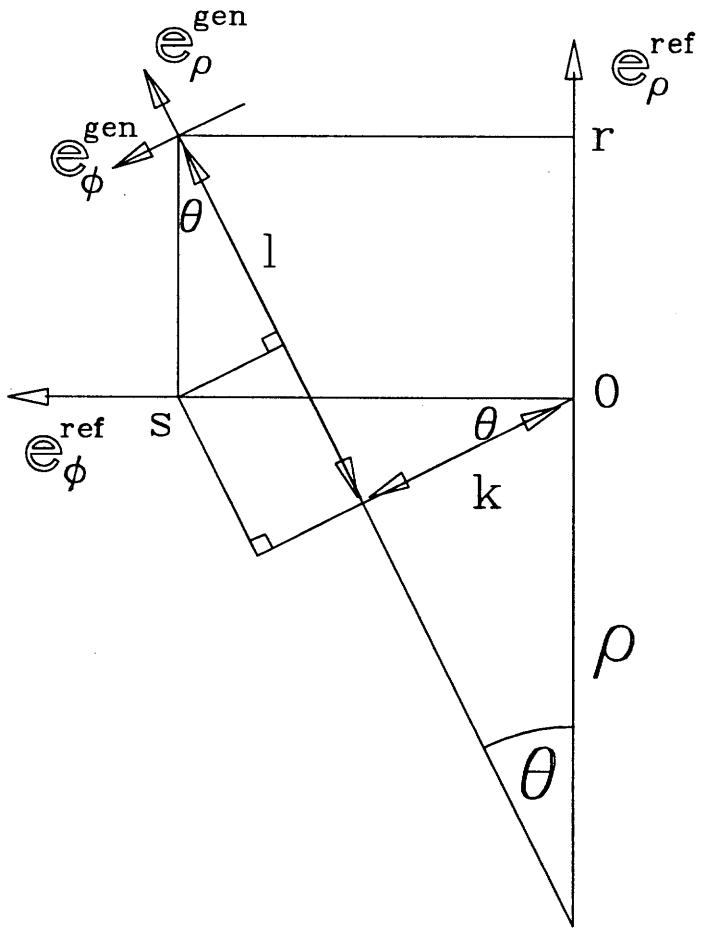

Figure 5: Reference and general unit vectors and the relation of local coordinates r, s to the displacements l, k .

form of circles in the median plane, which property lends itself to finding self-consistent matched charge distributions under internal space-charge forces. Also, the set of equations (15) allow steady state position offsets. Further, these equations are formally identical with those presented in the Kleeven Thesis[6], pages 101-103, and devised by Hamiltonian methods. In the Kleeven notation[4], x replaces our r and the motion is written:

$$\dot{s} + \nu \cdot x = p_s = const \ , \qquad \ddot{x} - \nu \cdot \dot{s} = 0 \ .$$

Finally, we note the focusing force can be explicitly written in a form that looks like the Lorentz form "$\mathbf{v} \wedge \mathbf{B}$". Let $\mathbf{v} = \mathbf{e}_\phi \Delta v_\phi + \mathbf{e}_\rho \Delta v_\rho$, where the extra azimuthal velocity [at point (r, s)] is $\Delta v_\phi = v_\phi - \omega_c l = \dot{s}\cos\theta - \dot{r}\sin\theta$, and the extra radial velocity at (r, s) is $\Delta v_\rho = v_\rho - \omega_c k = \dot{r}\cos\theta + \dot{s}\sin\theta$. By 'extra' we mean that we subtract off those parts of the speed (like $r\omega_c$, $s\omega_c$) which are due merely to the relative displacement from the reference particle. Further, let the extra magnetic field be $\mathbf{B} = \mathbf{e}_z m B_z^0$, then

$$(q/m_0)(\mathbf{v} \wedge \mathbf{B}) \ = \ \omega_c^2 m[\mathbf{e}_\rho \Delta v_\phi - \mathbf{e}_\phi \Delta v_\rho] \ .$$

8 Starting ensemble for PICN

The 'particle' ensemble used in the code PICN follows closely that in PICS; and, as before, each particle point $[s, r, s', r']$ may carry a different charge.

The overall density of points will be the convolution of an *elementary disc* composed of concentric rings of short vertical rods convolved with the distribution of disc centres. Once charges have been assigned to the individual rods, the disc can be loosely referred to as a 'charge cloud'. The disc centres are distributed over the nodes of a *rectangular grid.*

In the earlier code PICS, the cloud centres are distributed homogeneously over a rectangular grid. We shall follow this practise even though there is a small error incurred in not using a curved grid to follow the curvature of the on-momentum equilibrium orbit. In this spirit, 'azimuthal' grid lines are straight and parallel to $\mathbf{e}_s$, and 'radial' grid lines are parallel to $\mathbf{e}_r$. In the coordinate system (r, s) the radial (r) extent of the grid corresponds to the initial azimuthal momentum width of the bunch.

8.1 Aspect ratio of grid, etc.

In order that the overall density of points is roughly uniform in the occupied position space, the size of a 'cell' containing one rod in the disc must be roughly equal to the cell size containing one point/node in the grid of centres. If this were not done, then over-populated bands and under-populated voids would occur in the ensemble. For example, this defect would occur with a grid composed of small cells and a large elementary disc containing only a few representative radii.

Ideally, the grid-cell should be square in real space so that there is a uniform density of cloud centres. However, we had to violate this principle because the initial beam momentum spread is

[4]The coordinates used actually assume a curvilinear basis, rather than cartesian, but this difference is immaterial when terms are only retained to first order.

so narrow that the grid would collapse to a line along $r = 0$. To avoid this, we have introduced a weight YWEIGHT which measures the preference for azimuthal over radial grid lines.

It is essential that the grid cell and rod cell are not identical nor are they simple integer multiples of one another, else Moiré fringes will emerge in the population of points due to interference between the grid pattern and the repeated disc pattern. To avoid this, we have introduced a weight CWEIGHT which measures the preference for increased disc centres over increased particle-points per disc; and whose use results in varying the relative cell size.

8.1.1 Subroutine SQHIST – a distribution diagnostic

The routine SQHIST was written as an aid to finding the optimal value of CWEIGHT. The optimum will depend on the total number of particles NPUSD, the cloud radius BEAMRAD, and YWEIGHT, etc.. However, the optimum is somewhat broad and, fortunately, the final ensemble is somewhat insensitive to the exact value of CWEIGHT. Ordinarily, CWEIGHT=1.0

SQHIST bins the particle positions onto a diagnostic grid (i.e. not the grid of centres). Ideally, if the distribution of points is totally homogeneous and uniform, SQHIST should display a histogram with a single spike; indicating that all cells of the diagnostic grid contain an equal number of particle-points.

8.2 Matched elementary disc

The final particle distribution is composed of a sum of elementary ensembles which are circular in position and in velocity space. We now show how to make such a disc, in the absence of space charge. We take the usual rectangular cartesian coordinates s, r and corresponding unit vectors $\mathbf{e}_s, \mathbf{e}_r$. For a particle whose betatron motion (in the median plane) is centred at $(s = 0, r = 0)$ the integrals of motion are:

$$(s' + \nu \cdot r) = 0 \ , \qquad (r' - \nu \cdot s) = 0 \ .$$

Step 1
Generate a uniformly populated disc of unit radius, and then scale the points by the radial betatron amplitude BEAMRAD.

Step 2
Generate the initial correlated velocities for betatron motion according to

$$r'(0) = \nu s(0) \ , \qquad s'(0) = -\nu r(0) \ .$$

The arguments of r, s indicate time $t = 0$.

8.3 Matched disc under space-charge

It is desirable that the elementary charge clouds be stationary under the action of the internal space-charge force; this facilitates comparison with the earlier ball model used in the code PICS.

The mutual forces between the elementary charge clouds, however, are not compensated for and will cause a perturbation of the matched circular betatronic motion and also cause the cloud centres to move.

We now describe a numerical recipe for stationarity. The cloud consists of concentric rings, and our matching scheme will be to (numerically) adjust the velocity coordinates of particles on each ring so as to give a self-consistent distribution under the space-charge forces caused by the totality of rings.

We take the usual local rectangular cartesian coordinates s, r and (for sake of simplicity) place the centre of the elementary circular charge cloud coincident with the reference particle. The space-charge force is here denoted by $\mathbf{F} = \mathbf{e}_s F_s + \mathbf{e}_r F_r$. The equations of motion are:

$$(s' + \nu r)' = F_s \ , \qquad (r' - \nu s)' = F_r \ .$$

In the computer code PICN, subroutine EULERSTEP, $F_s \equiv FX$ and $F_r \equiv FY$. Let us now take a system of local[5] polar coordinates (ρ, θ) with centre at the reference particle $(s = 0, r = 0)$, such that $s = \rho\cos\theta$ and $r = \rho\sin\theta$. In the new coordinates, the equations of motion are:

$$\rho'' - \rho(\theta')^2 + \nu(\rho\theta') = F_\rho \qquad \text{and} \qquad -2\rho'\theta' - \rho\theta'' + \nu\rho' = -F_\theta \ . \tag{16}$$

$$\text{Here} \quad \begin{pmatrix} F_\rho \\ F_\theta \end{pmatrix} = \mathrm{M}(\theta) \begin{pmatrix} F_s \\ F_r \end{pmatrix}$$

are the radial and azimuthal (with respect to a centre at the reference particle) components of the space-charge force. $M(\theta)$ is the usual rotation matrix, as introduced earlier. For a charge distribution with circular symmetry $F_\theta \equiv 0$. Hence we may look for an equilibrium solution of (16) with $\theta' = constant$, $\rho = constant$. For the charged rods at a particular radius, ρ, the angular velocity θ' is given by:

$$2\theta'(\rho) = \nu \pm \sqrt{\nu^2 - 4F_\rho/\rho} \ .$$

For a tune-shifted betatron motion we should take the positive sign before the square root. The solution with negative sign before the square root tells us that there is no longer a static solution to the equations, and is not used. The matched ensemble is found as follows.

Step 1
Generate a uniformly populated disc of unit radius, and then scale the points by the radial betatron amplitude to give $s(0)$, $r(0)$. The arguments of s and r indicate initial time $t = 0$.

Step 2
Numerically solve for the space-charge forces F_s, F_r and transform to polar coordinates F_ρ, F_θ.

Step 3
Ring by ring evaluate $\theta'(\rho)$. Generate the correlated velocities for betatron motion according to

$$r'(0) = \theta'\, s(0) \ , \qquad s'(0) = -\theta'\, r(0) \ .$$

Note, when we apply this algorithm, we find that the occupied velocity space shrinks compared with the case of no space-charge. Hence the total emittance in $s \cdot s' \cdot r \cdot r'$ space is diminished. This is not an undesirable feature, and is a consequence of the fact that we freely chose (within the constraint of circular symmetry) the position space distribution.

[5]These coordinates have nothing to do with the global polar set with origin at the cyclotron centre.

9 Final transformations of coordinates

There are two further transformations that must be made to go from the motion equations presented previously to the equations appearing in the computer code. First a transform to the PSI standard 'top-down' local cartesians, and second a transform to the 'grid-units' of the space-charge evaluator.

9.1 PSI top-down view of cyclotron orbit

Imagine looking down on to the reference orbit in the cyclotron. The PSI standard, for some 20 years, for the local rectangular cartesian set is a right-handed set (s, r, z) with s directed opposite to the reference motion and r directed radially outward from the cyclotron centre; z is directed up toward the observer. This convention differs from that in the Cazoll thesis and from the notation used in this report so far. To go between the two conventions, one must make the replacements $s \Rightarrow -s$ and $F_s \Rightarrow -F_s$. For example the flat-field equations of motion

$$\text{Cazoll notation} \qquad (s' + r)' = F_s \ , \qquad (r' - s)' = F_r$$

will become

$$\text{PSI notation} \qquad (s' - r)' = F_s \ , \qquad (r' + s)' = F_r \ .$$

Viewed from above, the reference motion is anti-clockwise, while the relative betatron is performed clockwise in the median plane.

9.2 Use of grid coordinates

To facilitate the quick functioning of the space-charge algorithm, the particle positions are expressed in units of the grid used to calculate the space-charge electric field. This grid is not the same as the one used for generating the starting ensemble, and is usually significantly larger than the extent of the ensemble.

Let g_s and g_r be the grid-cell dimensions along directions s and r respectively. We shall now express the equations of motion in terms of the grid-coordinates $\sigma \equiv s/g_s$ and $\rho \equiv r/g_r$. The flat field equations

$$(s' + r)' = F_s^{sc}(s, r) \qquad \text{and} \qquad (r' - s)' = F_r^{sc}(s, r)$$

when expressed in grid-coordinates will become

$$\sigma'' + (g_r/g_s)\rho' = (1/g_s)F_s^{sc}(\sigma, \rho) \qquad \text{and} \qquad \rho'' - (g_s/g_r)\sigma' = (1/g_r)F_r^{sc}(\sigma, \rho) \ .$$

10 Elementary electric force law between two needles

Consider two parallel charged rods of length $2b$ separated by a distance a. The electric field (perpendicular to the rod) at some general point A along the second rod due to charge per unit length $\lambda_1 = Q_1/2b$ on the first rod is

$$E_A = \frac{\lambda_1}{4\pi\epsilon_0}\frac{1}{a}[\sin\theta_2 - \sin\theta_1]$$

where θ_1 and θ_2 are the (radian) angles subtended by the ends of the first rod at the point A. To find the total force on the second rod, we have to integrate over all points A. Let s be the distance of point A from the rod centre measured along the rod. Let $\lambda_2 = Q_2/2b$ be the charge density on the second rod. Then the total force is

$$F = \frac{\lambda_1 \lambda_2}{4\pi\epsilon_0 a} \int_{-b}^{+b} \left[\frac{(b-s)}{\sqrt{(b-s)^2 + a^2}} + \frac{(b+s)}{\sqrt{(b+s)^2 + a^2}} \right] ds \quad \text{or}$$

$$F(a,b) = \frac{Q_1 \times Q_2}{2\pi\epsilon_0 \, a \, (2b)^2} \left[\sqrt{(2b)^2 + a^2} - a \right] .$$

The force between unit charges is derivable from a potential $-\Phi(a) = \int_0^a F(x,b)dx$. The integral is evaluated to give:

$$\Phi(a,b) = \frac{1}{2\pi\epsilon_0} \frac{1}{(2b)^2} \left[a - \sqrt{(2b)^2 + a^2} + 2b \ln\left(2b + \sqrt{(2b)^2 + a^2} \right) - 2b \ln(a) \right] .$$

Within the code PICN, the elementary potential Φ is computed by subroutine POSTDF, from the analytic form given above, at points on a two dimensional grid to give a discretized or sampled form for the potential. Then the derivative is evaluated numerically from the discretized potential by differencing of values on the grid. This is essential so as to give a discretized elementary electric field which has numerically zero divergence; that is to say the Maxwell equation $\nabla \cdot E = 0$ is obeyed in discretized form.

11 Scaling factors and physical units for space-charge

The scaling factors and physical units for the new space charge model of moving vertical needles is derived in the steps to follow.

The electrostatic force between two uniformly charged needles of height $2 \cdot h$ at a distance d is given by the formula:

$$F_{1,2} = [const.] \cdot Q_1 \cdot Q_2 \cdot \frac{2(\sqrt{4h^2 + d^2} - d)}{d \cdot h^2} \quad .$$

In the limiting case, where $h/d \to 0$ this becomes the Coulomb law

$$F_{1,2} = [const.] \cdot Q_1 \cdot Q_2 \cdot \frac{1}{d^2} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q_1 \cdot Q_2}{d^2} \quad .$$

For our purpose, it is better to consider the electric field produced by the charge Q_1 at the place of Q_2:

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q_1}{d^2} \quad .$$

This electric field gets the SI-units [V/m] when Q_1 is in [A·s] and d is in [m]. In the part of the program, where the kernel for the convolution (the force field of a single needle with unit charge) is calculated, the distances are in [mm] and the charge is expressed in units of pico-Coulomb $= 10^{-12}$[As].

The constant needed to calculate the electric field in units of [V/m] from Q_1 in 10^{-12}[As] and d in [mm] is therefore:

$$\frac{1}{4\pi\varepsilon_0} \cdot 10^{-12} \cdot 10^6 = c_{Light}^2 \cdot 10^{-7} \cdot 10^{-12} \cdot 10^6 = 8.987551 \cdot 10^9 \cdot 10^{-6}$$

$$CFELFI = 8.987551 \cdot 10^3 \quad .$$

When the electric field in [V/m] is known, then the acceleration imposed on to a charged particle in this field is defined as $\ddot{x} = F/m = E \cdot Q/m$; thus the factor Q/m for the type of particle in consideration (for this study always the proton) must be evaluated.

The rest-mass of a particle is usually given in [MeV], a value that expresses the mass in terms of energy. In order to get the mass in SI units, this energy has to be divided by c_{Light}^2. For Q/m in SI units we get therefore:

$$\frac{Q}{m} = \frac{Qpart \cdot e_0[As] \cdot c_{Light}^2}{(E_0)_{particle} \cdot e_0[As] \cdot 10^6[V]} \quad .$$

The cancellation of the elementary charge directly yields a conversion factor which has the units of $[m^2/(s^2 \cdot V)]$:

$$\frac{Q}{m} = \frac{Qpart}{(E_0)_{particle}} \cdot \frac{c_{Light}^2}{10^6} = \frac{Qpart}{(E_0)_{particle}} \cdot c_{Light}^2 \cdot 10^{-6} \quad .$$

$$CFQDIVM = \frac{Qpart}{(E_0)_{particle}} \cdot c_{Light}^2 \cdot 10^{-6} = \frac{Qpart}{(E_0)_{particle}} \cdot 8.987551 \cdot 10^{10}$$

For protons, using E_0 =938.256 [MeV] and $Qpart$ =1, this constant gets the value:

$$CFQDIVM = 9.578997 \cdot 10^7 \quad .$$

Multiplying the two conversion factors CFELFI · CFQDIVM, a new conversion factor CFXDDT is derived which yields the acceleration $\ddot{x}$ of a particle (needle) in the field obtained from the convolution of the elementary force field with the charge density. The values for the acceleration $\ddot{x}$ are in SI-units $[m/s^2]$.

$$CFXDDT = CFELFI \cdot CFQDIVM$$

$$CFXDDT = 8.987551 \cdot 10^3 \cdot \frac{Qpart}{(E_0)_{particle}} \cdot 8.987551 \cdot 10^{10}$$

$$CFXDDT = c_{Light}^4 \cdot \frac{Qpart}{(E_0)_{particle}} \cdot 10^{-19}$$

For protons the value of CFXDDT is

$$CFXDDT = 8.6091735 \cdot 10^{11} \quad .$$

The next scaling comes from the fact that the derivative in the differential equation for the evolution of the charge distribution is not taken with respect to time, but with respect to rotation angle of the beam around the cyclotron centre. The number of turns are the values

used to specify the calculation steps and the output, but internally in the integration of the differential equation, the running variable is the angle of rotation measured in radians. A value of 2π for this variable corresponds to one full revolution of the beam which then corresponds to (HARM/FREQ [MHz]) $\cdot 10^{-6}$ seconds. The time conversion factor CFSTORAD based on this is therefore:

$$CFSTORAD = \frac{1}{2\pi} \cdot \frac{HARM}{FREQ\,[MHz]} \cdot 10^{-6} = \frac{1}{2\pi} \cdot 1.974996 \cdot 10^{-7}$$

$$CFSTORAD = 3.143304626 \cdot 10^{-8} \quad .$$

The values of $\ddot{x}$ being in SI-units $[m/s^2]$ whereas the values of x'' are in [Grid-units/(Turn-radians)^2]. The grid-units along the two coordinate axes are not necessarily the same. Therefore the common scaling factor CFXTORAD converts units of $[m/s^2]$ into units of [mm/(Turn-radians)^2]. In order to get the second derivatives x''_1 or x''_2, the "force" components calculated with the common scaling factor have to be divided by $\mathsf{GCELLSIZ}_1$ or $\mathsf{GCELLSIZ}_2$ respectively. Both scalings, the multiplication with the common factor and the division by the value for the cell size in the longitudinal or the radial direction, are included in the subroutine ATRAN As the common factor uses [mm] instead of [m], another factor of 1000 on top of the factor (CFSTORAD)^2 has to be included.

Summing up the above, the general scaling factor CFXDDRAD converting units of [pico-Coulomb/(mm)^2] into [mm/(Turn-radians)^2] becomes

$$\begin{aligned} CFXDDRAD &= CFXDDT \cdot CFXTORAD \\ &= CFXDDT \cdot (CFSTORAD)^2 \cdot 1000 \quad . \end{aligned}$$

$$CFXDDRAD = c_{Light}^4 \cdot \frac{Qpart}{(E_0)_{particle}} \cdot 10^{-19} \cdot \frac{1}{4\pi^2} \cdot \left(\frac{HARM}{FREQ\,[MHz]}\right)^2 \cdot 10^{-12} \cdot 1000$$

$$CFXDDRAD = c_{Light}^4 \cdot \frac{Qpart}{(E_0)_{particle}} \cdot \frac{1}{4\pi^2} \cdot \left(\frac{HARM}{FREQ\,[MHz]}\right)^2 \cdot 10^{-28}$$

$$CFXDDRAD = 0.850617676$$

11.1 An elementary sample case

As a simple case that is useful to test the scaling factors and the principal functioning of the program, two charged balls at a distance of 20 mm are considered. The total current is assumed to be 10 mA. That yields 197.4996 $\cdot 10^{-12}$ [As] of charge per bunch and 98.749827 $\cdot 10^{-12}$ [As] per ball.

The electric field E in [V/m] at a distance of 20 mm from a charge of 98.7498 pico-Coulomb is:

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q_1}{d^2}$$

$$E = c_{Light}^2 \cdot 10^{-7} \cdot \frac{98.7498}{0.02^2} = 2218.797[V/m] \quad .$$

At an average radius of 751.8 mm, the circumference of $2\pi \cdot$Rav of 4723.7 mm is the path a leading particle travels per turn, while it constantly sees the accelerating space charge field of 2.218797 [kV/m]. This results in an energy gain per turn of 10.48 kV

The average radius of an isochronous circular orbit is $\beta \cdot$RCU . For the Injector II , RCU is 9423.4 mm. Using this value , the average radius for 3.00 MeV is 751.7666817 mm and the same for 3.01 MeV is 753.01257 mm. The difference is 1.2459 mm/10 keV. As the rotation of the two balls occurs symmetrically around the common centre the initial rotation speed is $1.2459\,mm \cdot 10.48\,keV/10\,keV\ /\ 10\,mm = 0.130$ radians per turn.

Adapting the above calculation to a distance of 22 mm instead of 20 mm yields an electric field of 1.833717 [V/m], an energy gain per turn of 8.662 [MeV/Turn] and an initial rotation speed of 0.1079 [radians/Turn].

The second sample case with a distance of 22 mm between the two charged vertical needles (and, again, with a current of 10 mA) was directly compared to the values obtained in the program PICN. In this program, the scaling factor CFXDDRAD of 0.850617676 and a grid size of 0.33756 mm in both directions was applied and an acceleration x'' of 0.514 [Grid-units/(Turn-radians)^2] was given out. From this, the value Q/d^2 could be calculated backwards as $x'' \cdot GCELLSIZ/CFXDDRAD$. This gave the same result (0.2040 pico-Coulomb/mm^2) as is obtained when $98.74 \cdot 10^{-12} As$ is directly divided by $d^2 = (22\,\mathrm{mm})^2$

The value of Q/d^2 multiplied by the scaling factor CFELFI yields the electric field of 1832 [V/m]. In this electric field, protons get an acceleration of $\ddot{x} = 1.755 \cdot 10^{14} mm/s^2$. This corresponds to an acceleration x'' of 0.514 [Grid-units/(Turn-radians)^2] as has been found by the program PICN. The factor to convert $\ddot{x}$ into x'' is:

$$\left(\frac{HARM}{FREQ[MHz]}\right)^2 \cdot 10^{-12} \cdot \frac{1}{4\pi^2} \cdot \frac{1}{GCELLSIZ}$$

12 Numerical integration schemes

The code PICS used a 'leap frog' scheme to integrate the equations of motion.

12.1 Leap frog

The leap frog integrator, widely used in plasma physics, is described in reference[9] chapter 4 and is known to be a canonical[10] or symplectic integrator. When the forces acting on particles are of the Lorentz form, involving positions and velocities, a "double leap frog" must be used: two sets of particle positions and velocities are maintained (but separated by half a time-step), and each set is integrated under the forces calculated from the other set. Let the time-step be $\Delta\theta$ and F stand for the forces. The motion is integrated as follows:

$$\begin{aligned}
s_n &\Rightarrow s_{n-1} + s'_{n-1/2}\,\Delta\theta \\
r_n &\Rightarrow r_{n-1} + r'_{n-1/2}\,\Delta\theta \\
s'_{n+1/2} &\Rightarrow s'_{n-1/2} + F_s(s_n, r_n, r'_n) \cdot \Delta\theta \\
r'_{n+1/2} &\Rightarrow r'_{n-1/2} + F_r(s_n, r_n, s'_n) \cdot \Delta\theta
\end{aligned}$$

Neither set of coordinates is more fundamental, and the time evolution of both sets is only an approximation to the true motion. The double leap frog has two difficulties: (i) a starting algorithm has to push one set $\frac{1}{2}$ time-step forward in time; and (ii) it is a low order scheme, requiring small time-steps for accuracy.

12.2 Runge-Kutta

In the code PICN it was decided to replace the leap-frog with a high order scheme: a fourth order time explicit Runge-Kutta integrator. The overhead in storage of large arrays is even greater than that required for a double leap-frog, but can be somewhat reduced by adopting the special low-storage variant given in reference[9] section 4-7-3. This has been coded as subroutine RUKU, which subdivides the given time-step into four and calls the elementary routine EULERSTEP within each of the substeps. The Runge-Kutta is not a symplectic integrator, but there is no cause for concern because in a cyclotron we usually track the particles for only a few decade turns.

To understand the time explicit Runge-Kutta, let us consider the simple second order variant: the mid-point Runge-kutta. Let the vector $\mathbf{x} = (s, s', r, r')$, and $\mathbf{y}$ be an intermediate storage vector, then the algorithm is:

$$\begin{aligned} \mathbf{y} &\Rightarrow \mathbf{x}_n + \mathbf{F}(\mathbf{x}_n) \cdot \Delta\theta/2 \\ \mathbf{x}_{n+1} &\Rightarrow \mathbf{x}_n + \mathbf{F}(\mathbf{y}) \cdot \Delta\theta \, . \end{aligned}$$

The evaluation of $\mathbf{F}$ includes solving Maxwells equations for the internal electromagnetic fields, and calculating the Lorentz forces due to both internal and applied electric and magnetic fields. In words the mid-point integrator operates as follows: we find an approximation to the average $\mathbf{F}$ during the time-step by finding out what $\mathbf{F}$ would become if the particles were to move according to the velocities and forces at the beginning of the time-step, and then apply this better approximation of the average $\mathbf{F}$ to integrate the motion from beginning to end of the time-step.

13 PICN Input data set

The codes PICS and PICN have been modified such that NAMELIST extension to ANSI Fortran is used for specifying the run. Namelist forces us to explicitly associate an input value with a Fortran variable name. If the variable names are chosen judiciously, then we have a 'self-describing' input data set which is easily read and modified by novices. Namelist has other strengths: the number and order of elements in a namelist group has few restrictions, reading a partially empty or null namelist causes no new assignments to be made; and so so new program versions can easily read data-sets intended for earlier program versions (and *vice-versa*). The total NAMELIST is composed of the following groups.

Namelist group RUNSPEC

Variable name	Meaning and example
RUNTITL	The title for a specific run, up to 80 characters. e.g. 'Reference Data Set for 5 MeV Injector II'
PLOTUNITS	This control string determines whether the scatter-plots are in real units (i.e. mm) or grid units. Recognized values are 'REAL' and 'GRID'.
EKINET	The initial kinetic energy of the reference particle (MeV), e.g. 5.0
RAV	The average radius of the reference orbit (mm). Setting the value 0.0, causes RAV to be calculated from EKINET.
CURRAVRG	The average beam current (mA), e.g. 1.0
NPUSDI	The number of macro-particles to be used, e.g. 10000. NPUSDI must be less than NPART, the dimension of particle arrays. NPUSDI will be ignored if NCENTRS and NPCLOUD are non-zero.
MULTBUNCH	The number of sub-bunches to consider. For code PICN, use 0

Namelist group RUNGO

Variable name	Meaning and example
DTSTEP	The basic integration time-step (turns), for example 0.05
DTPLOT	The time between graphical outputs (turns), e.g. 0.5 gives two plots per turn.
TEND	The turn at which the calculation should terminate, e.g. 10.0
PLOTSCALE	A plot scaling parameter used by routine OUTCN, which determines how much of the plot is occupied by the bunch. Example value 5.0

Namelist group GRIDPAR

Variable name	Meaning and example
NXGRIDU	The maximum azimuthal extent of the FFT grid which may occupied by the bunch (in grid units). Ideally, NXGRIDU is less than NX/2. Example value 62, when NX=128
NYGRIDU	The maximum radial extent of the FFT grid which may occupied by the bunch (in grid units). Ideally, NYGRIDU is less than NY/2. Example value 31, when NY=64

Namelist group INITSHAPE

Variable name	Meaning and example
YSHAPECOEF	The initial shape of the bunch, in the median plane and excluding the betatron motion, may be described by a series of (up to 4) polynomial coefficients.
NSHAPECOEF	The number of polynomial coefficient used, for example 0
PHWIDEFF	The effective phase width of the bunch (deg); the actual length will depend on the choice of FMBINX. Example value 15.0
PHINSERT	The length of a uniform current insertion in the bunch (deg), e.g. 5.0
YLENEFF	The effective momentum width of the bunch (mm); the actual width will depend on the choice of FMBINY. Example value 0.5
YINSERT	The width of a momentum insertion in the bunch (mm), e.g. 0.0
BEAMRAD	The maximum amplitude of betatron oscillations in the median plane (mm), for example 2.5
BEAMZRAD	The ensemble average amplitude vertical betatron oscillations (mm), e.g. 2.0
FMBINX	Binomial parameter for azimuthal charge distribution. A value 1.5 will give a parabolic section.
FMBINY	Binomial parameter for radial charge distribution. A value 1.5 will give a parabolic section.
CWEIGHT	Weight which measures the preference for charge centres over charge clouds, for example 1.72
YWEIGHT	Weight which measures preference for azimuthal over radial grid lines, e.g. 5.0
DRNXTBUNCH	The radial distance to the next bunch (mm). For PICN use 0.0

Namelist group NROUND

Variable name	Meaning and example
DENSTY	Control string, soon to be redundant. For PICN use value 'UNIFORM'
PARITY	Controls the polar moments present in the elementary particle cylinder. 'ODDSS' gives only odd moments, 'EVENS' gives only even moments, and 'MIXED' imposes no restriction.
ANGLE1	Control string, which determines the relationship between successive rings of the elementary cylinder. Recognized values are 'DECORR' and 'CORREL'
COPY	Control string, determines whether the elementary cylinders are exact copies of one another, or merely similar. Recognized values 'SIMIL' and 'EXACT'
RADMIN	Inner radius of the elementary cylinder (dimensionless), for example 0.0
RADMAX	Outer radius of the elementary cylinder, and must be less than or equal unity.
RINDEX	Value soon to be redundant. For PICN use 0.0
QINDEX	Value determines the radial distribution of charge within the elementary cylinder (disc formed of short vertical rods), according to a binomial law. The value 0.5 will give an elliptic radial density, with a parabolic projection.
NCENTRS	The number of charge cloud centres. The value 0 results in a calculated default which attempts to produce a uniform density of centres.
NPCLOUD	The number of needles per cloud. The value 0 results in a calculated default.
ISTABILIZ	Control variable which enables or disables subroutine STABILIZE. The value 1 causes the elementary charge clouds to be matched under internal space-charge forces.

Namelist group CYCLOTRON

Variable name	Meaning and example
FREQ	The radio-frequency (MHz), for example 50.633
HARM	The harmonic of r.f. operation, for example 10.0
GAP	The gap width (mm), for example 40.0
EZERO	The rest mass energy (MeV), for example 938.2561 for protons.
QPART	The basic charge state, for example 1.0 for H^+.
BFINDEX	The local magnetic field index. This variable is soon to be redundant and will be replaced by TUNE, the number of oscillations per turn.

Namelist group ACCEL

Variable name	Meaning and example
VOLTOFF	Constant offset part of the voltage shape (kV) Two values expected for 1st and 2nd harmonic, e.g. 188.221, 23.57
VOLTCOS	The amplitude of the cosine part of the voltage shape (kV) Two values expected, e.g. 61.52, 41.45
RFPHASE	The r.f. phase of the bunch centre (deg) with respect to 1st and 2nd harmonic waveforms. Two values expected, e.g. 0.0, 180.0
RFCOSMID	Two values expected, e.g. 3719.0, 2145.0
RFCOSLEN	Two values expected, e.g. 3332.0, 1881.0

14 PICN Parameter variables

Variable name	Meaning and example
CLIGHT	speed of light/10^8.
NPART	The dimension of particle arrays.
NX	The space-charge FFT grid extends from -NX to +NX.
N2	Hence the longitudinal dimension of the grid is N2=2×NX
NY	The space-charge FFT grid extends from -NY to +NY.
N1	Hence the transverse dimension of the grid is N1=2×NY

In order that the space-charge FFT arrays within the code PICN may be quickly redimensioned in a consistent manner throughout many subroutines, as might be needed when NXGRIDU and/or NYGRIDU are increased, we have placed the relevant parameters in a file PIC_FFTDIM.FOR which is Fortran included at the appropriate locations. The file will look roughly as follows.

```
INTEGER NY, NX, N1, N2, KK
PARAMETER (NY=128,NX=128,N1=2*NY,N2=2*NX,KK=N1*N2)
```

References

[1] M.M. Gordon: The Longitudinal Space Charge Effect and Energy Resolution; Proceedings of the 5th International Cyclotron Conference, Oxford 1969, pgs 305-317.

[2] C. Chasman and A.J. Baltz: Space Charge Effects in a Heavy Ion Cyclotron; Nuclear Instruments and Methods in Physics Research 219 (1984), pgs 279-283.

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[5] V. Cazoll: Theoretical and Numerical Study of a Sector Focused Cyclotron under Space-Charge; Doctoral Thesis presented at the University Paris-Sud, November 1988.

[6] W.J.G.M. Kleeven: Theory of Accelerated Orbits and Space-Charge Effects in an AVF Cyclotron; Doctoral Thesis presented at the Technical University of Eindhoven, Holland, August 1988.

[7] J.D. Jackson: Classical Electrodynamics; Published by John Wiley & Sons, 1975.

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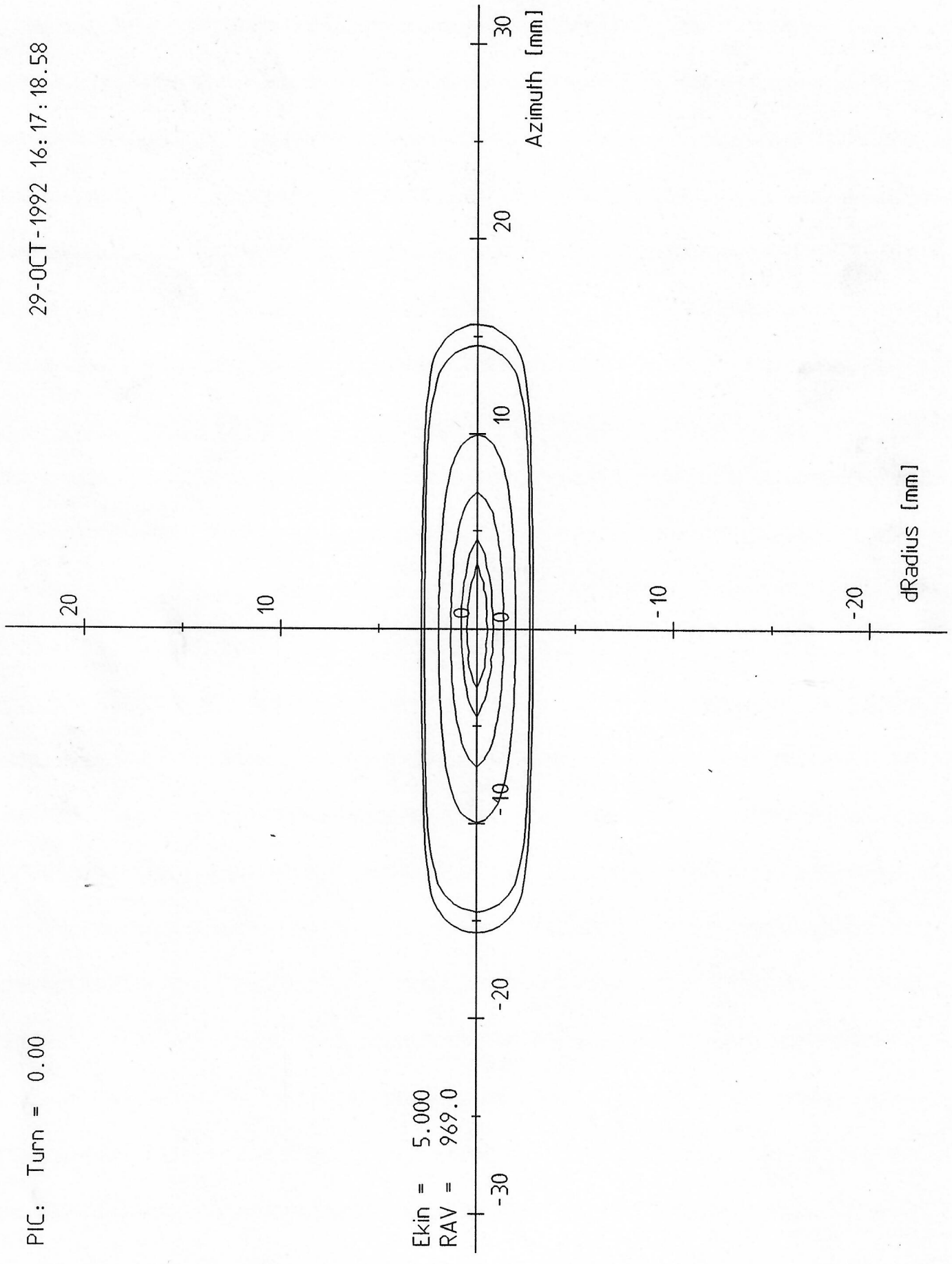
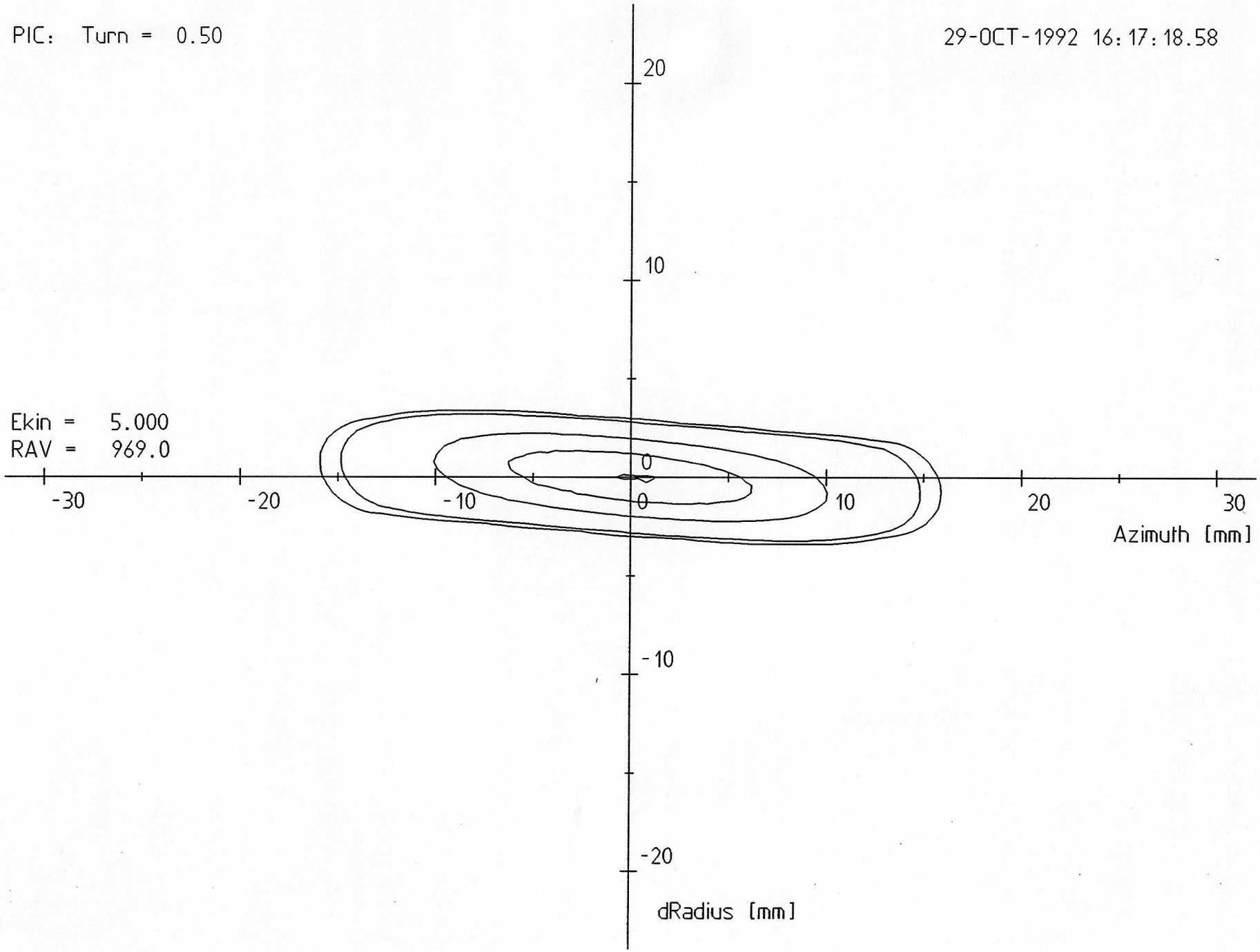
[9] R.W. Hockney and J.W. Eastwood: Computer Simulation Using Particles; Published by Adam Hilger, 1988.

[10] R.D. Ruth: A Canonical Integration Technique; IEEE Trans. on Nuclear Science, Vol NS-30, No.4, August 1983, pg 2669.

First fully functional run of PICN.



First fully functional run of PICN. Qr = 1.26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RAV = 969.0
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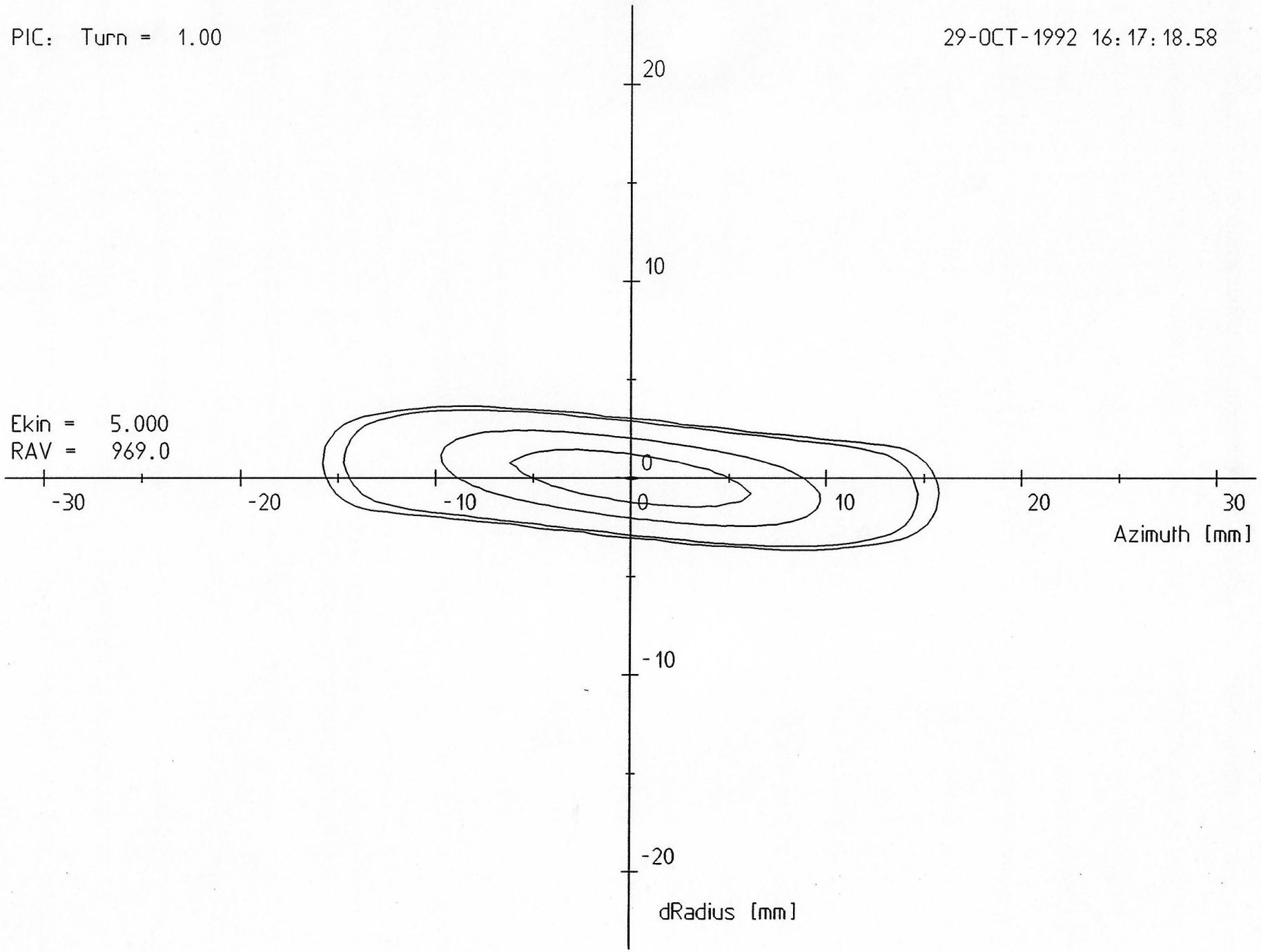
Azimuth [mm]
-10
-20
dRadius [mm]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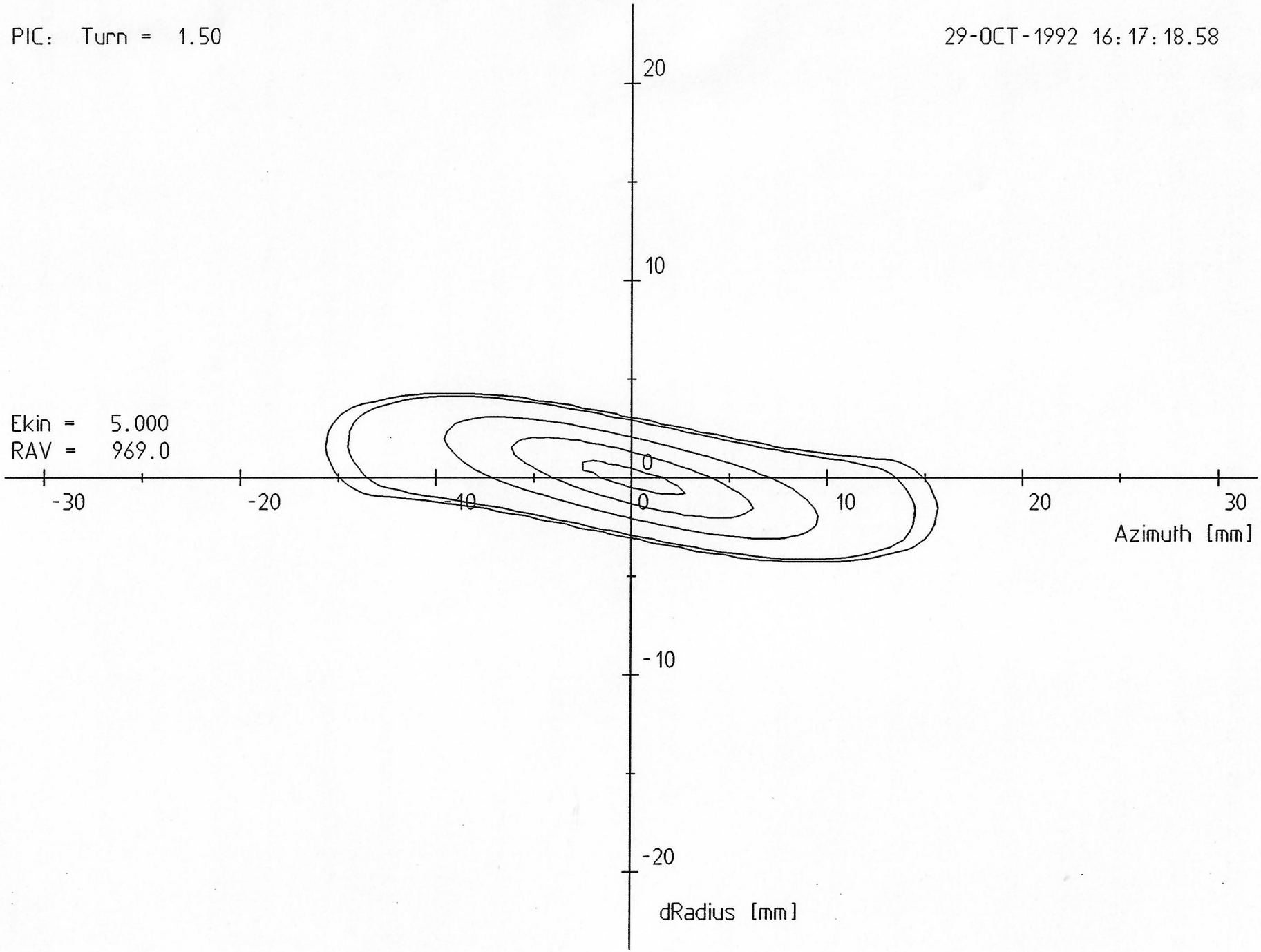
dRadius [mm]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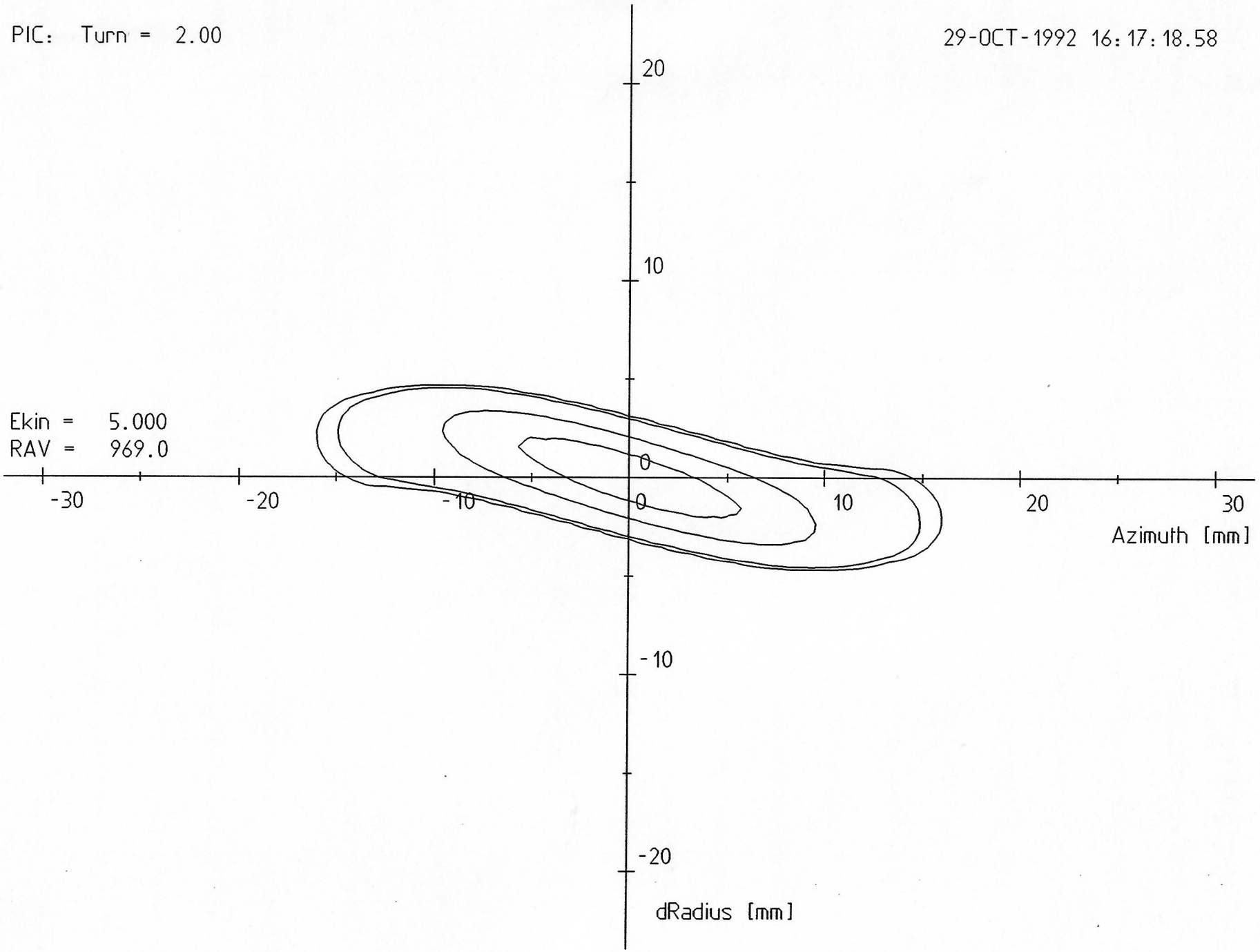
Azimuth [mm]
dRadius [mm]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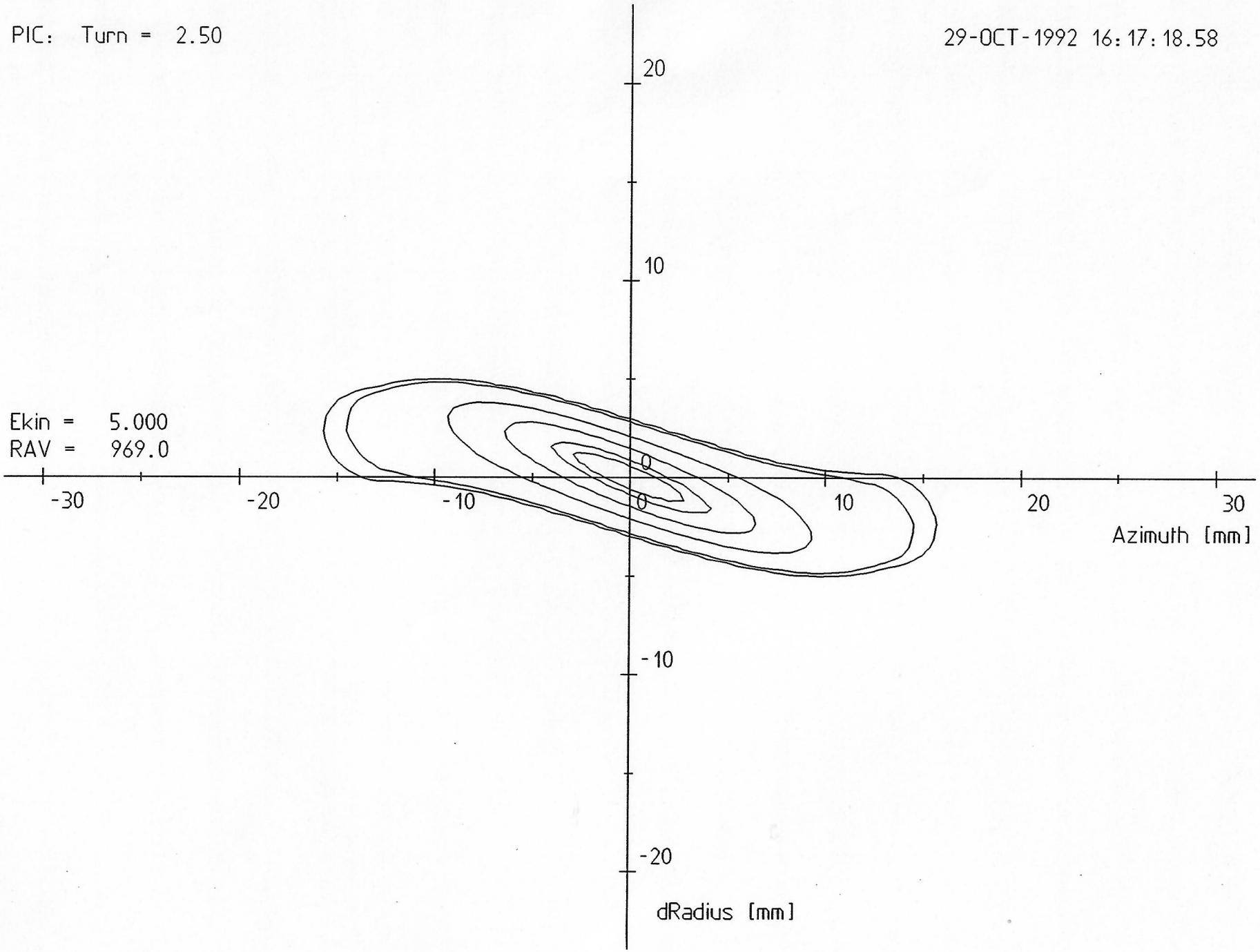
Azimuth [mm]
dRadius [mm]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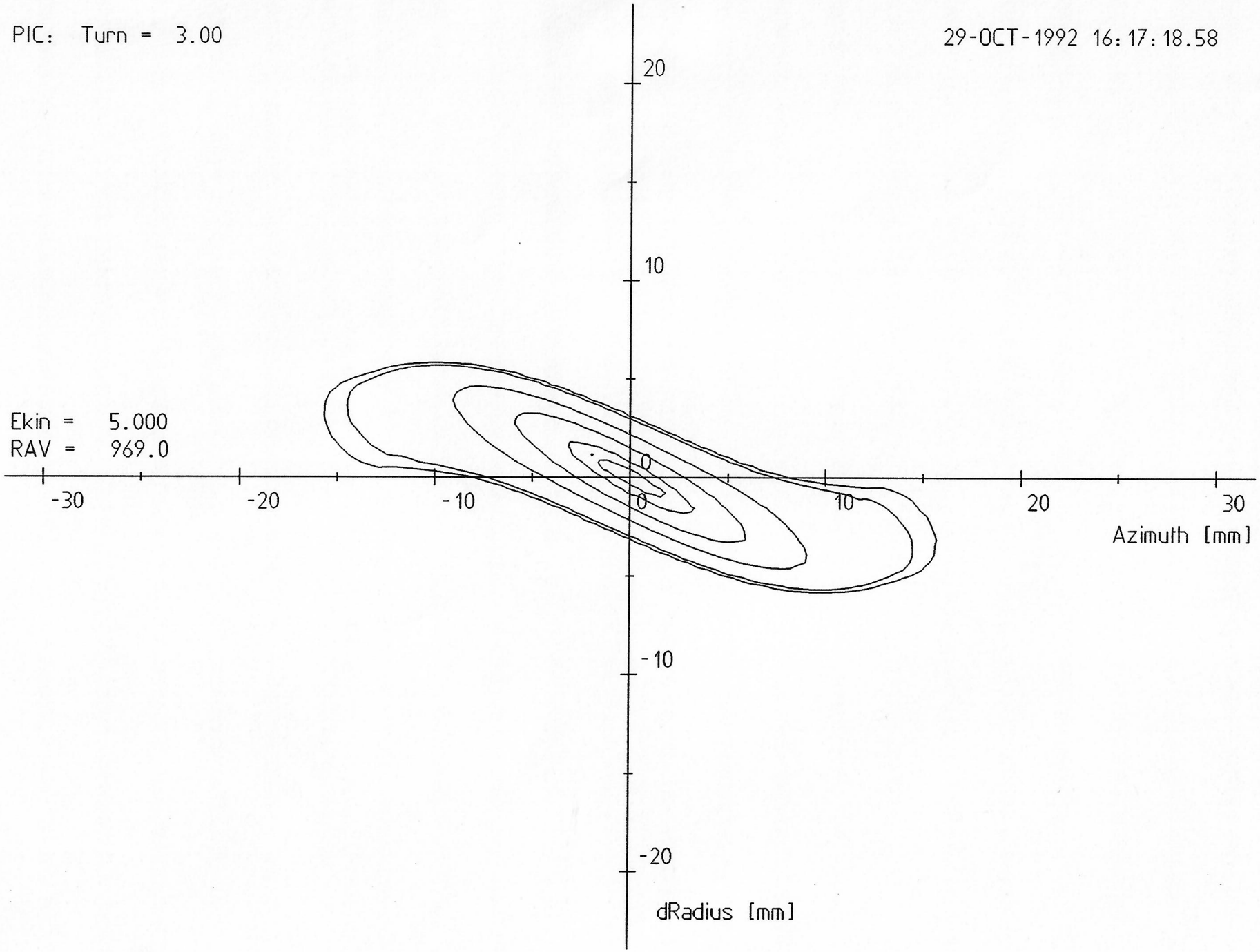
Azimuth [mm]
dRadius [mm]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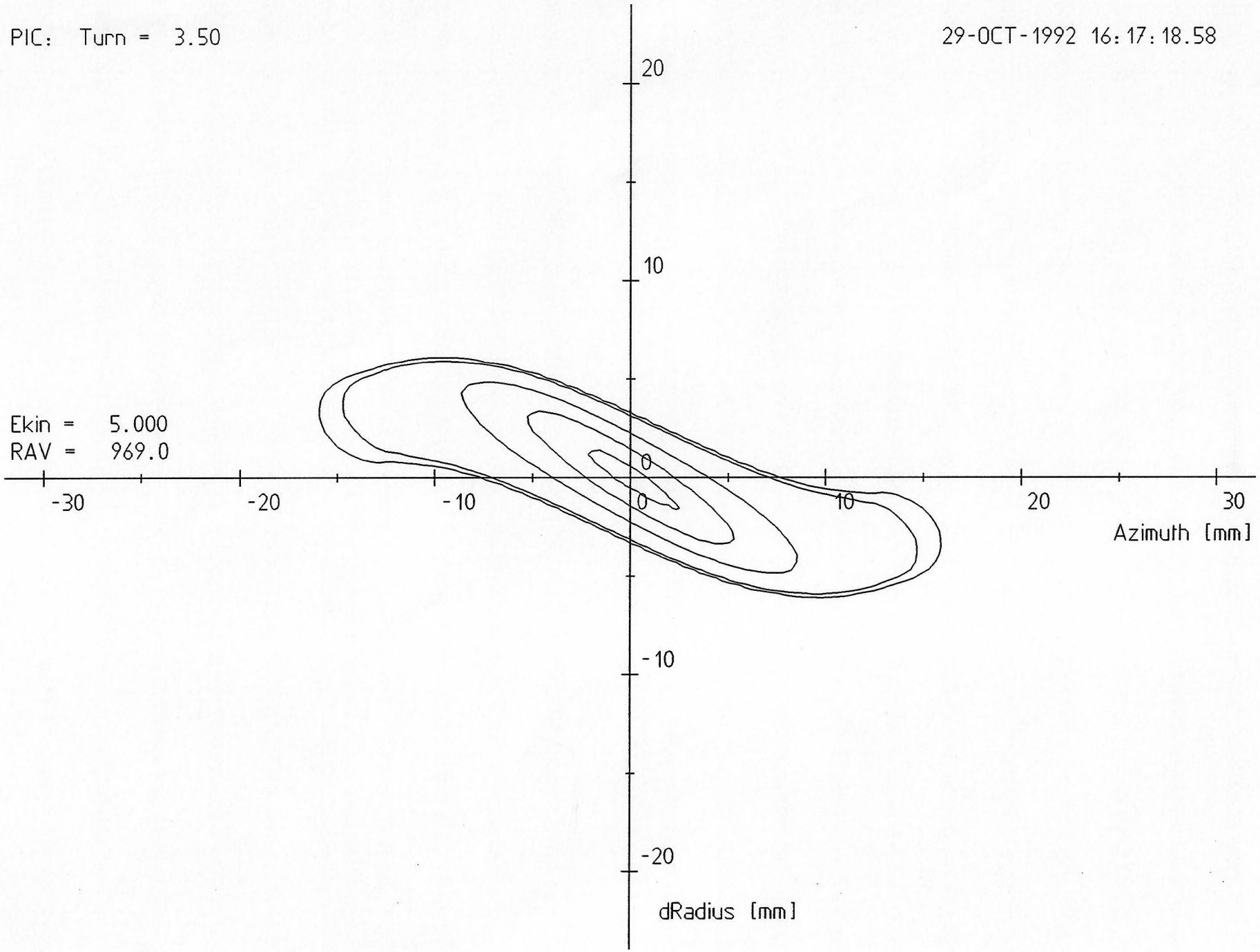
30
Azimuth [mm]
dRadius [mm]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RAV = 969.0
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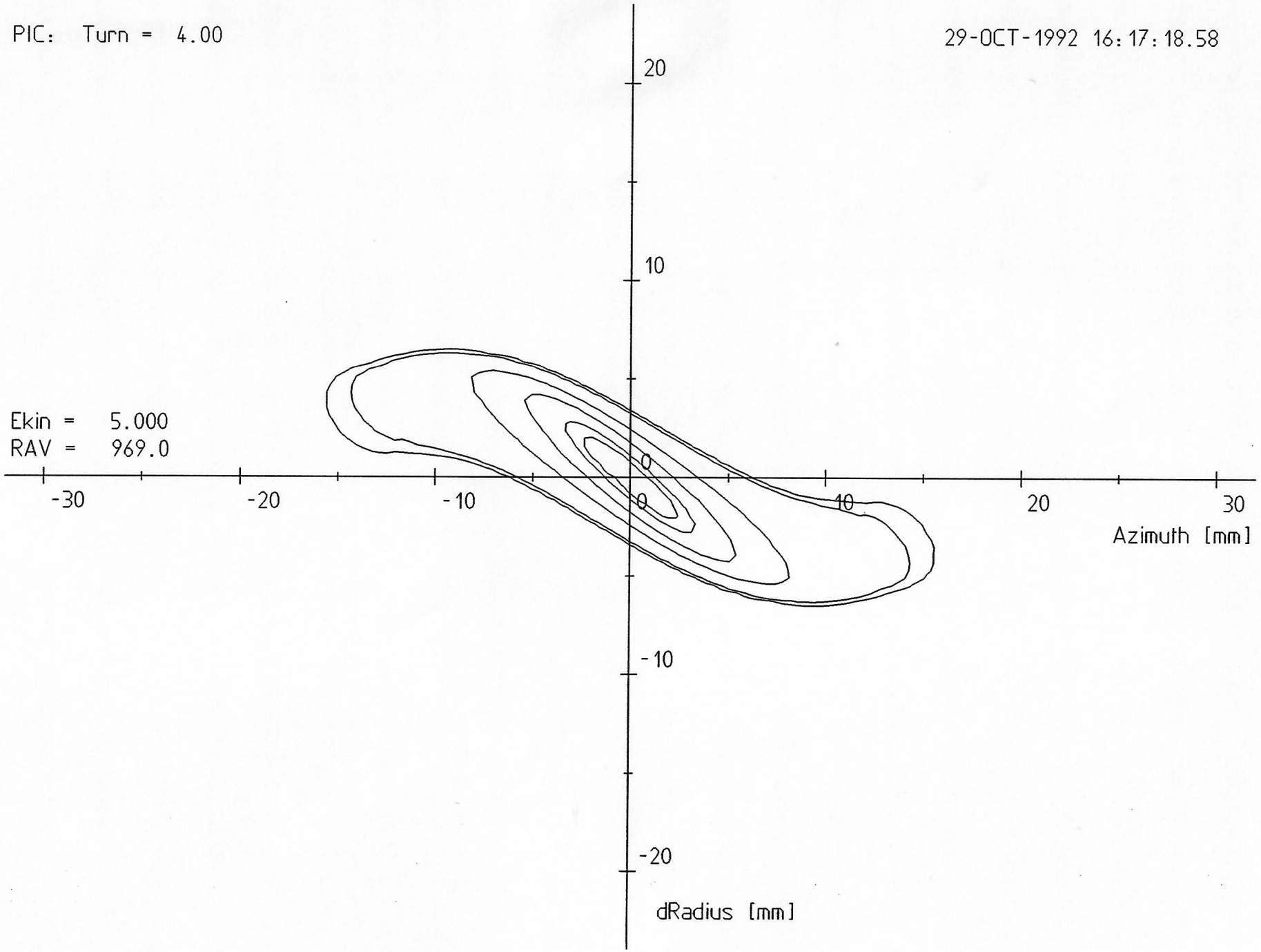
Azimuth [mm]
dRadius [mm]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RAV = 969.0
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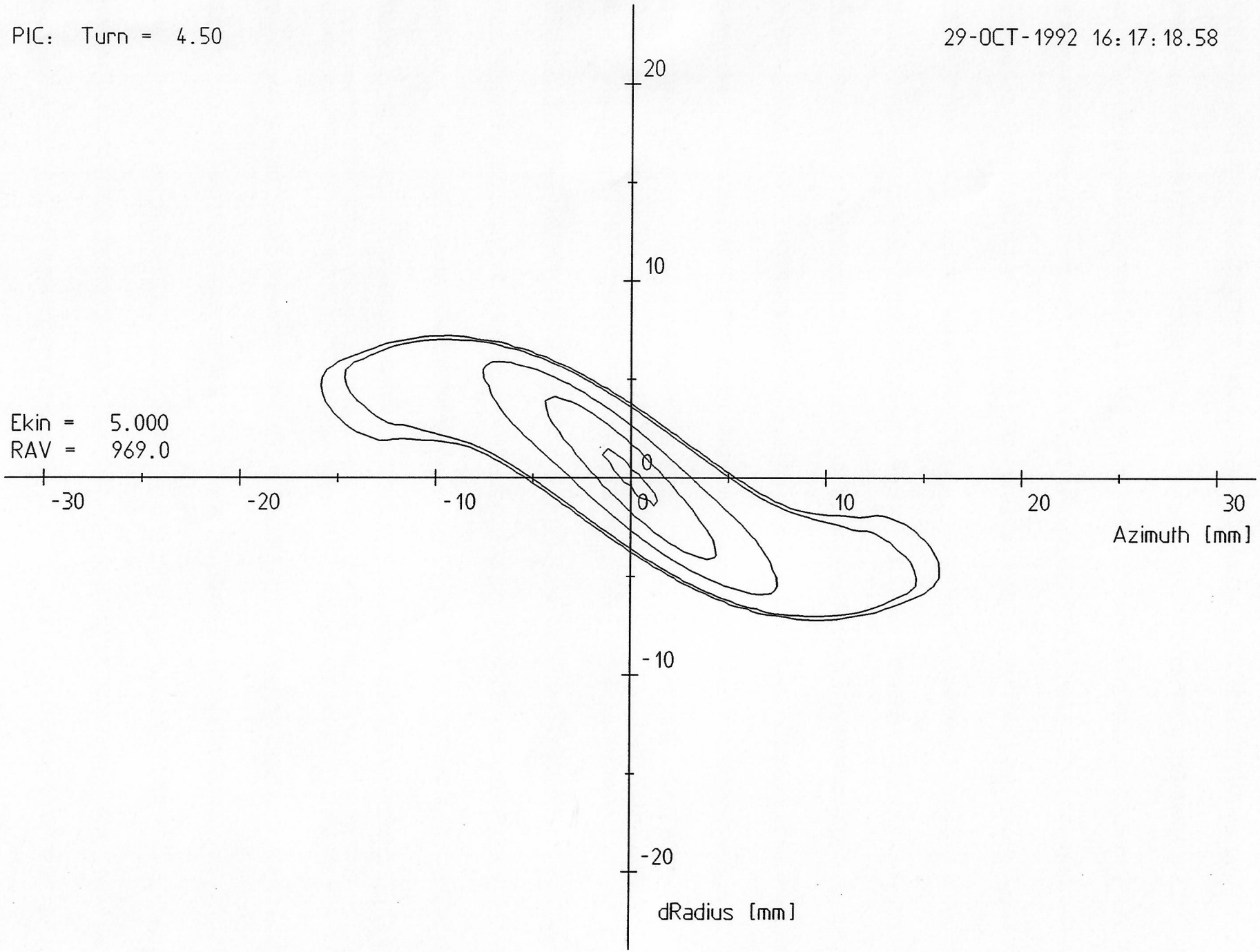
0
10
20
30
Azimuth [mm]
dRadius [mm]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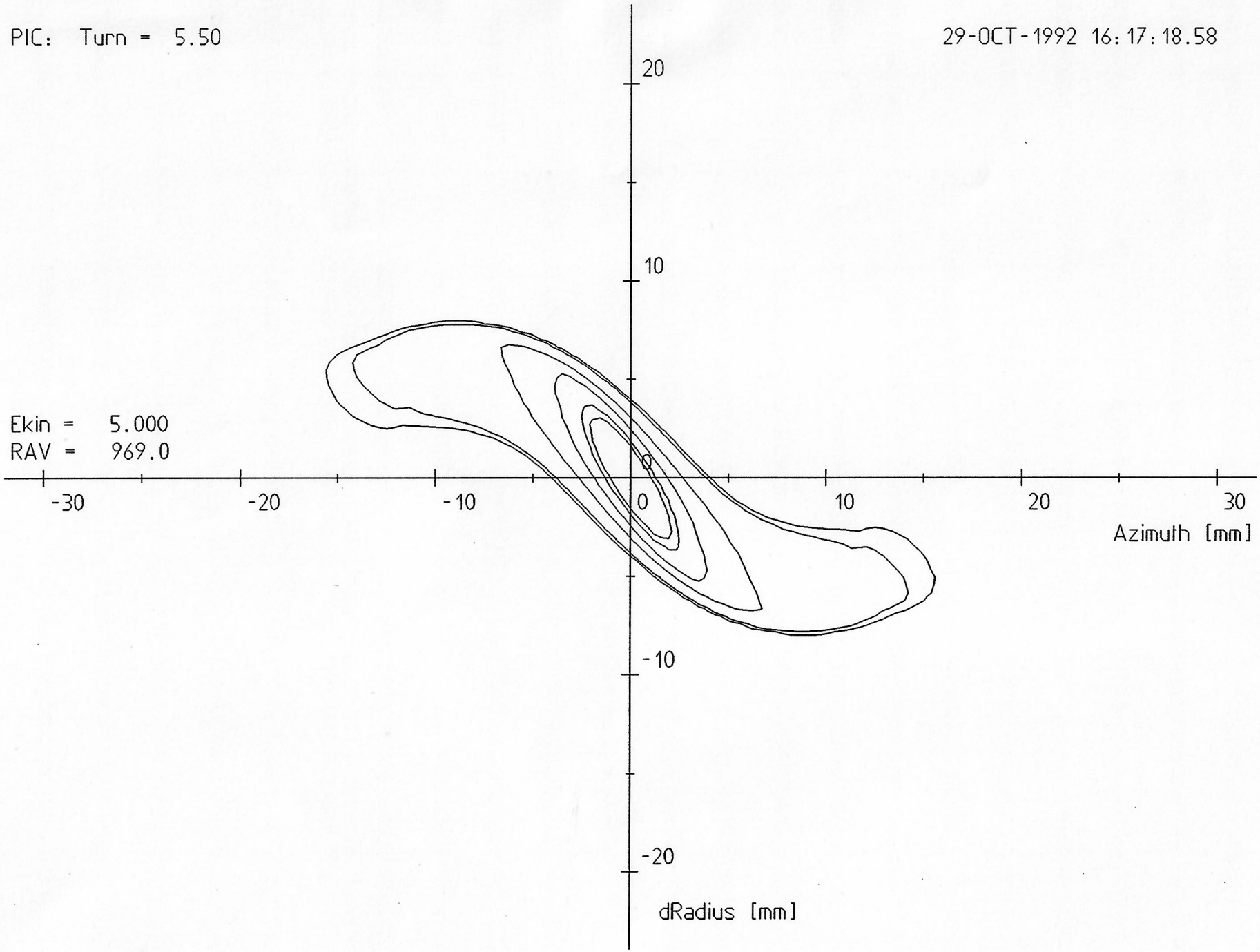
dRadius [mm]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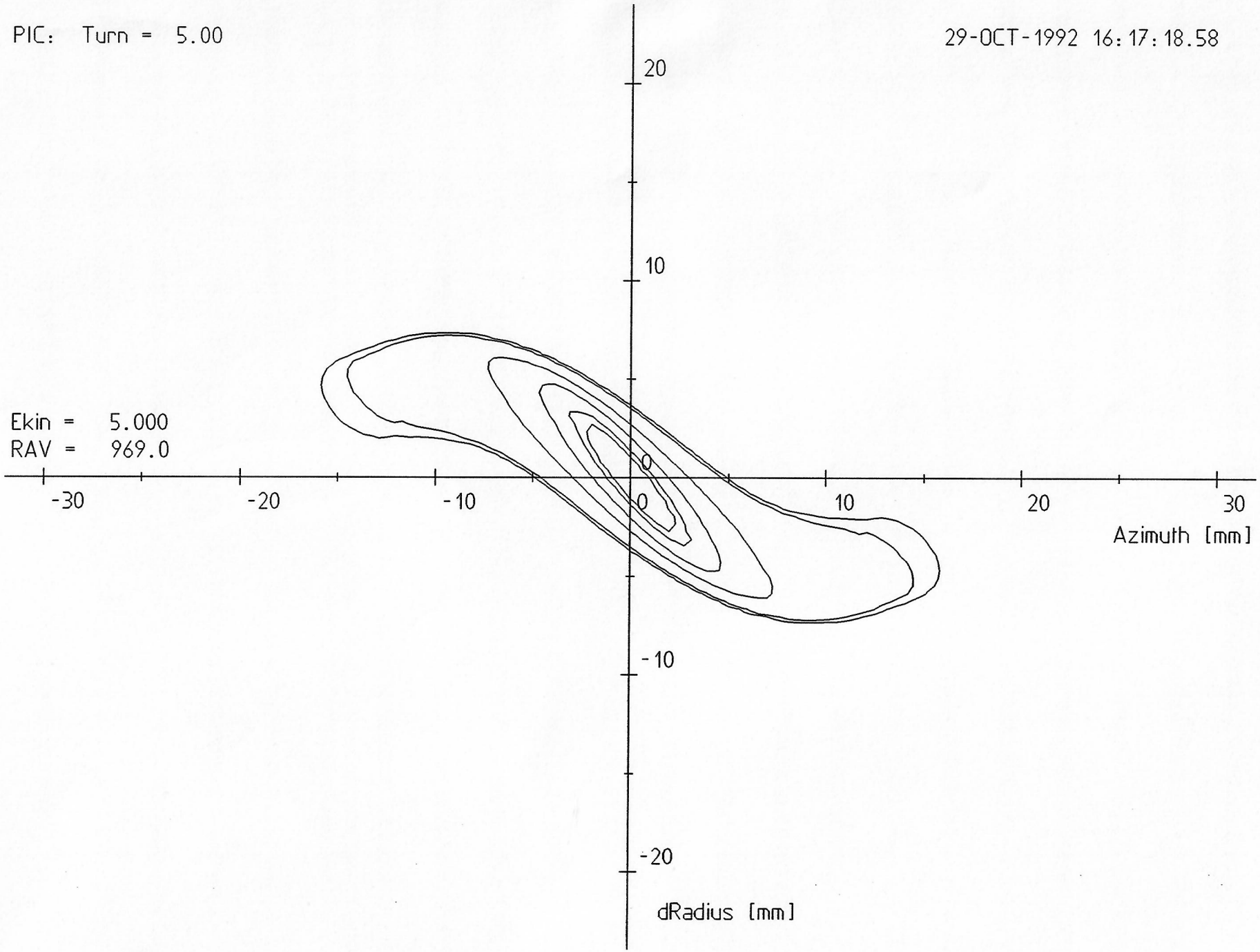
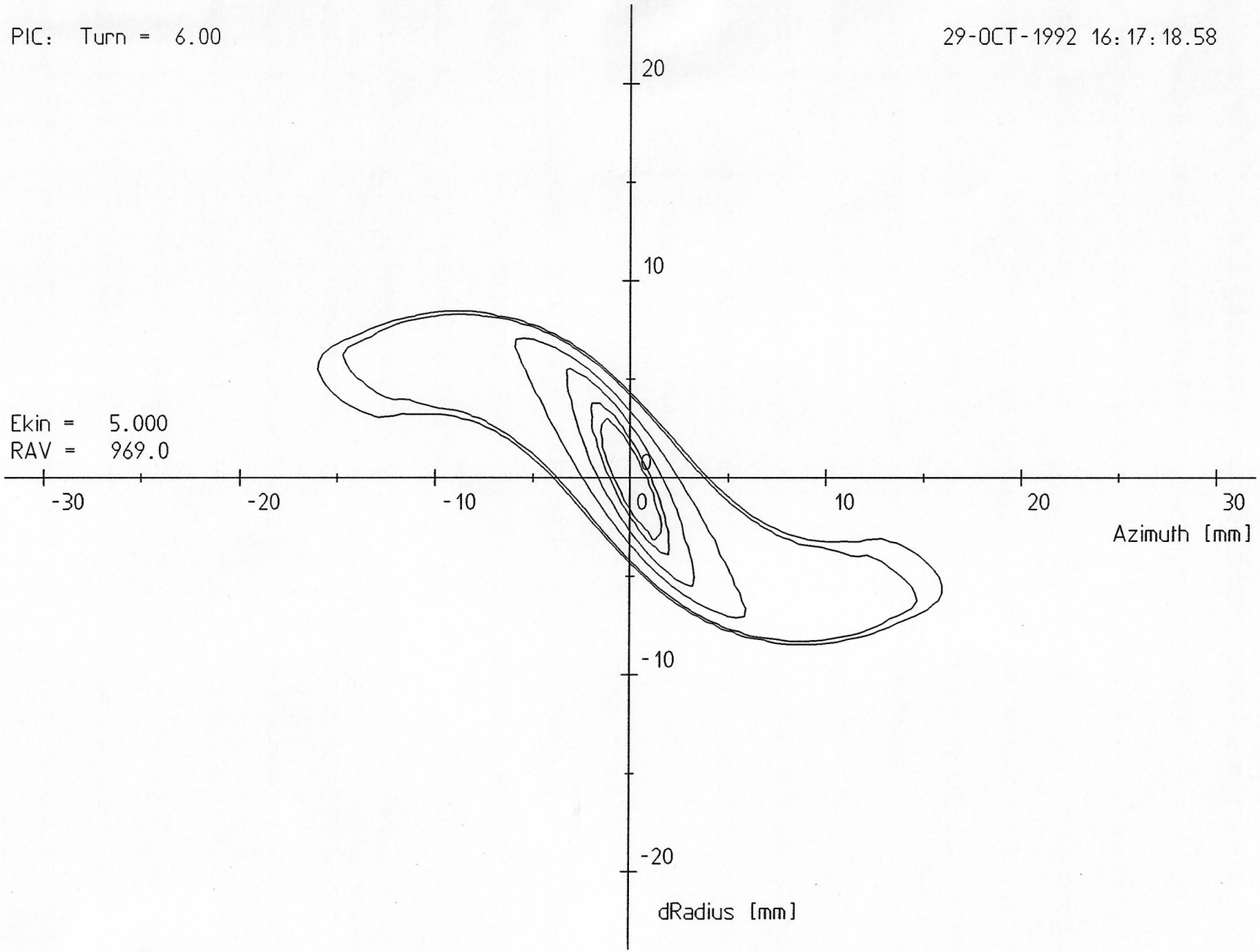
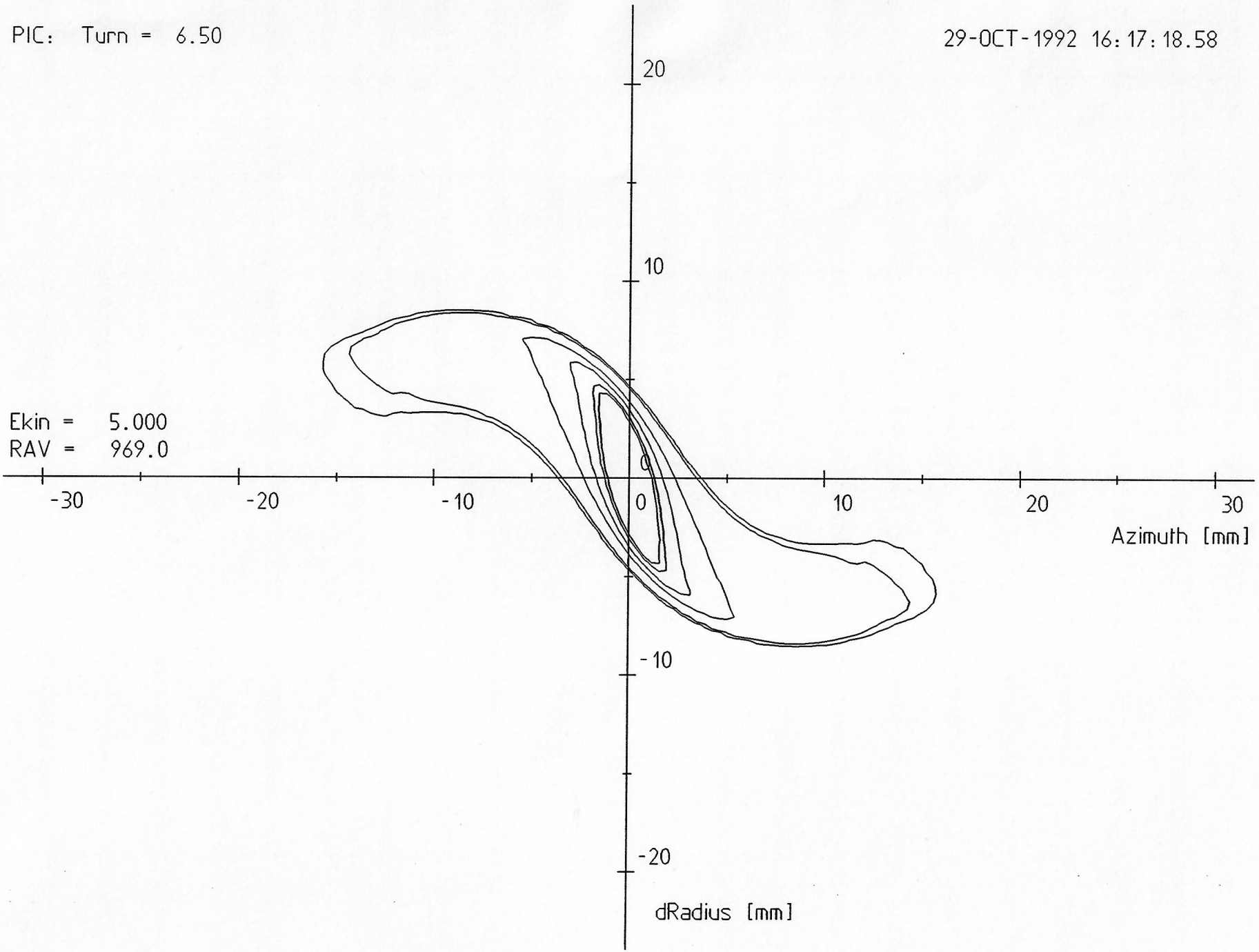
Azimuth [mm]
dRadius [mm]


PIC: Turn = 5.00
29-OCT-1992 16:17:18.58
Ekin = 5.000
RAV = 969.0
20
10
0
-10
-20
-30
-20
-10
0
10
20
30
Azimuth [mm]
dRadius [mm]


PIC: Turn = 6.00
29-OCT-1992 16:17:18.58
Ekin = 5.000
RAV = 969.0
20
10
0
-10
-20
-30
-20
-10
0
10
20
30
Azimuth [mm]
dRadius [mm]


PIC: Turn = 6.50
29-OCT-1992 16:17:18.58
Ekin = 5.000
RAV = 969.0
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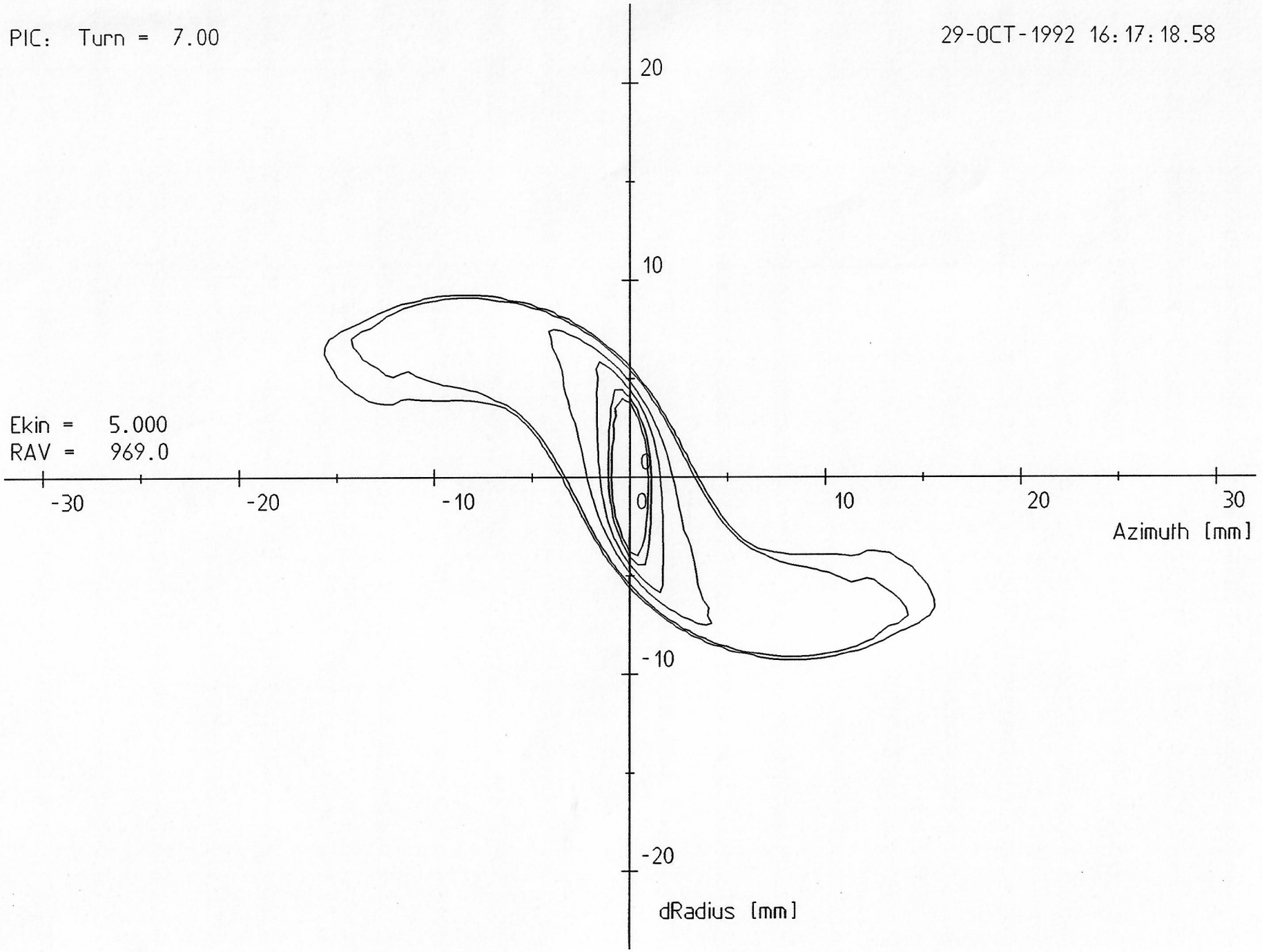
10
0
-10
-20
-30
-20
-10
0
10
20
30
Azimuth [mm]
dRadius [mm]


PIC: Turn = 7.00
29-OCT-1992 16:17:18.58
Ekin = 5.000
RAV = 969.0
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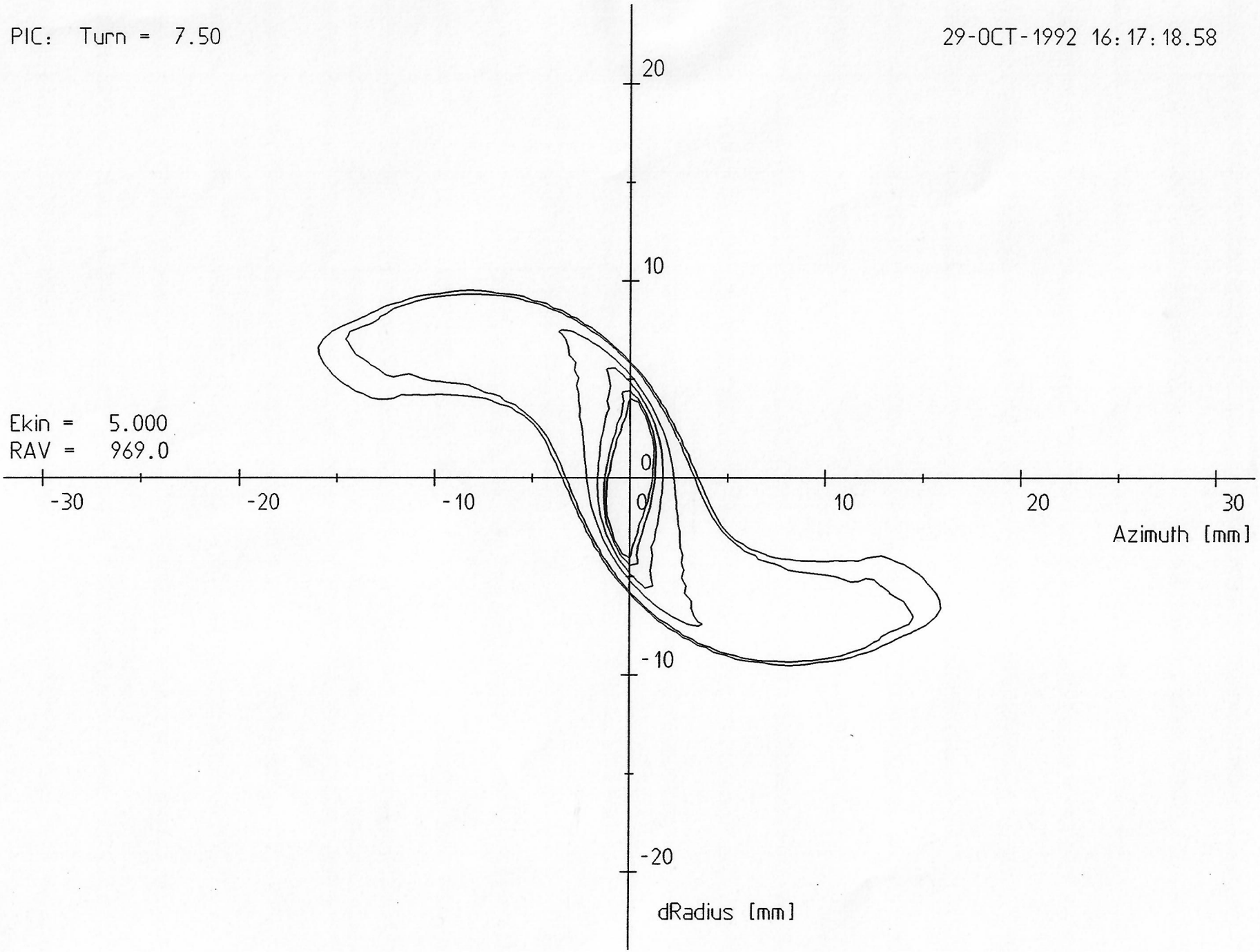
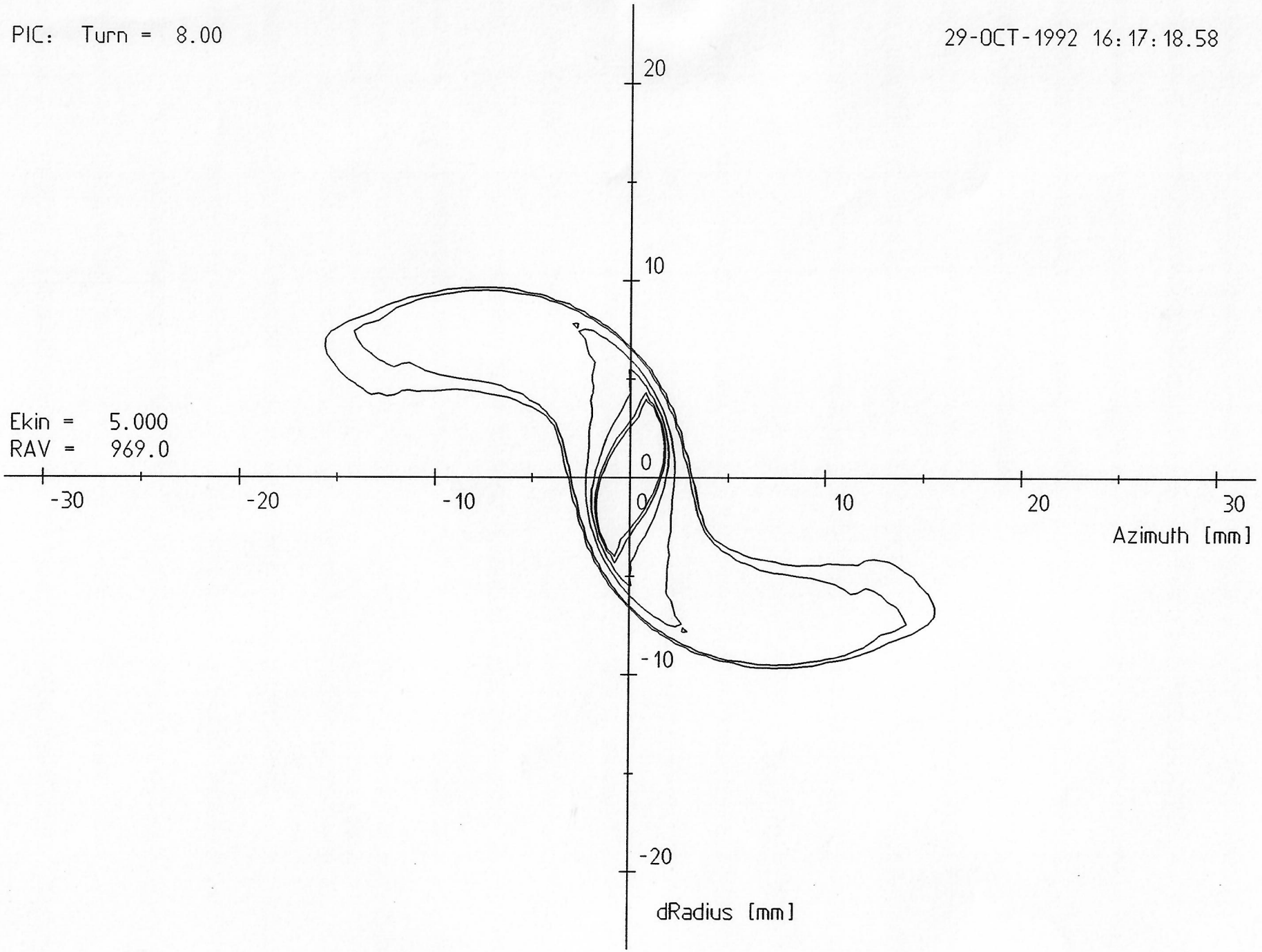
-10
-20
-30
-20
-10
0
10
20
30
Azimuth [mm]
dRadius [mm]


PIC: Turn = 7.50
29-OCT-1992 16:17:18.58
Ekin = 5.000
RAV = 969.0
20
10
0
-10
-20
-30
-20
-10
0
10
20
30
Azimuth [mm]
dRadius [mm]


PIC: Turn = 8.00
29-OCT-1992 16:17:18.58
Ekin = 5.000
RAV = 969.0
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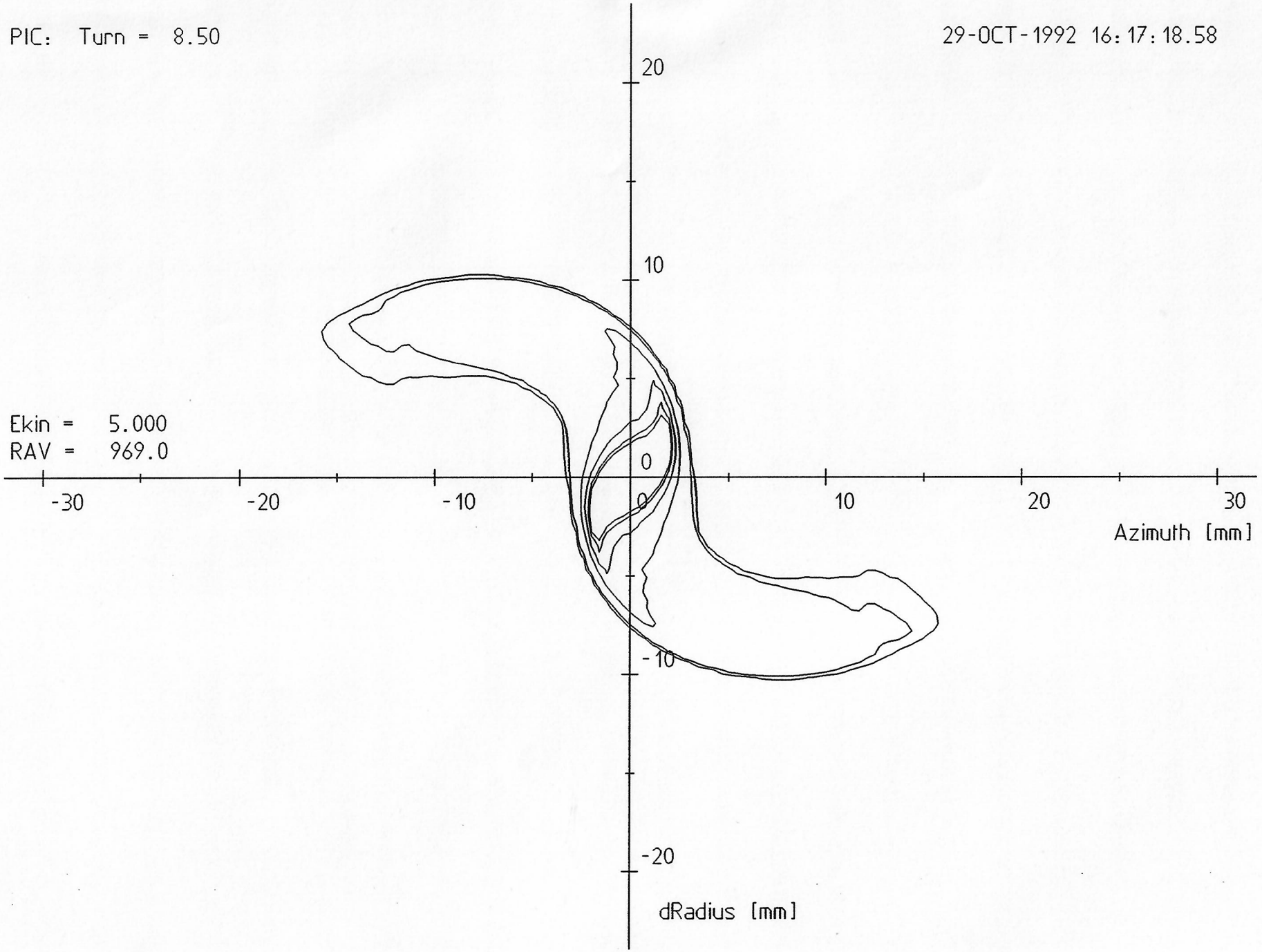
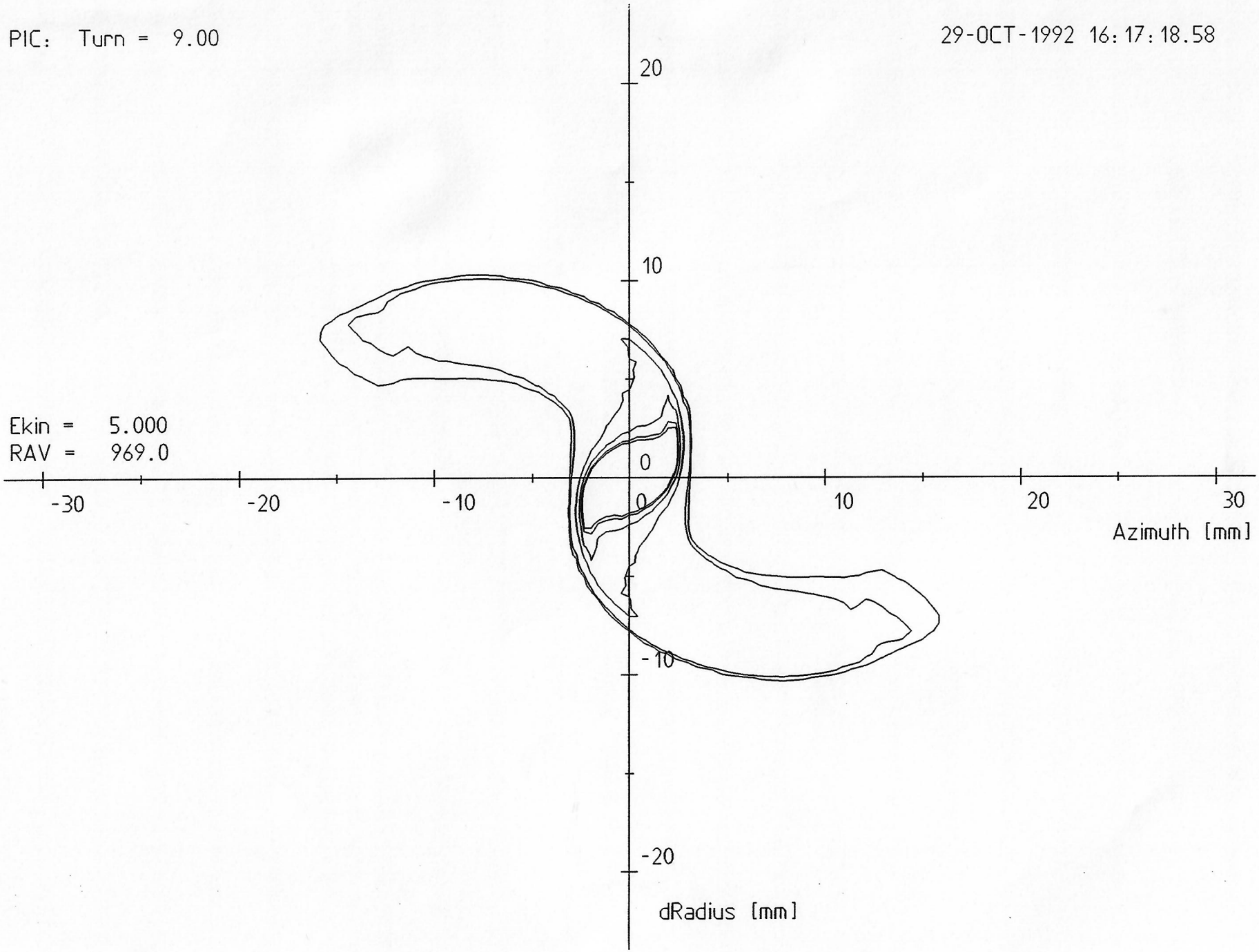
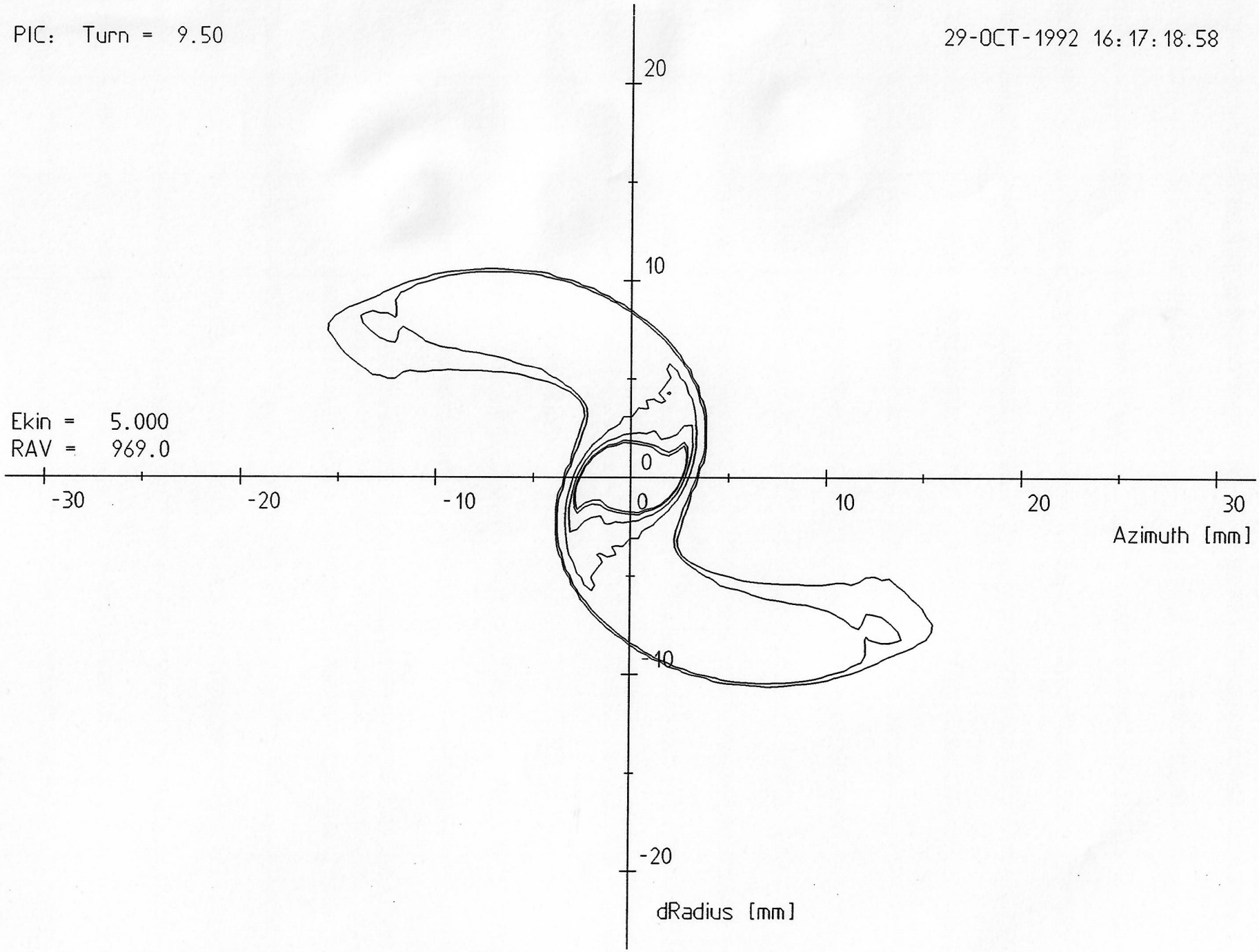
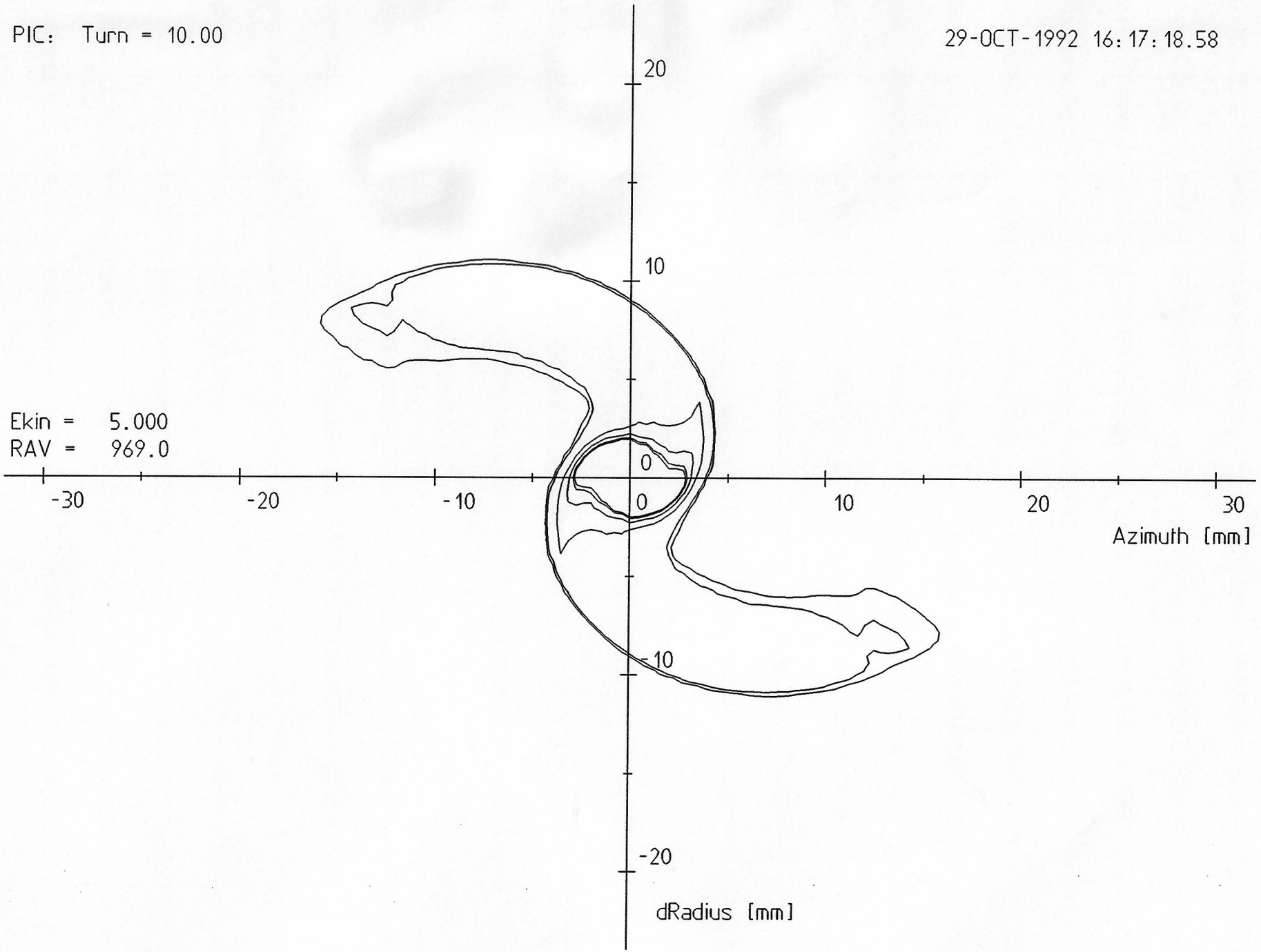
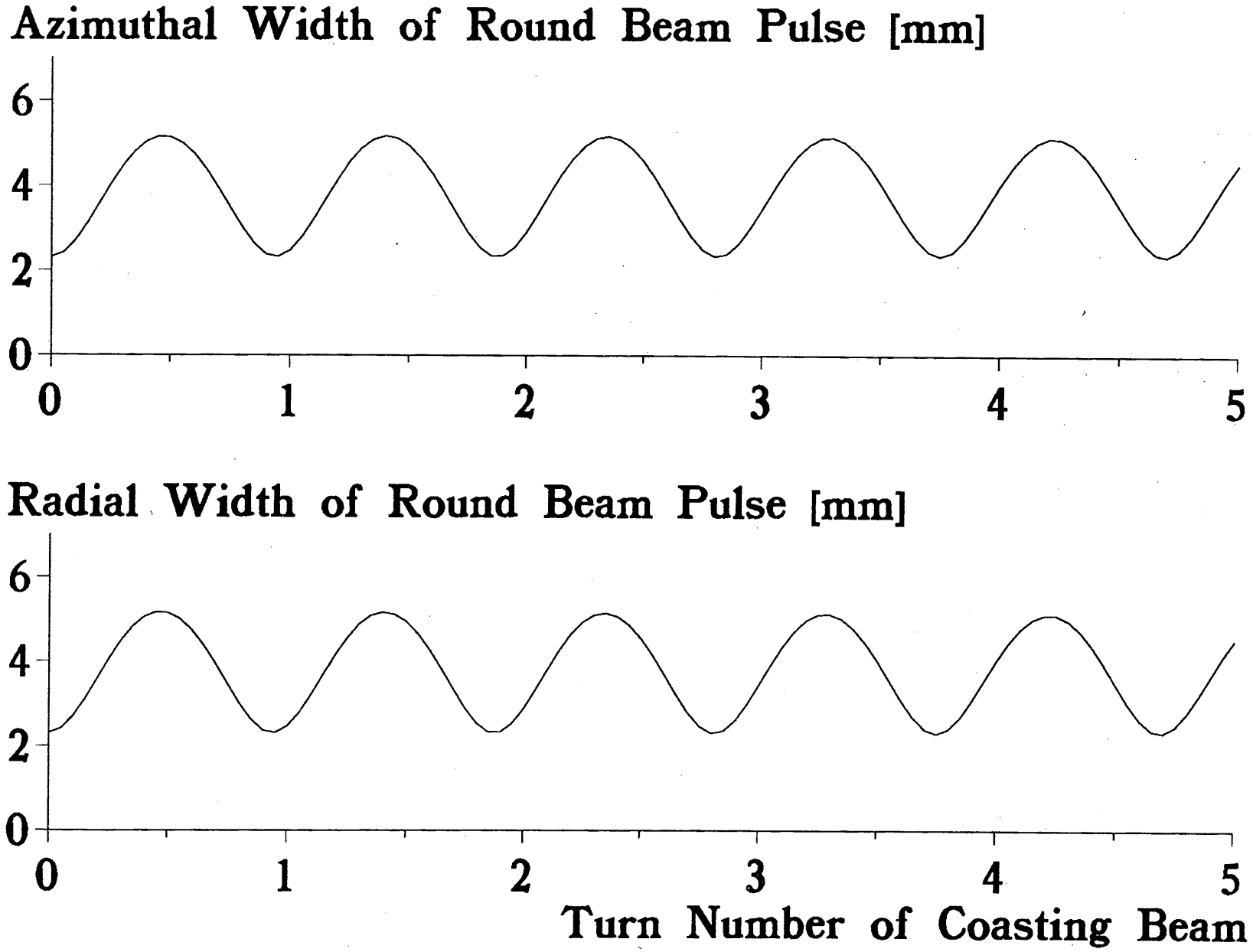
-10
-20
-30
-20
-10
0
10
20
30
Azimuth [mm]
dRadius [mm]


PIC: Turn = 8.50
29-OCT-1992 16:17:18.58
Ekin = 5.000
RAV = 969.0
20
10
0
-10
-20
-30
-20
-10
0
10
20
30
Azimuth [mm]
dRadius [mm]

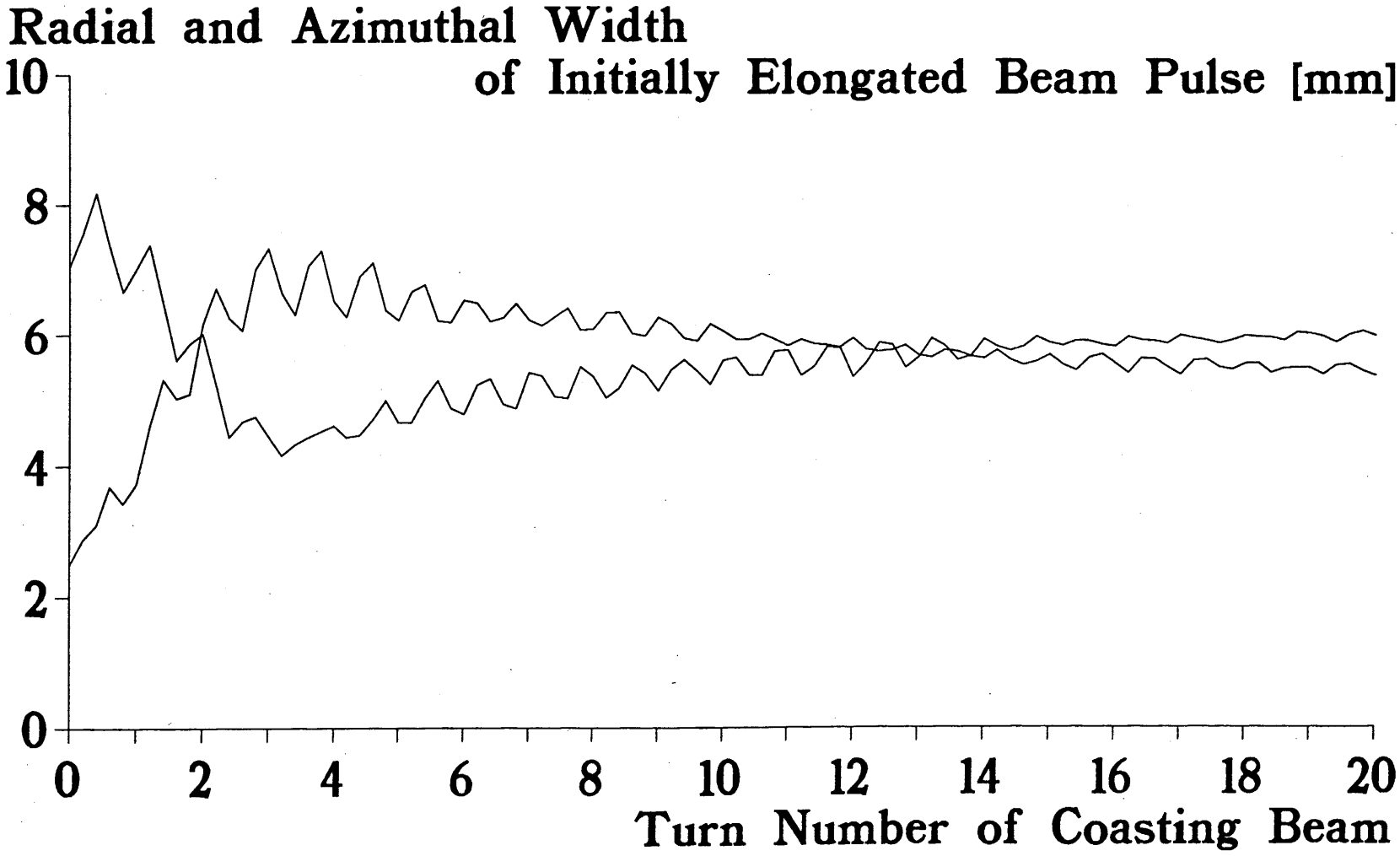

PIC: Turn = 9.00
29-OCT-1992 16:17:18.58
Ekin = 5.000
RAV = 969.0
20
10
0
-10
-20
-30
-20
-10
0
10
20
30
Azimuth [mm]
dRadius [mm]


PIC: Turn = 9.50
29-OCT-1992 16:17:18.58
Ekin = 5.000
RAV = 969.0
20
10
0
-10
-20
-30
-20
-10
0
10
20
30
Azimuth [mm]
dRadius [mm]

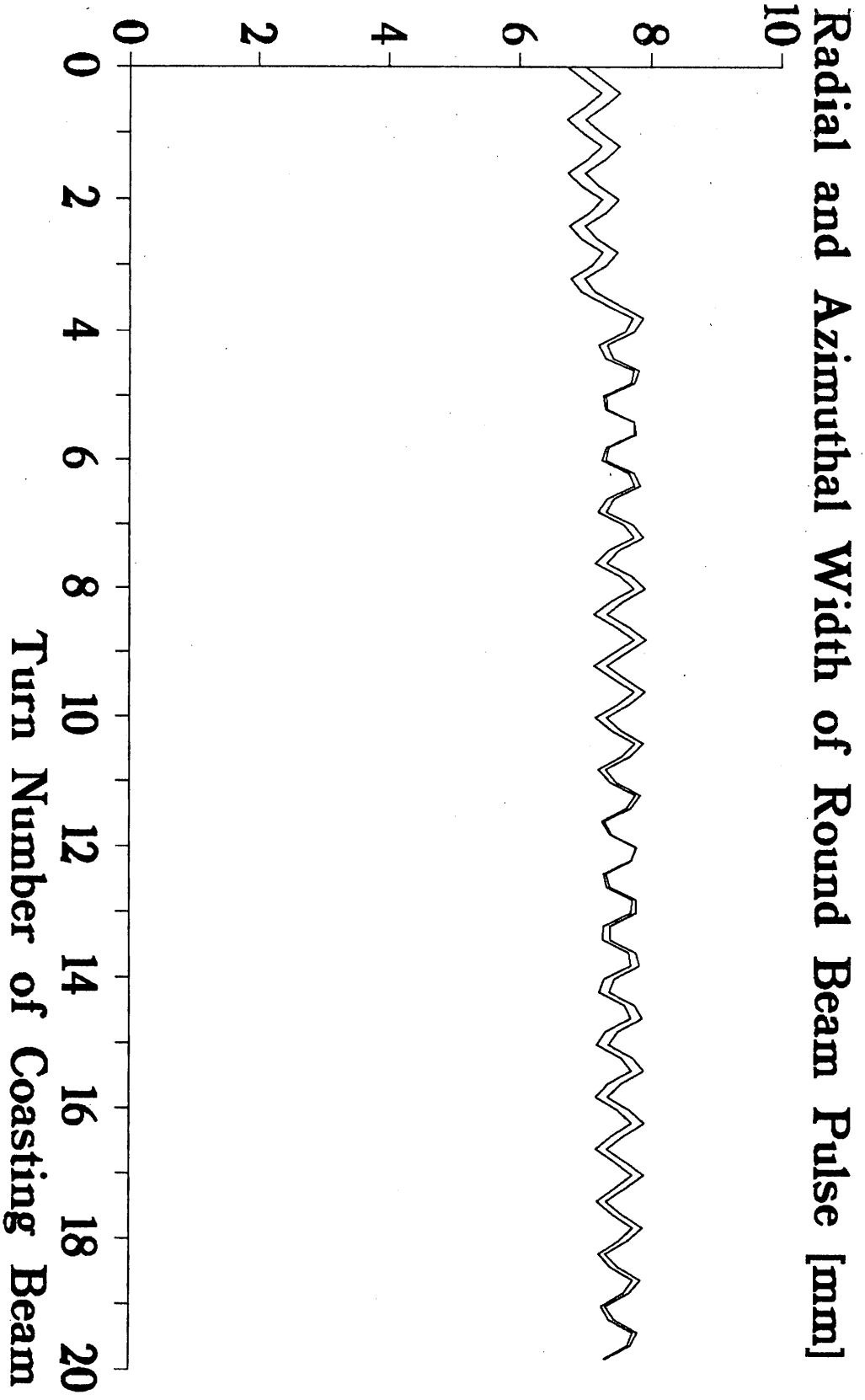

PIC: Turn = 10.00
29-OCT-1992 16:17:18.58
Ekin = 5.000
RAV = 969.0
20
10
0
-10
-20
-30
-20
-10
0
10
20
30
Azimuth [mm]
dRadius [mm]



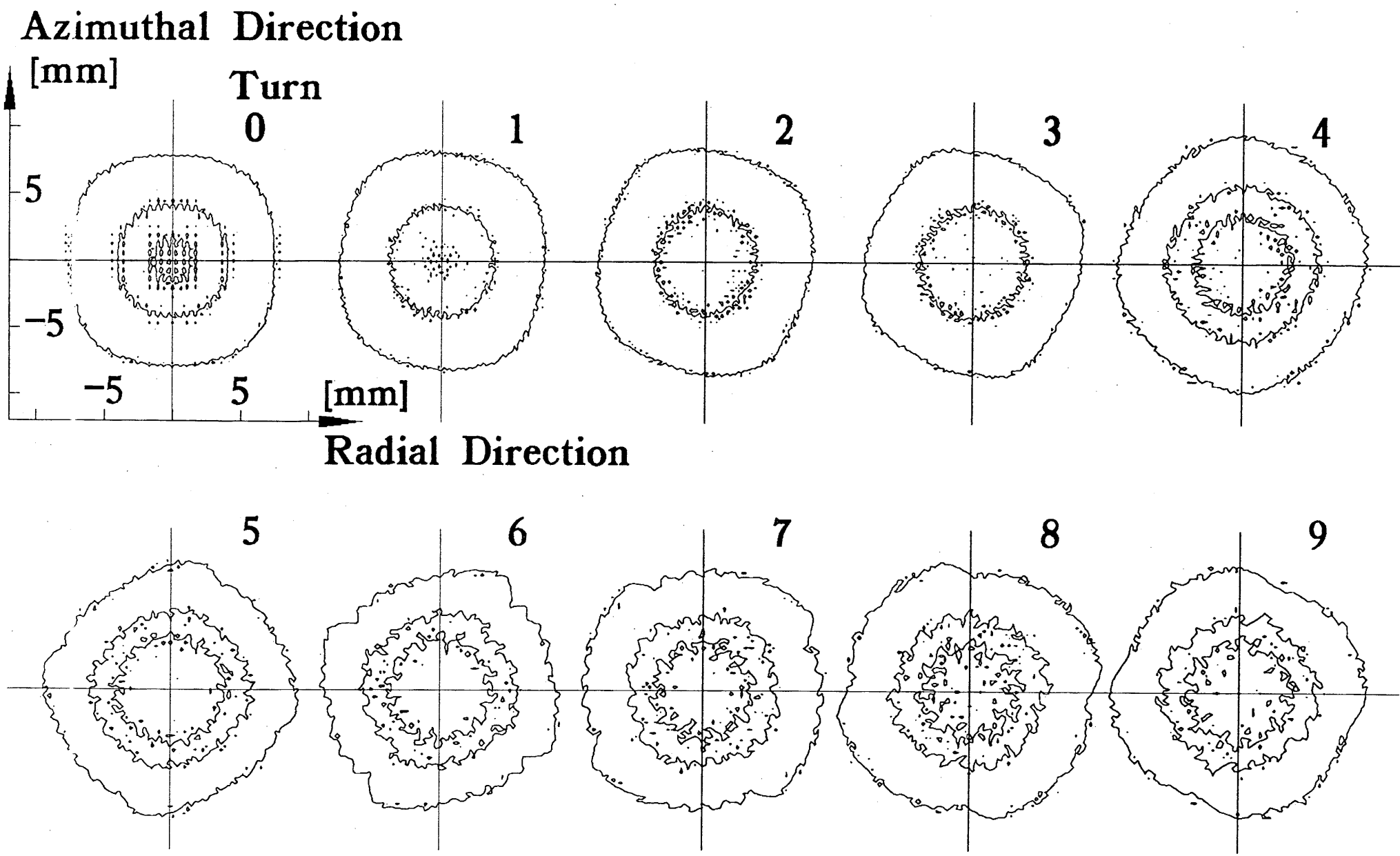
Injector II, 5 MeV, 1 mA, 5000 needles in a circular charge cloud with no matching of space-charge. Breathing mode oscillations ensue with frequency lower than the betatron tune.



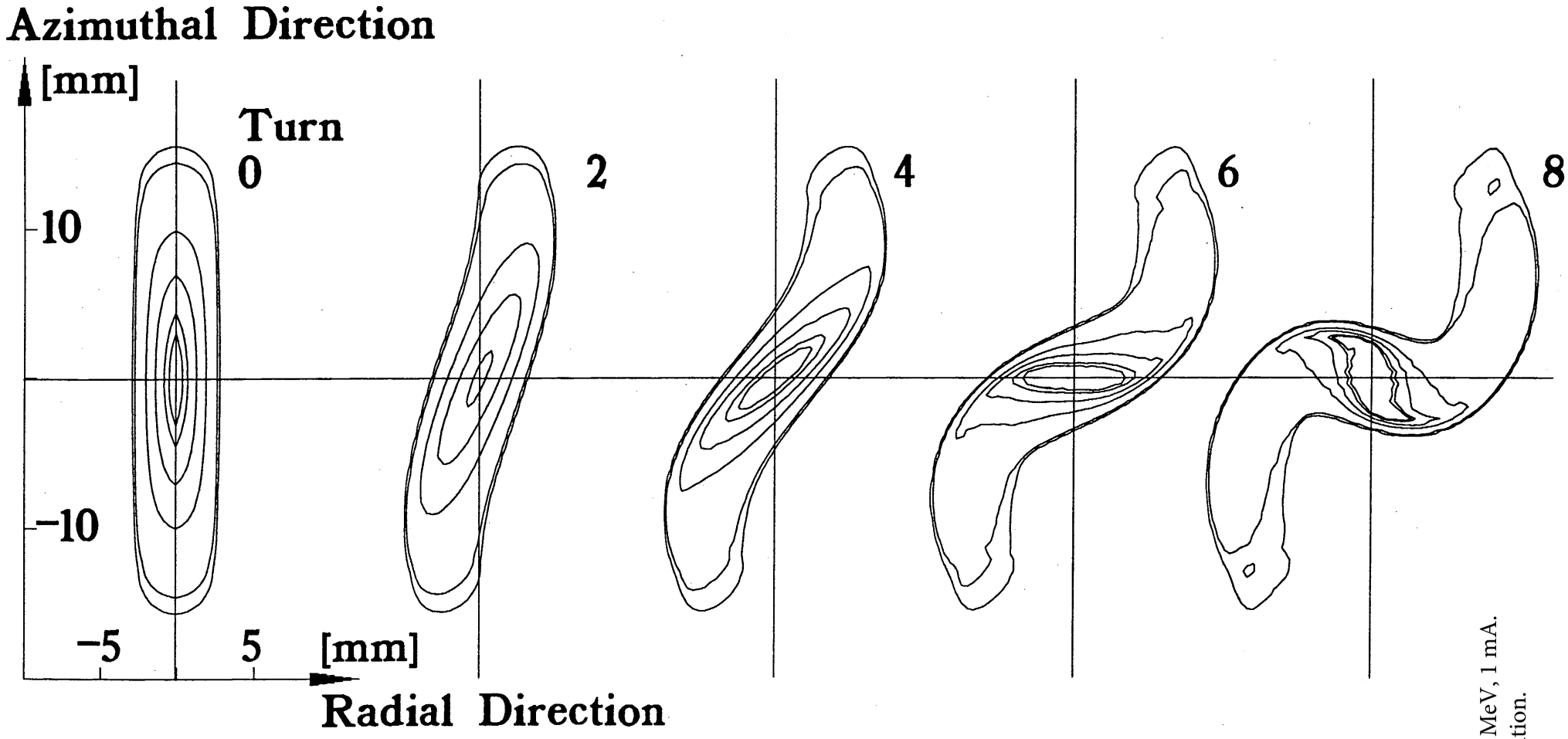
Injector II, 5 MeV,1 mA, initial unmatched elongated ensemble. Radial and azimuthal r.m.s. width of a bunch of 7.5 deg initialRF-phase width; showing the increase of the radial width and decrease of the azimuthal width, followed by a small, slow rise towards a common value for both widst at a round beam.



Injector II, 5 MeV, 1 mA, circular disc matched under space-charge. Almost constant radial and azimuthal width over many turns. From the way the needle centres are initialized, the distribution is a little "squarish" at the start; and so this case probably did not have precise matching

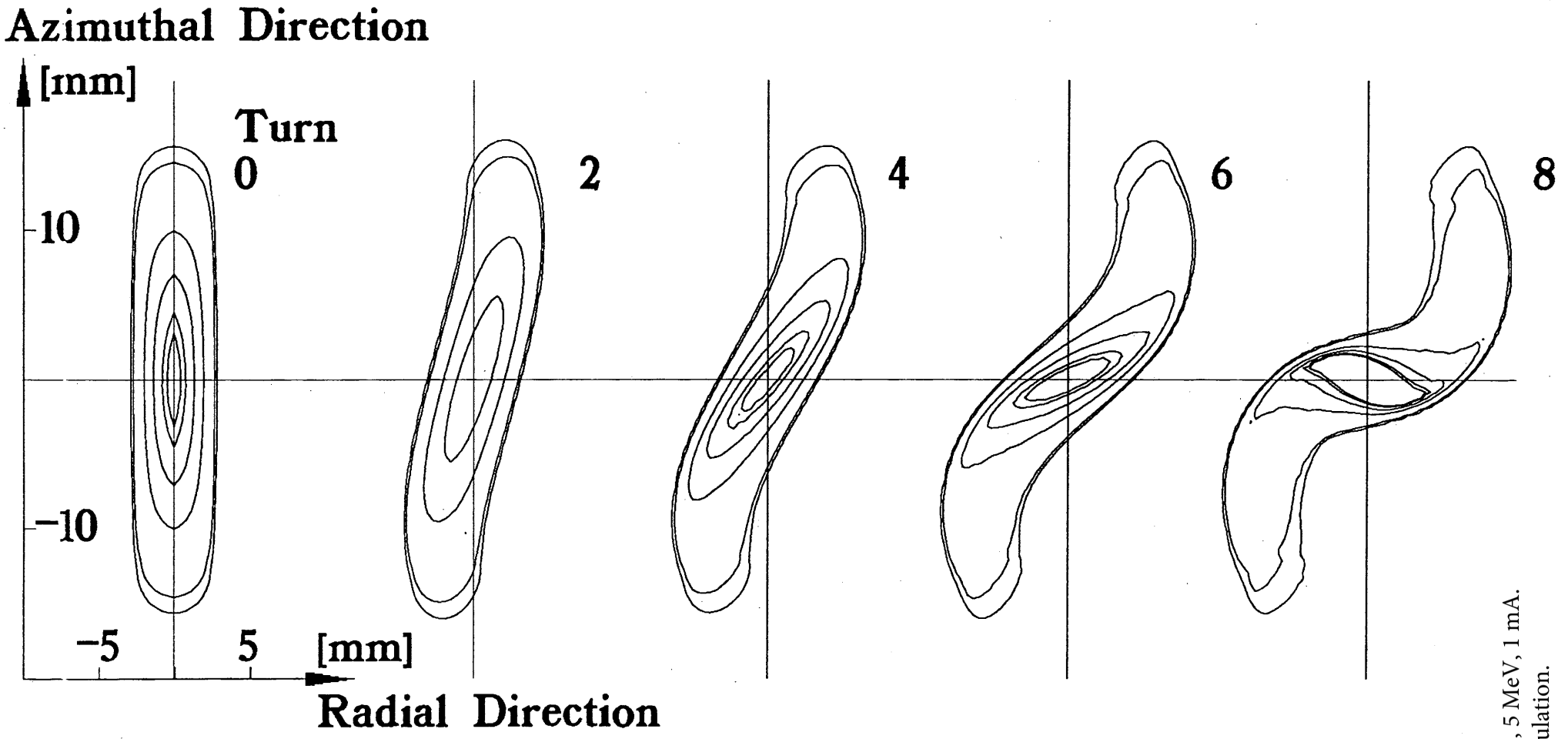


Circular disc matched under space-charge. Almost constant radial and azimuthal width over many turns. From the way the needles are initialized, the distribution is a little "squarish" at the start; so the matching is imprecise



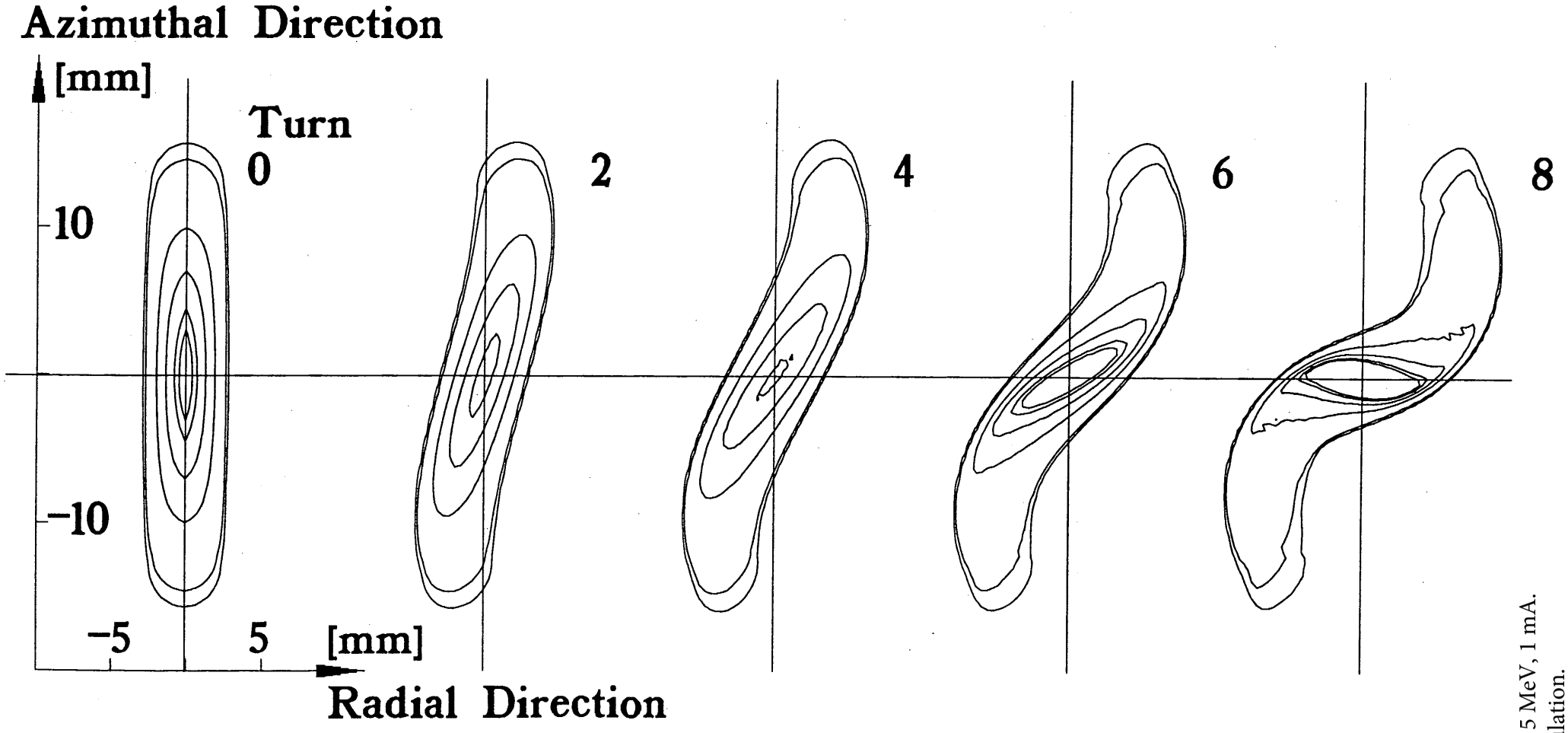
Simulation with Qr = 1.0

Injector II, 5 MeV, 1 mA.
PICN simulation.



Simulation with Qr = 1.26

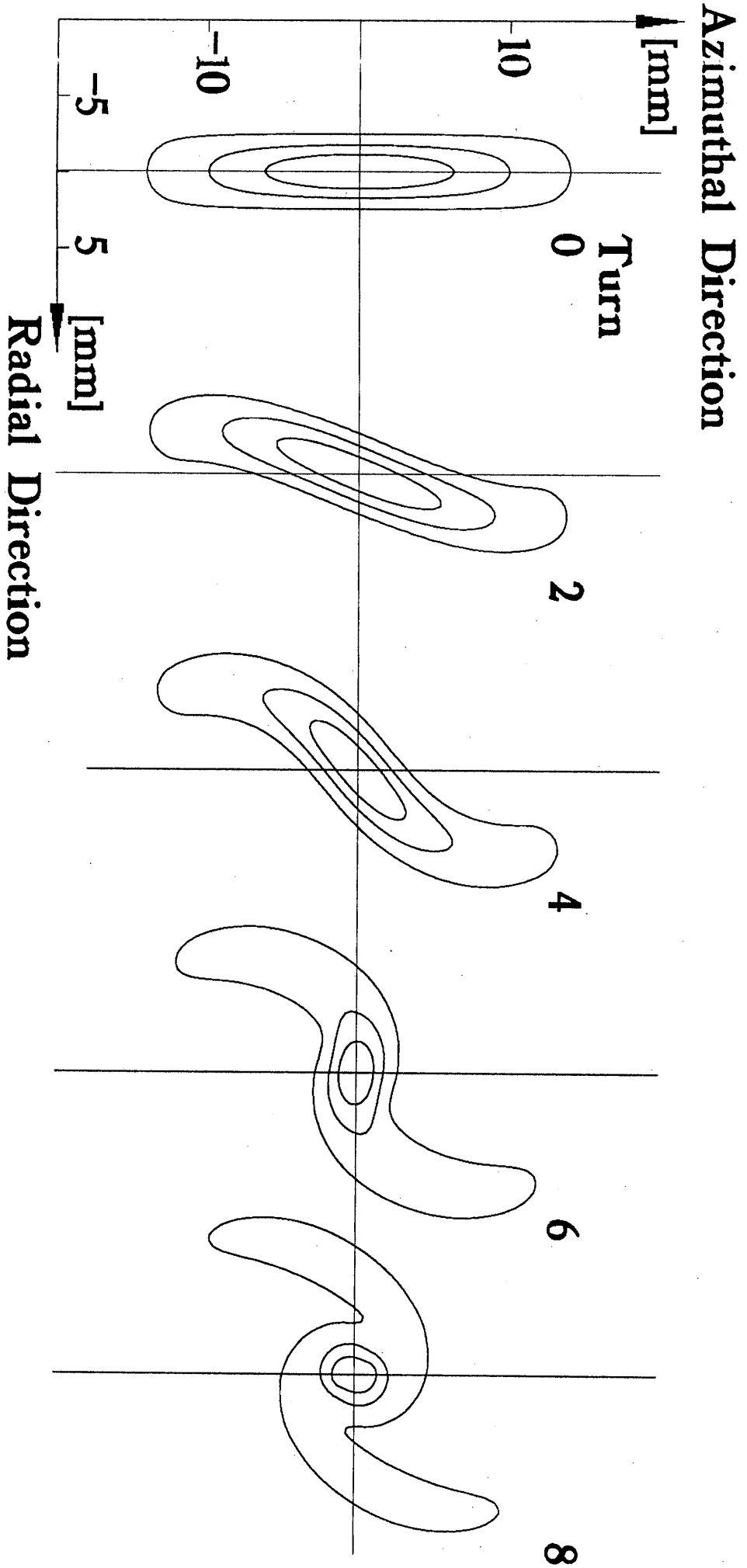
Injector II, 5 MeV, 1 mA.
PICN simulation.



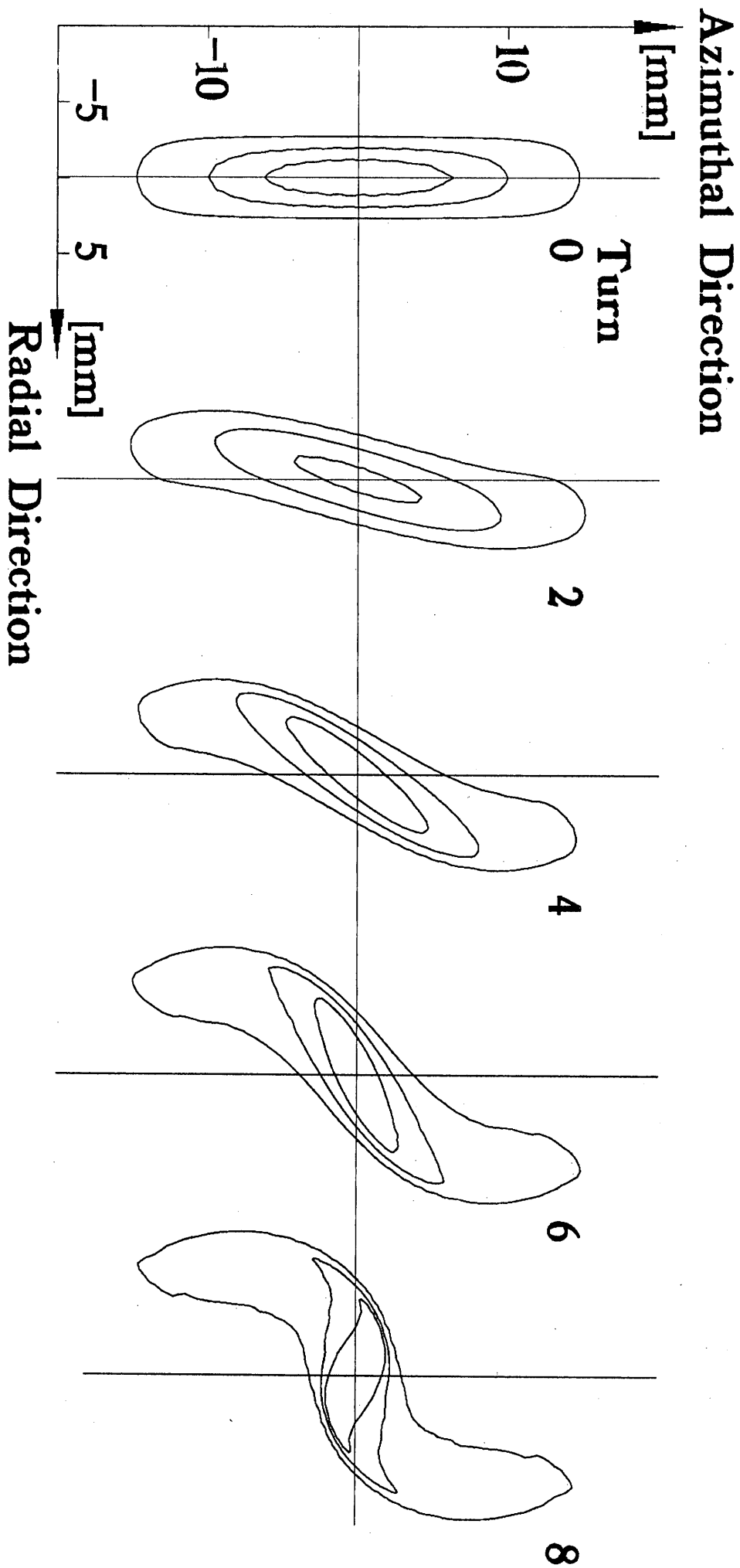
Simulation with Qr = 1.38

Injector II, 5 MeV, 1 mA.
PICN simulation.

Injector II, 5 MeV, 1 mA. PICS Simulation of beam with 15 deg RF-phase width



Injector II, 5 MeV, 1 mA. PICN Simulation of beam with 15 deg RF-phase width



Simulation of Space-charge dominated Beam dynamics in an isochronous AVF cyclotron

Stefan R. Adam, PSI, Villigen, Switzerland
Shane R. Kosciuniak, TRIUMF, Vancouver BC, Canada

2 of 2

PICN Simulation graphics created by Stefan Adam for poster presentation at 1993 Particle Accelerator Conference

Beam bunch has 15 deg RF-phase width

Injector II, 5 MeV, 1 mA, tune
Qr = 1.26. PICN simulation

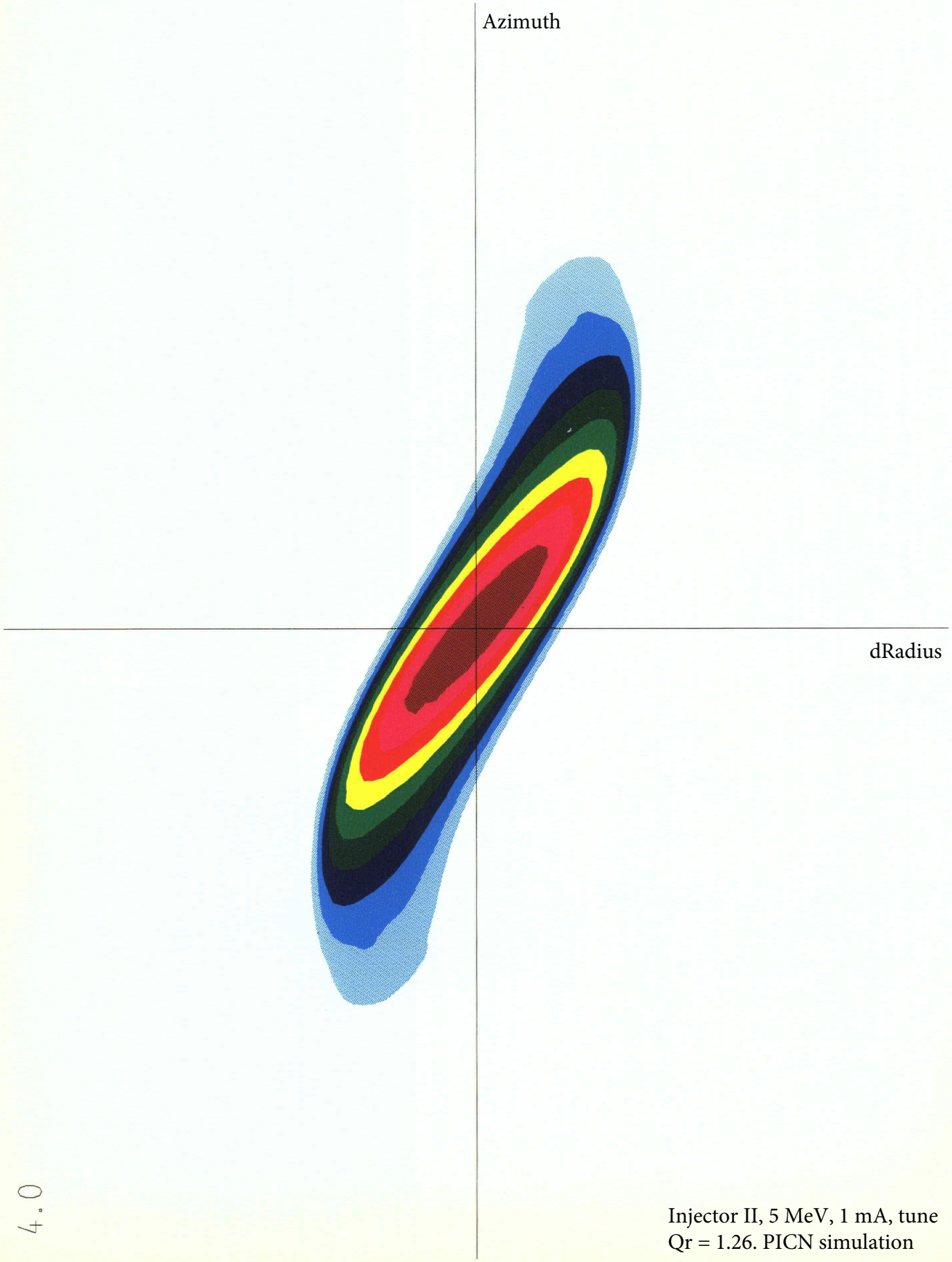
Injector II, 5 MeV, 1 mA, tune
Qr = 1.26. PICN simulation

2.0

Injector II, 5 MeV, 1 mA, tune
Qr = 1.26. PICN simulation

3.0

Injector II, 5 MeV, 1 mA, tune
Qr = 1.26. PICN simulation



4.0

Injector II, 5 MeV, 1 mA, tune
Qr = 1.26. PICN simulation

5.0

Injector II, 5 MeV, 1 mA, tune
Qr = 1.26. PICN simulation

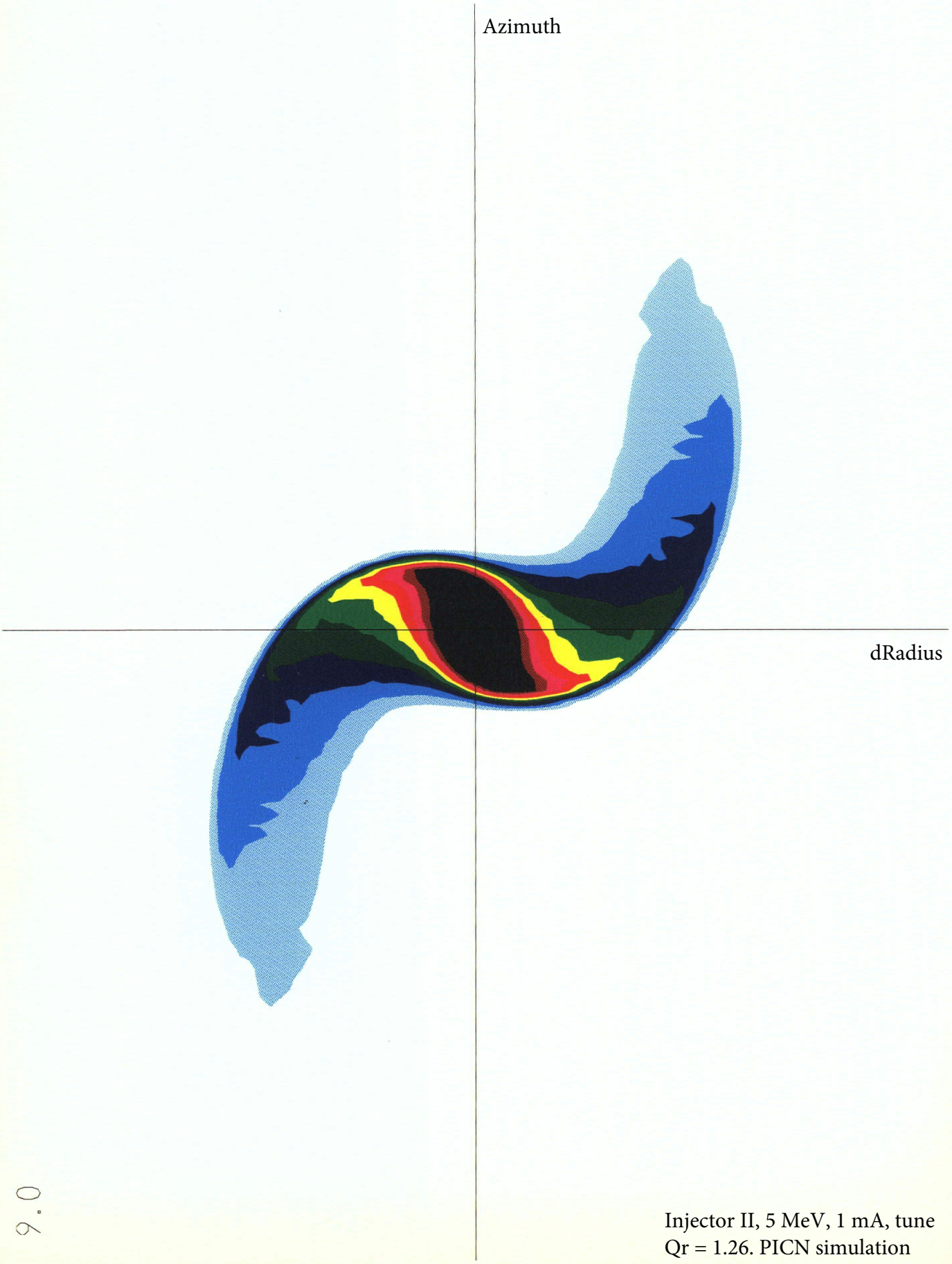
6.0

Injector II, 5 MeV, 1 mA, tune
Qr = 1.26. PICN simulation

7.0

Injector II, 5 MeV, 1 mA, tune
Qr = 1.26. PICN simulation

Injector II, 5 MeV, 1 mA, tune
Qr = 1.26. PICN simulation



Injector II, 5 MeV, 1 mA, tune
Qr = 1.26. PICN simulation

Injector II, 5 MeV, 1 mA, tune
Qr = 1.26. PICN simulation

PICS

PICS Simulation graphics created by Stefan Adam for poster presentation at 1993 Particle Accelerator Conference

Beam bunch has 15 deg RF-phase width

Injector II, 5 MeV, 1 mA
PICS simulation

Injector II, 5 MeV, 1 mA
PICS simulation

2.0

Injector II, 5 MeV, 1 mA
PICS simulation

3.0

Injector II, 5 MeV, 1 mA
PICS simulation

4.0

Injector II, 5 MeV, 1 mA
PICS simulation


Azimuth
dRadius
5.0

6.0

Injector II, 5 MeV, 1 mA
PICS simulation

7.0

Injector II, 5 MeV, 1 mA
PICS simulation

Injector II, 5 MeV, 1 mA
PICS simulation

Injector II, 5 MeV, 1 mA
PICS simulation

Injector II, 5 MeV, 1 mA
PICS simulation