Motion In Dual-Harmonic Potential Well

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Chapter 1

Motion In Dual-Harmonic Potential Well

"This book is an attempt to restore the Jacobian functions to the elementary curriculum by exhibiting them as functions constructed on a lattice", E.H. Neville[17] (1944).

1.1 Prologue

Let t, x and $p = \dot{x}$ be time, position and momentum; and dots denote time derivatives ($dx/dt \equiv \dot{x}$). This article concerns the motion of a particle having the equation of motion (EOM) $\ddot{x} + \omega^2[\sin x \quad \pm \sin 2x] = 0$. Ensembles of particles that homogeneously populate equi-density contours of the corresponding Hamiltonian $H(x, \dot{x})$, constitute bunches. The negative sign gives long bunches, while the positive sign produces short bunches. The restoring force in the EOM can be derived from a dual-harmonic[1, 2, 3, 4] (i.e. periodic) potential well. Motion that is confined within the well is termed *libration*, whereas motion that streams over the well is termed *rotation*.

For the long bunch case, Elena Shaposhnikova[1] wrote down a solution for the motion in the dual-harmonic well and used it as basis for the calculation of the beam transfer function[6] for amplitude- and phase- modulations of the RF-waveform (that maintains the well), as a precursor to particle-beam stability analysis.[2] No references or details of the derivation are provided, only the result. It occurred to me that there are similar (or related) results for the short bunch case, and for motion that escapes the periodic potential well. But it is necessary to either find (in the literature) the derivation, or to discover the derivation oneself. Further, it transpires that the same derivation solves the general case of the EOM $\ddot{x} + \omega^2[\sin x + q \sin 2x] = 0$ with arbitrary q.

Shaposhnikova's solution is given in terms of the Jacobi elliptic functions: $\mathrm{cn}(z, m)$, $\mathrm{sn}(z, m)$, $\mathrm{dn}(z, m)$. So it is natural to look in texts about those functions. Whereas there are many sources that state the mathematical properties of these functions, there are rather fewer that demonstrate their application to physical problems. An account of the use of elliptic function in classical mechanics is given in Sec. 1.9. Thus far the author has not found a text containing a derivation of Elena's solution; so we have prepared one, and then applied it to a variety of conditions including the bunch-shortening regime.

[1] An alternative transliteration from Russian to English of this name is "Chapochnikova".

[2] Shaposhnikova developed these ideas, including the loss of Landau damping, in series of papers[7, 8, 9, 10].

1.2 Basic properties of Jacobi elliptic functions

The elliptic functions may be defined either by differential equations or as the inverse of incomplete integrals. Here, in the manner of Meyer[11], we introduce them from a dynamics point of view. There are four sets of 3 functions, giving a total of twelve. However, the properties of all may be derived from the principal set: cn, sn, dn. Each function has two arguments (z, m); z is the phase in radian, and m is the Jacobi parameter. $k = \sqrt{m}$ is called the elliptic modulus. The Jacobi elliptic functions are solutions of the system of equations and initial conditions

$$\dot{X} = Y\,Z, \qquad \dot{Y} = -Z\,X, \qquad \dot{Z} = -m\,X\,Y\ . \tag{1.1}$$

$$\mathrm{sn}(0,m) = X(0) = 0, \qquad \mathrm{cn}(0,m) = Y(0) = 1, \qquad \mathrm{dn}(0,m) = Z(0) = 1\ . \tag{1.2}$$

Alternatively, we may think of them as defined by the following relations:

$$\mathrm{cn}(z,m)^2 + \mathrm{sn}(z,m)^2 = 1 \quad \text{and} \quad \mathrm{dn}(z,m)^2 + m.\mathrm{sn}(z,m)^2 = 1 \tag{1.3}$$

$$\frac{d\mathrm{cn}(z,m)}{dz} = -\mathrm{dn}(z,m)\mathrm{sn}(z,m)\,, \quad \frac{d\mathrm{sn}(z,m)}{dz} = +\mathrm{dn}(z,m)\mathrm{cn}(z,m)\,, \quad \frac{d\mathrm{dn}(z,m)}{dz} = -m.\mathrm{cn}(z,m)\mathrm{sn}(z,m) \tag{1.4}$$

Let $X = \mathrm{sn}(t,m)$, $Y = \mathrm{cn}(t,m)$ and $Z = \mathrm{dn}(t,m)$. From the properties above, it follows that:

$$(\dot{X})^2 = (1-X^2)(1-mX^2)\ , \quad (\dot{Y})^2 = (1-Y^2)[(1-m)+mY^2]\ , \quad (\dot{Z})^2 = (1-Z^2)[Z^2-(1-m)]\ . \tag{1.5}$$

It is critical to note that each of the functions cn, sn and dn satisfies a *different* equation; and therefore cn and sn do not form a basis (to satisfy initial conditions) in the same way that sine and cosine are two independent solutions of the harmonic oscillator equation $\ddot{x} + \omega^2 x = 0$. To find a second linearly independent solution, we advance the phase by a quarter period $K(m)$. We may term these quadrature solutions:

$$\mathrm{cn}(z+K,m) = -\sqrt{1-m}\,\mathrm{sd}(z,m), \quad \mathrm{sn}(z+K,m) = \mathrm{cd}(z,m), \quad \mathrm{dn}(z+K,m) = \mathrm{nd}(z,m)\ . \tag{1.6}$$

The period of oscillation (for pure real value of the argument z) is given by $z = 4K(m)$ where K is the complete elliptic integral of the first kind. The period of repetition (for pure imaginary value of the argument z) is given by $z/\sqrt{-1} = 4K'(m) \equiv 4K(1-m)$.

The twelve Jacobi elliptic functions are generally written as $\mathrm{pq}(z,m)$ where p and q are any of the letters c, s, n, d. From the multiplication rule[3] it follows that function pq $\equiv$ pr/qr, (with arguments suppressed). For example cn/dn $\equiv$ cd; and sn/cn $\equiv$ sc. Note that in some older texts the ratio $\mathrm{sn}(z,m)/\mathrm{cn}(z,m)$ is written $\mathrm{tn}(z,m)$ in analogy to $\sin(z)/\cos(z) \equiv \tan(z)$ for the circular trigonometric functions; but this is not recommended.

By convention, the Jacobi functions are defined for the Jacobi parameter $0 \le m \le 1$. When m is extended into the range $m \ge 1$, the functions become other functions. This property is called the Jacobi real transformation:

$$\mathrm{sn}(z,m) \to \mathrm{sn}(z\sqrt{m},1/m)/\sqrt{m}\,, \;\; \mathrm{cn}(z,m) \to \mathrm{dn}(z\sqrt{m},1/m)\,, \;\; \mathrm{dn}(z,m) \to \mathrm{cn}(z\sqrt{m},1/m)\,, \;\; m \ge 1\,. \tag{1.7}$$

The Jacobi elliptic functions provide a basis to find the solutions to the equation $\ddot{x} = (a-x)(b-x)(c-x)$. We shall not discuss this further, but note that depending on the values of the constants a, b, c a wide variety of non-linear oscillators may be studied. See Whittaker[24], or Brizard[25] for the equivalent form $(\dot{x})^2 = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$.

[3]The multiplication rule follows immediately from the identification of the elliptic functions with the Neville[17] theta functions.

1.3 Single Harmonic

The simple case $\ddot{x} + \omega^2 \sin x(t) = 0$ describes the motion of a simple pendulum. Here ω is angular frequency. To illustrate how solutions are found, we shall suppose for a moment that we do not know the solution is $x(t) = 2\arcsin[\sqrt{m}\,\mathrm{sn}(\omega t, m)]$. We multiply the EOM by $\dot{x}$ and integrate with respect to time, leading to the Hamiltonian:

$$(1/2)(\dot{x})^2 \;+\; \omega^2[1 - \cos x] \;=\; 2J\omega^2 \;. \tag{1.8}$$

Here the potential $U = (1 - \cos x)$ has been chosen to be zero at the centre of the well, $x = 0$. The constant of integration J is chosen to satisfy the initial conditions, and is dimensionless. Employing the identity $\cos x = \cos^2 x/2 \;-\; \sin^2 x/2$ leads to:

$$(1/2)(\dot{x})^2 \;+\; 2\omega^2(\sin x/2)^2 \;=\; 2J\omega^2 \;. \tag{1.9}$$

We make the key transformation $\sin[x(t)/2] \to X(t)$. We form the time derivative:

$$\dot{X} = (1/2)\cos[x(t)/2]\,\dot{x} = (1/2)\sqrt{1 - X^2}\,\dot{x} \;. \tag{1.10}$$

Substitute the expressions for $\sin(x/2)$ and $\dot{x}$ in terms of X and $\dot{X}$ into Eq. 1.9, and solve for $\dot{X}$:

$$(\dot{X})^2 = \omega^2(1 - X^2)(J - X^2) \;. \tag{1.11}$$

Eq. 1.11 is satisfied by $X \propto \mathrm{sn}(\omega t, m)$.

The next step is to substitute a trial function $X(t) = \sqrt{r}\,\mathrm{sn}(at, m)$ with adjustable constants (a, r, m, J) into Eq. 1.11, leading to

$$(a^2 r - J\omega^2) \;+\; r[(1+J)\omega^2 - a^2(1+m)]\mathrm{sn}(z,m)^2 \;+\; r(a^2 m - r\omega^2)\mathrm{sn}(z,m)^4 = 0 \;. \tag{1.12}$$

The coefficients of the time-varying functions must all be zero. This condition is sometimes called *harmonic balance*.[4] This leads to simultaneous non-linear equations for the adjustable constants. The number of equations is reduced by half when we take $X = \sqrt{r}\,\mathrm{sn}(z,m)$ rather than $X = r\,\mathrm{sn}(z,m)$. These equations are purely algebraic equations, and may produce parameter values that are incompatible with the solutions X and $\dot{X}$ being pure real. Moreover, the values may be incompatible with the expected arguments of the Jacobi functions; phase z real and parameter $0 \le m \le 1$. Therefore, the simultaneous solutions, and the results of their adoption, must be carefully inspected.

In this case, the coefficients of $\mathrm{sn}^0, \mathrm{sn}^2, \mathrm{sn}^4$ all zero leads to three simultaneous non-linear equations for the adjustable parameters. There are 3 equations and 4 adjustable constants. Therefore, we may choose one quantity as the control parameter; and find the remainder constants in terms of that parameter. If we treat r as the parameter, we find the solution $[a = \omega,\ J = r,\ m = r]$; or

$$X = \sqrt{r}\,\mathrm{sn}(\omega t, r)\,, \qquad \dot{X}/\omega = \sqrt{r}\mathrm{cn}(\omega t, r)\mathrm{dn}(\omega t, r)\,, \qquad 0 \le r \le 1\,. \tag{1.13}$$

$$\text{and} \qquad \sin[x(t)/2] = X\,, \qquad \dot{x}/\omega = 2\sqrt{r}\mathrm{cn}(\omega t, r)\;. \tag{1.14}$$

This $(x, \dot{x})$ is a *libration*, confined within the well. The Hamiltonian value is $2J\omega^2 = 2r\omega^2$. The period of oscillation $\tau(r)$ is given by the condition $a\tau = 4K[m]$. We substitute the solutions for a and m to find $\tau(r) = 4K(r)/\omega$.

If we treat m as the parameter, we find two solutions. The first $[a = \omega,\ J = m,\ r = m]$ is just another way of writing the libration solution, with r replaced by m. The second is a new solution: $[a = \omega/\sqrt{m},\ J = 1/m,\ r = 1]$, or

$$\sin[x(t)/2] = X = \mathrm{sn}(\omega t/\sqrt{m}, m)\,, \qquad \dot{x}/\omega = 2\,\mathrm{dn}(\omega t/\sqrt{m}, m)/\sqrt{m}\,, \qquad 0 \le m \le 1\,. \tag{1.15}$$

[4] The terminology is often reserved for driven systems wherein the harmonic content of the drive and response must be equated. But the principle holds also for free oscillations.

This $(x, \dot{x})$ is a *rotation*, traversing the well. The Hamiltonian value is $2J\omega^2 = 2\omega^2/m$. The period of oscillation $\tau(r)$ is given by the condition $a\tau = 2K[m]$. We substitute the solutions for a and r to find $\tau(m) = 2K(m)\sqrt{m}/\omega$.

The fact that there are two solutions for the same value of m is potentially the source of ambiguity. For that reason, it is customary to write

$$\sin[x(t)/2] = \mathrm{sn}(\omega t\sqrt{n}, 1/n)\,, \qquad \dot{x}/\omega = 2\sqrt{n}\,\mathrm{dn}(\omega t\sqrt{n}, 1/n) \qquad \text{with } n \geq 1 \text{ and } n = 1/m\,. \tag{1.16}$$

It may be noted that if we take the libration Eq. 1.14 and extend into the range $r \geq 1$ by use of the Jacobi real transformation, we obtain the rotation Eq. 1.16 with n replaced by r. However, the behaviour $m \to 1/m$ transforms a libration into a rotation, is not a general property of non-linear oscillations described by combinations of Jacobi elliptic functions.

It may be confirmed that $\mathrm{cn}(z, m)$ is not a solution of the pendulum oscillator EOM by substituting $X = \sqrt{r}\,\mathrm{cn}(at, m)$ and repeating the steps above. The conditions for harmonic balance yield three equations for the adjustable constants, one of which is $a^2 m + r\omega^2 = 0$. Imaginary values of a or ω, and negative values of m or r, are not permitted; therefore cn is not permissible.

An alternative approach to the problem is to work directly with the EOM: a second order differential equation. To proceed, we substitute the trial function and its second derivative into the EOM. This leads again to an equation containing powers of the Jacobi elliptic function, and simultaneous equations for the adjustable constants. Naturally, the outcome is the same. However, the details of the calculation are more laborious and protracted (because, at the very least, an additional derivative must be calculated and simplified). Contrastingly, working only with the first derivative is a simpler route because it appeals to the fact that there is an integral of motion $J\omega^2$ that is a conserved quantity. Consequently, the value of the Hamiltonian is given automatically (by the equations for harmonic balance); and there is no need to substitute explicit expressions for x and $\dot{x}$ into Eq. 1.9 to calculate the total energy.

1.4 Dual Harmonic - bunch lengthening

For the long-bunch mode of operation, the second harmonic opposes the fundamental waveform. The equation of motion (EOM) is $\ddot{x} + \omega^2[\sin x - \frac{1}{2}\sin 2x] = 0$. The energy function (or Hamiltonian) is obtained by multiplying the EOM throughout by $\dot{x}$, and integrating with respect to time. The Hamiltonian is

$$(1/2)(\dot{x})^2 \;+\; 2\omega^2 \sin(x/2)^4 \;=\; 2J\omega^2\,. \tag{1.17}$$

The potential function $U = 2\sin^4(x/2)$ is chosen to be zero at the bunch centre $x = 0$. We begin the investigation by trying the transformation used for the pendulum oscillator: $\sin[x(t)/2] \to Y(t)$ and $\dot{x} = 2\dot{Y}/\sqrt{1 - Y^2}$. We substitute into Eq. 1.17, yielding

$$2\omega^2 Y^4 \;+\; \frac{2(\dot{Y})^2}{1 - Y^2} \;=\; 2J\omega^2\,. \tag{1.18}$$

This may be solved for $(\dot{Y})^2 = \omega^2(1 - Y^2)(J - Y^4)$. The expression contains powers of Y up to the sixth; and this is not compatible with differential equations that define cn, sn and dn.

Evidently, a new trial function is needed. We shall take a combination of Jacobi functions: $\sin[x(t)/2] = F(\mathrm{cn}, \mathrm{sn}, \mathrm{dn})$. There are some constraints: F must be anti-symmetric about $x(t) = 0$ and symmetric about $x = \pm\pi$; the modulus of $|F| \leq 1$; and F is chosen to cancel high powers of the Jacobi functions. Another constraint is that any combinations of the Jacobi functions must be co-periodic, which implies they all have the same arguments (z, m). The strength of the restoring force varies during the oscillation, and this provides the hint that the motion is modulated at twice the fundamental oscillation frequency. One possibility is $\sin[x(t)/2] = Y\sqrt{1 - X^2}$ where X, Y are a doublet chosen from the triplet cn, sn, dn. It transpires that such a choice leads to an analog of Eq. 1.18 that contains even higher powers of Y.

Another possibility is $\sin[x(t)/2] \to Y(t)/\sqrt{1-X^2(t)}$. Forming the derivative we find:

$$\frac{1}{2}\cos[x(t)/2]\,\dot{x} \equiv \frac{1}{2}\sqrt{1-\frac{Y^2}{1-X^2}}\;\dot{x} \;=\; \frac{(1-X^2)\dot{Y}+Y\,X\,\dot{X}}{(1-X^2)^{3/2}}\;. \tag{1.19}$$

The relationships (Eq. 1.1) between cn sn, dn imply the possibility that $Y \times X \times \dot{X}$ is proportional to $\dot{Y}$ and that the equations may close. The expressions for $\sin(x/2)$ and $\dot{x}$ are substituted in the Hamiltonian, Eq. 1.17, with the result:

$$\omega^2 Y^4 \;+\; \frac{[(1-X^2)\dot{Y}+Y\,X\,\dot{X}]^2}{1-Y^2-X^2} \;=\; (1-X^2)^2 J\omega^2\;. \tag{1.20}$$

The next step is to insert trial functions in the place of X and Y. One such possibility is $Y(t) = \sqrt{r}\,\mathrm{cn}(at,m)$ and $X(t) = \sqrt{p}\,\mathrm{sn}(at,m)$ where a, r, p, m and J are adjustable constants. The result is an expression in powers of $\mathrm{sn}(at,m)$ up to the sixth that is equated to zero. This is generally zero if the coefficients of all powers $\mathrm{sn}^0, \mathrm{sn}^2, \mathrm{sn}^4, \mathrm{sn}^6$ are individually zero. There results four simultaneous non-linear equations.

$$\mathrm{sn}^0 \to\; (1-r)(J-r^2)\omega^2 = 0\,, \qquad \mathrm{sn}^2 \to\; a^2(1-p)^2 r - [(2+p-3r)r^2 + J(-3p+r+2pr)]\omega^2 = 0$$

$$\mathrm{sn}^4 \to\; -a^2 m(1-p)^2 r + [(1+2p-3r)r^2 + Jp(p(r-3)+2r)]\omega^2 = 0\,, \quad \mathrm{sn}^6 \to (p-r)(Jp^2-r^2)\omega^2\,. \tag{1.21}$$

1.4.1 Solution types

In general there are five types of solution: (1) trivial solutions such as $r = 0$ or $p = r = 1$; (2) $p = r$ and $r \neq 1$; (3) $r = 1$ and $p \neq 1$; (4) other. The trivial solutions correspond to the particle being stuck at $x = 0$ or $x = \pm\pi$. The second and third types are the libration and rotation solutions, respectively. The fourth type are accidental duplicates of types (2) and (3) for special conditions between p and r; typically they are valid only for very restrictive ranges of p or r. The exact form of the solutions depends on which of the variables we take as the control parameter.

Trivial

$$p \to 1 \quad \text{yields} \quad \sin[x/2] = \sqrt{r} \;\;\&\;\; \dot{x} = 0; \qquad r \to 1 \;\;\&\;\; p \to 1 \quad \text{yields} \quad \sin[x/2] = 1 \;\;\&\;\; \dot{x} = 0\,. \tag{1.22}$$

Libration: $p \to r$ and $r \neq 1$

$$\sin[x/2] = \frac{\sqrt{r}\,\mathrm{cn}(at,m)}{\sqrt{1-r\,\mathrm{sn}(at,m)^2}} \qquad\qquad \dot{x} = \frac{2a\sqrt{1-r}\sqrt{r}\,\mathrm{dn}(at,m)\,\mathrm{sn}(at,m)}{-1+r\,\mathrm{sn}(at,m)^2} \tag{1.23}$$

Rotation: $r \to 1$ and $p \neq 1$

$$\sin[x/2] = \frac{\mathrm{cn}(at,m)}{\sqrt{1-p\,\mathrm{sn}(at,m)^2}} \qquad\qquad \dot{x} = \frac{2a\sqrt{1-p}\,\mathrm{dn}(at,m)}{-1+p\,\mathrm{sn}(at,m)^2} \tag{1.24}$$

Special case: $p \to m$

$$\sin[x/2] = \sqrt{r}\frac{\mathrm{cn}(at,m)}{\mathrm{dn}(at,m)} = \sqrt{r}\,\mathrm{cd}(at,m) = \sqrt{r}\,\mathrm{sn}(at+K[m],m) \qquad \text{is not a solution.}$$

1.4.2 Treat r as parameter

We may find the adjustable constants in terms of r. There are two distinct solutions. The first solution is $[a = \omega\sqrt{r(1-r)},\; m = (1+r)/(r-1),\; p = -1,\; J = r^2]$; but it is incompatible with $0 < m \leq 1$ and is rejected. The second is the *libration* solution:

$$[a = \omega\sqrt{2r},\;\; m = (1+r)/2,\;\; p = r,\;\; J = r^2] \quad \text{valid for }\; 0 < r \leq 1\,. \tag{1.25}$$

These values are to be substituted in Eq. 1.23; this is the solution given by Shaposhnikova. The Hamiltonian value is $2J\omega^2 = 2r^2\omega^2$. The period of oscillation $\tau(r)$ is given by the condition $a\tau = 4K[m]$. We substitute the solutions for a and m to find $\tau(r) = 4K\left[\frac{1}{2}(1+r)\right]/[\omega\sqrt{2r}]$.

1.4.3 Treat p as parameter

We may find the adjustable constants in terms of p. There are two distinct solutions. The first is the *rotation* solution

$$[a = (\omega/p)\sqrt{1+p},\ \ m = 2p/(1+p),\ \ r = 1,\ \ J = 1/p^2] \quad \text{valid for } \ 0 < p \leq 1;. \tag{1.26}$$

These values are to be substituted in Eq. 1.24. The Hamiltonian value is $2J\omega^2 = 2(\omega/p)^2$. The period of oscillation $\tau(p)$ is given by the condition $a\tau = 2K[m]$. We substitute the solutions for a and m to find $\tau(p) = 2K\,[2p/(1+p)]\,(p/\omega)/\sqrt{1+p}$.

The second is the libration solution with r substituted by p. We may also treat m as the parameter. However, this leads either to duplicate solutions of those above or to solutions incompatible with the adjustable constants having real values.

1.4.4 Libration & Rotation, $q = -1/2$

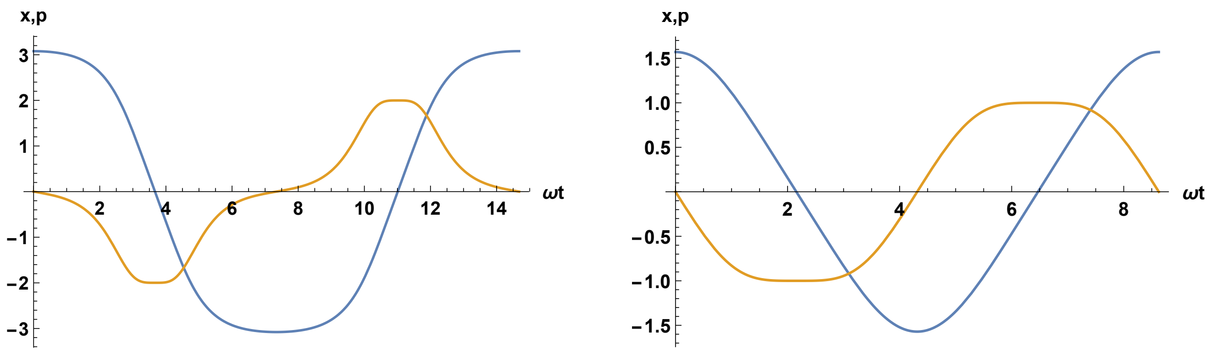

Figure 1.1: Libration: Position (blue) and momentum (yellow) trajectories versus phase. Left: $r = 0.999$. Right: $r = 0.5$

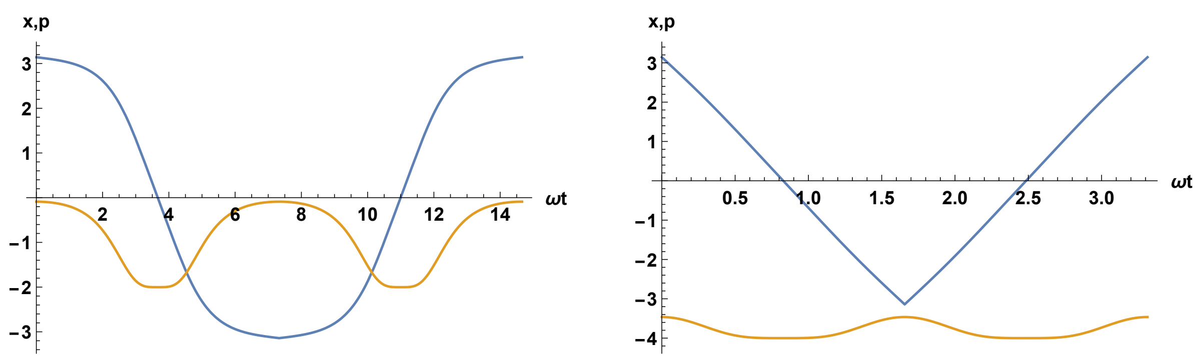

Figure 1.2: Rotation: Position and momentum trajectories versus phase. Left: $p = 0.999$. Right: $p = 0.5$. The x-motion (blue) has been folded modulo π; really it goes $\pi \to -\pi \to -3\pi$ and so on. Hence the period is that of the momentum p shown yellow.

Whereas the phase space portraits Figs. 1.3 & 1.6 could have been constructed from the Hamiltonian by solution for $\dot{x}$ in terms of x and J, the trajectories $x(t)$ and $\dot{x}(t)$ versus time can only be graphed if explicit solutions to the EOM have been obtained.

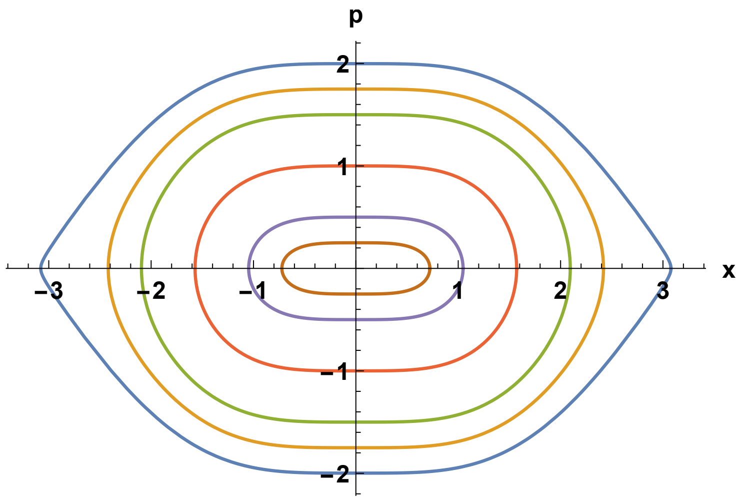

Figure 1.3: Contours of constant Hamiltonian for libration in bunch-lengthening mode. Blue: $r = 0.999$; gold: $r = 7/8$; green: $r = 3/4$; coral: $r = 1/2$; purple: $r = 1/4$; brown: $r = 1/8$. Abscissa: position (radian). Ordinate: momentum (dimensionless).

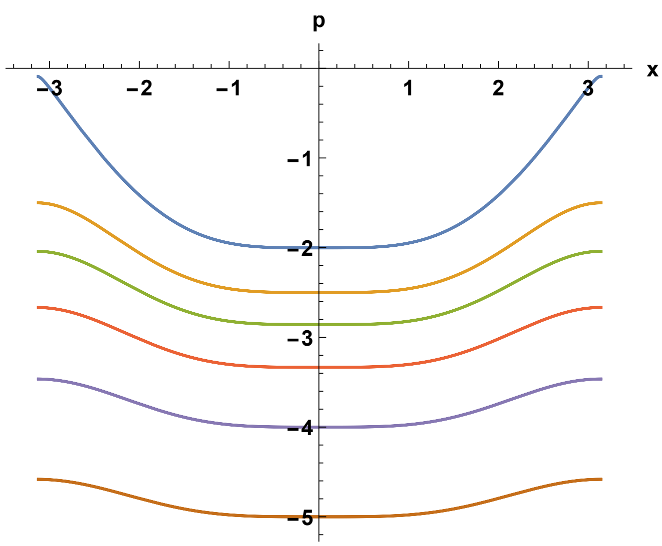

Figure 1.4: Contours of constant Hamiltonian for rotation in bnch-lengthening mode. Blue: $p = 0.999$; gold: $p = 0.8$; green: $p = 0.7$; coral: $p = 0.6$; purple: $p = 0.5$; brown: $p = 0.4$. Abscissa: position (radian). Ordinate: momentum (dimensionless).

1.5 Dual Harmonic - bunch shortening

For the long bunch mode of operation, the second harmonic reinforces the fundamental waveform. The equation of motion (EOM) is $\ddot{x} + \omega^2[\sin x + \frac{1}{2}\sin 2x] = 0$. The energy function is obtained by multiplying the EOM throughout by $\dot{x}$, and integrating with respect to time. The Hamiltonian is

$$(1/2)(\dot{x})^2 \ + \ 2\omega^2 \left[1 - \cos(x/2)^4\right] \ = \ 2J\omega^2 \ . \tag{1.27}$$

The potential function $U = 2[1 - \cos^4(x/2)] \equiv 2\sin^2(x/2)[2 - \sin^2(x/2)]$ is chosen to be zero at the bunch centre $x = 0$. We begin the investigation by trying the transformation used for the pendulum oscillator: $\sin[x(t)/2] \to Y(t)$ and $\dot{x} = 2\dot{Y}/\sqrt{1 - Y^2}$. We substitute into Eq. 1.27, yielding

$$2\omega^2(2Y^2 - Y^4) \ + \ \frac{2(\dot{Y})^2}{1 - Y^2} \ = \ 2J\omega^2 \ . \tag{1.28}$$

This may be solved for $(\dot{Y})^2 = \omega^2(1 - Y^2)(J - 2Y + Y^4)$. The expression contains powers of Y up to the sixth; and is not compatible with differential equations that define cn, sn and dn.

We try again the solution $\sin[x(t)/2] \to Y(t)/\sqrt{1 - X^2(t)}$. The expressions for $\sin(x/2)$ and $\dot{x}$ are substituted in the Hamiltonian, Eq. 1.27, with the result:

$$2\omega^2 \left[1 - \frac{A^2}{B^2}\right] \ + \ 2\frac{[B\dot{Y} + Y\,X\,\dot{X}]^2}{A\,B^2} \ = \ 2J\omega^2 \quad \text{and} \ A = (1 - Y^2 - X^2)\,, \quad B = (1 - X^2)\ . \tag{1.29}$$

The next step is to insert trial functions in the place of X and Y. Again, we try $Y(t) = \sqrt{r}\,\mathrm{cn}(at, m)$ and $X(t) = \sqrt{p}\,\mathrm{sn}(at, m)$ where a, r, p, m and J are adjustable constants. The result is an expression in powers of $\mathrm{sn}(at, m)$ up to the sixth that is equated to zero. This is generally zero if the coefficients of all powers $\mathrm{sn}^0, \mathrm{sn}^2, \mathrm{sn}^4, \mathrm{sn}^6$ are individually zero. There results four simultaneous non-linear equations.

$$\begin{aligned}
&\mathrm{sn}^0 \ \to \quad \omega^2(r - 1)[J + r(r - 2)] = 0\,, \qquad \mathrm{sn}^6 \to \quad (p - r)[Jp^2 + r(r - 2p)]\omega^2 = 0\,,\\
&\mathrm{sn}^2 \ \to \quad a^2(1 - p)^2 r - [J(2pr - 3p + r) + r(2 + 4p - 6r - 3pr + 3r^2)]\omega^2 = 0\,,\\
&\mathrm{sn}^4 \ \to \quad -a^2 m(1 - p)^2 r + [Jp(p(r - 3) + 2r) + r(2p^2 + p(4 - 6r) + 3(r - 1)r)]\omega^2 = 0\ .
\end{aligned} \tag{1.30}$$

Because we have used the same trial function, the solutions are of the same form as Eqs. 1.22-1.24 but with different values for the adjustable constants.

1.5.1 Treat r as parameter

We may find the adjustable constants in terms of r. There are three distinct solutions. The first, $p = 1$ is a trivial solution (particle stuck at fixed point). The second is the *libration* solution:

$$[a = \omega\sqrt{2}\sqrt{1 - r}, \ \ m = r/2, \ \ p = r, \ \ J = (2 - r)r] \quad \text{valid for} \ \ 0 < r \leq 1;. \tag{1.31}$$

These values are substituted in Eq. 1.23. The Hamiltonian value is $2J\omega^2 = 2\omega^2(2-r)r$. The period of oscillation $\tau(p)$ is given by the condition $a\tau = 4K[m]$. We substitute the solutions for a and m to find $\tau = 4K[r/2]/(\omega\sqrt{2}\sqrt{1 - r})$. The third solution $[a = \omega\sqrt{2 - 3r + r^2}$, $m = r/(r - 2)$, $p = r(2 - r)$, $J = (2 - r)]$ and the conditions a real and $0 < m \leq 1$ leads to the result $r < 0$; which is incompatible with the trial function.

1.5.2 Treat p as parameter

We may find the adjustable constants in terms of r. There are three distinct solutions. The first $[a = \sqrt{p - 1}\,(\omega/p)$, $m = 2p$, $r = 1$, $(2p - 1)/p^2]$ and third $[a = \omega\sqrt{2}\sqrt{1 - p}/(1 + p)$, $m = -p$, $r = 2p/(1 + p)$, $J = 4p/(1 + p)^2]$ solutions are incompatible with a real and $0 < m \leq 1$. The second is the libration solution with r substituted by p.

We may also treat m as the parameter. There are six trivial solutions; and three distinct viable solutions. Of these, the first is an $r = 1$ rotation; the second is a $p(m) = r(m)$ solution; and the third is incompatible with $p > 0$ and $r > 0$.

1.5.3 Libration, $q = +1/2$

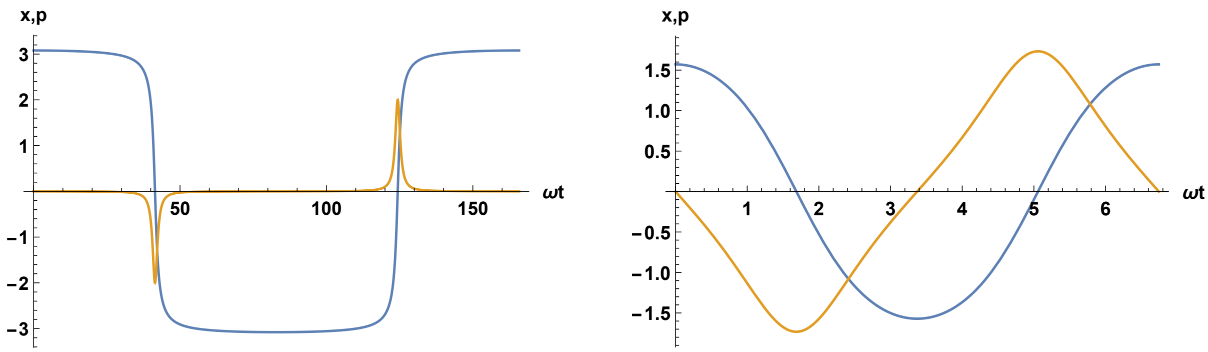

Figure 1.5: Libration: Position (blue) and momentum (yellow) trajectories versus phase. Left: $r = 0.999$. Right: $r = 0.5$

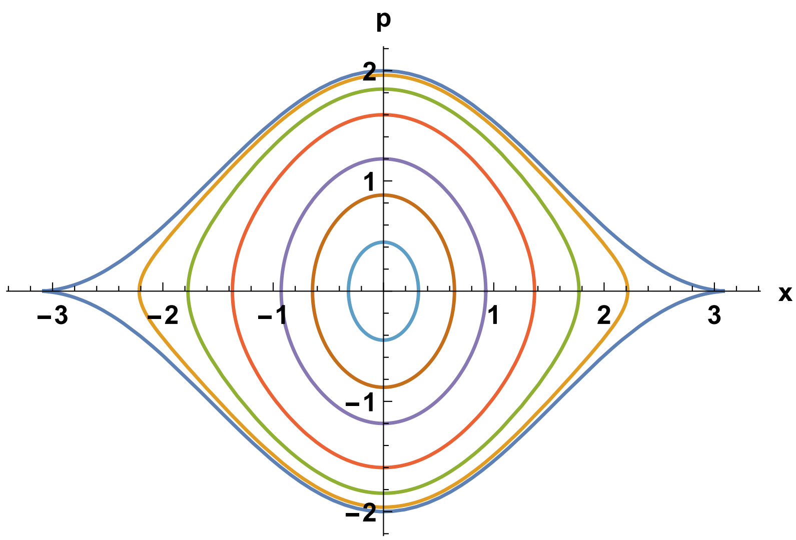

Figure 1.6: Contours of constant Hamiltonian for libration in bunch-shortening mode. Blue: $r = 0.999$; gold: $r = 0.8$; green: $r = 0.6$; coral: $r = 0.4$; purple: $r = 0.2$; brown: $r = 0.1$; light blue: $r = 0.025$. Abscissa: position (radian). Ordinate: momentum (dimensionless).

1.5.4 No rotation solution, $q = +1/2$

For the particular choice of the trial function in this section, there are no values of the adjustable constants that are compatible with the properties of the elliptic functions and a rotation solution. One may experiment with other trial functions, but to no avail. We shall come to understand why this is so in Sec. 1.7.

1.6 Double RF system

Gram and Morton[1] introduced the dual harmonic system. It is a special case of motion in the general *double RF system* which satisfies the equation $\ddot{x}+\omega^2[\sin x+q\sin(nx)]=0$. Here amplitude ratio q and integer n harmonic number are arbitrary. The system can be further generalized by inserting a phase difference ϕ between the harmonic components. The term *dual harmonic* is usually reserved for the case $n=2$. The equation of motion may be integrated once to give the energy equation:

$$(1/2)(\dot{x})^2+\omega^2 U(x)=H \quad \text{where} \quad U(x)=[1-\cos x]+(q/n)[1-\cos(nx)]\ . \tag{1.32}$$

For a specific value of the Hamiltonian (H_0), this may be solved for the momentum $p=\dot{x}=2\omega\sqrt{H_0-U(x)}$. We suppose $p=0$ when $x=X$ and the potential is symmetric. Hence it follows that the acceptance area A and oscillation period τ are given by

$$A=4\int_0^X p(x)dx \quad \text{and} \quad \tau=4\int_0^X dx/p(x) \quad \text{with} \quad p(x)=\omega\sqrt{2}\sqrt{U(X)-U(x)}\ . \tag{1.33}$$

These formulae were given in Ref.[1] and evaluated as a function of q for $n=2$.

Hofmann[2, 3] considers the general case $n=2,3,4,\ldots$ and $\phi\neq 0$. It is not possible to find closed-form large-amplitude solutions to the EOM. In the small amplitude limit[5] the force law is Taylor expanded up to cubic terms (leading to the Duffing equation) and the potential expanded up to terms in x^4. In Ref.[2] the acceptance and period are given in terms of elliptic integrals. In Ref.[3] the motion is given in terms of Jacobi elliptic functions. Hofmann was motivated by both the bunch-lengthening and bunch-shortening modes of operation of a Landau[6] cavity. The specific case $n=2$ has the distinction that the large-amplitude motion can be found in closed form.

Subsequent to the original version of this article, Shaposhnikova recommended to me Chap.3 Sec V.5 *Double rf Systems* of the accelerator physics book[12] by S.Y. Lee. Lee considers the dual-harmonic bunch-lengthening mode and arbitrary ratio of amplitudes $q<0$; and he evaluates analytically the acceptance and period (Eq.1.33) which are related to elliptic integrals. The present author was momentarily confused by Lee's choice to call the amplitude ratio $r\equiv -q$. Nevertheless, Lee's treatment inspires us to believe that the dynamics may be solved explicitly for arbitrary q.

1.7 Arbitrary Ratio of Amplitudes

We now consider the case of (almost) arbitrary ratio q of fundamental and 2nd harmonic components. The force law $F(x)$ and potential function $U(x)$, respectively, are

$$F(x)=\sin x\ +\ q\sin 2x \quad \text{and} \quad U(x)=[1-\cos x]\ +\ (q/2)[1-\cos 2x]\ . \tag{1.34}$$

Positive ratio $q>0$ increases the restoring force yielding the bunch shortening mode. Negative ratio $q<0$ reduces the restoring force, providing the bunch lengthening mode.

Lee's analysis teaches us three things: (i) the nature of the fixed points changes abruptly at $|q|=1/2$; (ii) we may expect to find analytic solutions $[x(t),\dot{x}]$ for a broad range of q; and (iii) the mathematical form of the Hamiltonian and of the solutions is (mostly) the same in all cases (although the details depend on q). Fig. 1.7 shows the potential function for several q values.

For $|q|<1/2$, there is a single well with stable fixed point (SFP) at $x=0$ and unstable fixed points (UFP) at $x=\pm\pi$. For $|q|>1/2$, a second, local well emerges; and the nature of the FPs

[5] Some authors call this the "short bunch limit". This terminology should be avoided, because small amplitudes occur both in bunch lengthening and shortening modes of operation of the harmonic cavity.

[6] A Landau cavity is any RF cavity operated with the purpose of increasing the synchrotron frequency spread so as to stabilize a particle beam against collective instability.

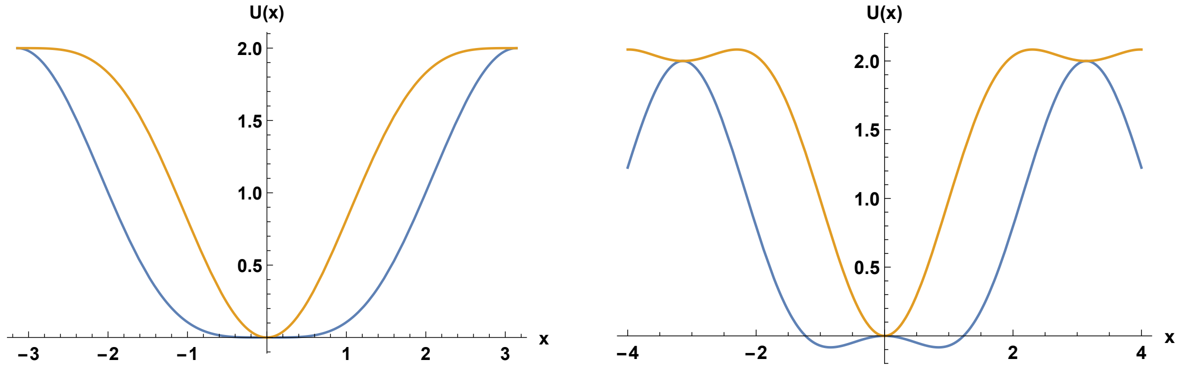

Figure 1.7: Potential $U(x)$ versus position x. Left: $|q| = 1/2$. Right: $|q| = 3/4$. Negative and positive q shown blue and yellow, respectively.

change. For $q > +1/2$, there are SFPs at $x = 0$ and $x = \pm\pi$, and local UFPs at $x = \pm\pi \mp \Phi$ where $\tan\Phi = \sqrt{-1 + (2q)^2}$. For $q < -1/2$, there are UFPs at $x = 0$ and $x = \pm\pi$, and local SFPs at $x = \pm\Phi$. For $q = +1/2$, there is a SFP at $x = 0$ and astable fixed points (AFP) at $x = \pm\pi$. For $q = -1/2$, there is an AFP at $x = 0$ and UFPs at $x = \pm\pi$. It is the emergence of the secondary potential wells, that prevented us from finding a rotation-type solution when $q = +1/2$. Adapting the method introduced in Sec. 1.4 and 1.5, we can find solutions $[x(t), \dot{x}]$ for all of these circumstances except the motions trapped in the asymmetric wells centred at $x = \pm\Phi$ when $q < -1/2$.

As before, we write the potential function in terms of $x/2$; leading to the Hamiltonian:

$$\omega^2 U(x) + (1/2)(\dot{x})^2 = 2J\omega^2 \qquad \text{where} \quad U(x) \;=\; 2[\sin(x/2)]^2\left\{1 + 2q - 2q[\sin(x/2)]^2\right\}\;. \tag{1.35}$$

We substitute the same trial function as above: $\sin[x(t)/2] \to Y(t)/\sqrt{1 - X^2(t)}$. The trial function may also be written:

$$\tan[x/2] = \frac{Y}{\sqrt{1 - Y^2 - X^2}} \;=\; \frac{\sqrt{r}\,\mathrm{cn}(at, m)}{\sqrt{1 - r\,\mathrm{cn}(at, m)^2 - p\,\mathrm{sn}(at, m)^2}}\;.$$

It follows that $0 < r \leq 1$ and $0 < p \leq 1$. Although the $\tan(x/2)$ form is more concise, the $\sin(x/2)$ version is better suited for substitution in the Hamiltonian; and yields:

$$2\omega^2 Y^2\left[\frac{(1 + 2q)}{B} - 2q\frac{Y^2}{B^2}\right] \;+\; 2\frac{[B\dot{Y} + Y\,X\,\dot{X}]^2}{A\,B^2} \;=\; 2J\omega^2\;. \tag{1.36}$$

We substitute the explicit functions $Y = \sqrt{r}\,\mathrm{cn}(at, m)$ and $X = \sqrt{p}\,\mathrm{sn}(at, m)$ where a, r, p, m, J are adjustable constants; leading to a cubic equation in powers of $[\mathrm{sn}(at, m)]^2$. As above, the coefficients of all powers must be zero; and this leads to a set of simultaneous equations to be solved for the adjustable constants.

$$\mathrm{sn}^0 \to (1-r)[J - r - 2qr(1-r)]\omega^2 = 0, \qquad \mathrm{sn}^6 \to (p-r)[Jp^2 + r(2qr - p(1+2q))]\omega^2 = 0 \tag{1.37}$$
$$\mathrm{sn}^2 \to \quad a^2(1-p)^2 r - [-3Jp + (1+2p)(1+J+2q)r - (2+p)(1+4q)r^2 + 6qr^3]\omega^2 = 0$$
$$\mathrm{sn}^4 \to \quad -a^2 m(1-p)^2 r + [-3Jp^2 + p(2+p)(1+J+2q)r - (1+2p)(1+4q)r^2 + 6qr^3]\omega^2 = 0$$

The *Mathematica* symbolic manipulation software finds solutions of these equations; and we may choose which of the triplet m, p, r is the independent parameter. The number of solutions is reduced by imposing the constraints $0 < m \leq$, $0 < r \leq 1$, $0 < p \leq 1$, $J > 0$ and $a > 0$. As above, there are four types of solution: (i) libration with $p = r$; (ii) rotation with $r = 1$ and $p \neq 1$; (iii) solutions with $p = p(r)$ or $r = r(p)$; and (iv) trivial solutions corresponding to the particle being stuck at a fixed point.

1.7.1 Treat r as parameter

There are two distinct *libration* solutions. The most useful and least constrained is

$$\left[a = \sqrt{1+2q(1-2r)}\omega, \quad m = [r(1-2qr)]/[1+2q(1-2r)], \quad p \to r, \quad J = r + 2qr(1-r)\right] \ . \tag{1.38}$$

This solution is valid over all q, provided r is chosen appropriately:

$$q \leq -\frac{1}{2} \ \& \ 0 < r < 1 + \frac{1}{2q} \quad \text{or} \quad -\frac{1}{2} < q \leq +\frac{1}{2} \ \& \ 0 < r < 1 \quad \text{or} \quad q > \frac{1}{2} \ \& \ 0 < r < \frac{1}{2q} \ .$$

The Hamiltonian is $H = 2J\omega^2$ and rotation period given by $\tau = 4K(m)/a$. The second libration solution is:

$$\left[a = \sqrt{1+2q(1-r-)}\sqrt{1-r}\,\omega, \ m = \frac{r(2qr-1)}{[1+2q(1-r)](1-r)}, \ p = \frac{2qr}{1+2q(1-r)}, \quad J = r + 2qr(1-r)\right] \ .$$

This solution duplicates (partially) the first; but is valid only over a very restricted range: $q > 1/2$ and $1/(2q) < r < [1 + 1/(2q)]/2$.

1.7.2 Treat p as parameter

There are three distinct solutions. The first has a imaginary when $q = +1/2$ and the third has $a = 0$ whe $q = -1/2$, signaling issues with the behaviour at the SFPs. The first is the *rotation* solution:

$$\left[a = \sqrt{p-2q}\,\omega/p, \quad m = p(p-4q+2pq)/(p-2q), \quad r = 1, \quad J = (p-2q+2pq)/p^2\right] \ . \tag{1.39}$$

This solution is valid over the ranges:

$$q < 0 \ \ \& \ \ 0 < p < 1 \quad \text{or} \quad 0 < q < 1/2 \ \ \& \ \ 4q/(1+2q) < p < 1 \ .$$

The Hamiltonian is $H = 2J\omega^2$ and rotation period given by $\tau = 2K(m)/a$. The second solution duplicates the libration solution above with the roles of p and r exchanged. The third solution is an accidental solution valid for $q > +1/2$ and $1/(2q) < p \leq 1$. It is a libration in the main potential well, and the analogue of the second libration solution found when r was the free parameter.

1.7.3 Treat m as parameter

When we attempt to find the adjustable constants a, p, r, J in terms of m, there are around 14 distinct solutions. The first 8 are trivial solutions. There are six remainder solutions. Of these the 2nd duplicates the rotation solution with $r = 1$ and is valid for $q < 1/2$. The 1st, 3rd and 4th are inconsistent with the constraints. The 5th and 6th duplicate the $p = r$ libration solution.

1.7.4 Fundamental solutions

The underlying properties of cn, sn, dn imply that there are two fundamental solutions: $p \to r$ (libration) and $r \to 1$ (rotation). Other solutions are accidental duplicates for special conditions between p and r. This implies that we could have short-cut the whole procedure by introducing $p = r$ or $r = 1$ at an early stage of the calculation. In this way, we dispose of the irrelevant solutions; and work with simper algebraic equations. When $p = r$, the sextic reduces to a quadratic in powers of sn^2; and when $r = 1$ the sextic reduces to a cubic in powers of sn.

1.7.5 Amplitude ratio $q > /2$

When the amplitude ratio satisfies $q > 1/2$, a hanging valley appears above the main valley floor. We move the centre of the coordinate system to $x = \pm\pi$ in order to symmetrize the potential with respect to the local SFP within the hanging valley. In the new coordinates, the fixed points move and the mathematical structure of the energy equation is modified. Therefore, we cannot assume to find all the relevant solutions merely by the substitutions $p = r$ or $r = 1$. (Such a manoeuvre fails to find the libration in the hanging valley.) So we must perform the complete analysis.

The potential takes on the form: $U(x) = 2 \;-\; 2(1+2q)[\sin(x/2)]^2 \;+\; 4q[\sin(x/2)]^4$. We insert U into the Hamiltonian and make the substitution $\sin(x/2) \to Y/\sqrt{1-X^2}$ to find

$$2\omega^2\, A\left[\frac{1}{B} + 2q\frac{Y^2}{B^2}\right] \;+\; 2\frac{[B\dot{Y} + Y\,X\,\dot{X}]^2}{A\,B^2} \;=\; 2J\omega^2\,. \tag{1.40}$$

We multiply throughout by AB^2 and insert explicit forms for $Y(t)$ and $X(t)$. There results an equation in powers of $\mathrm{sn}(at,m)$; and their coefficients must all be zero, leading to the simultaneous equations:

$$\mathrm{sn}^0 \to 2(1-r)[J+(r-1)(1+2qr)]\omega^2 = 0, \qquad \mathrm{sn}^6 \to 2(p-r)[(J-1)p^2+2qr^2+p(r-2qr)]\omega^2 = 0 \tag{1.41}$$

$$\mathrm{sn}^2 \to \;\; -2a^2(1-p)^2 r + 2[3(1-J)p + (1+2p)(J+2q-2)r - (2+p)(4q-1)r^2 + 6qr^3]\omega^2 = 0$$

$$\mathrm{sn}^4 \to 2a^2 m(1-p)^2 r - 2[3(1-J)p^2 + p(2+p)(J+2q-2)r - (1+2p)(4q-1)r^2 + 6qr^3]\omega^2 = 0$$

We may solve for the adjustable constants in terms of r or p or m. Note, thus far, q may take on any value.

Treat r as parameter

There are three distinct solutions. The first is a trivial solution. The second is inconsistent with constraints. The third is the *libration* solution for oscillations inside the secondary potential well:

$$\left[a = \sqrt{1+2q(r-1)}\sqrt{r-1}\,\omega,\;\; m = \frac{r(1+2qr)}{[1+2q(r-1)](r-1)},\;\; p = \frac{-2qr}{1+2q(r-1)},\;\; J = (1-r)(1+2qr)\right]. \tag{1.42}$$

This solution is valid for the range $q > +1/2$ and $0 < r < (2q-1)/(4q)$.

Treat p as parameter

There are three distinct solutions. The first is a re-write of the $q < 0$ *rotation* solution (with respect to the shifted coordinate) for bunch lengthening:

$$\left[a = \frac{\sqrt{-(p+2q)}}{p}\omega, \qquad m = \frac{p(p+4q-2pq)}{(p+2q)}, \qquad r = 1, \qquad J = \frac{(p-1)(p+2q)}{p^2}\right]. \tag{1.43}$$

This solution is valid for bunch-lengthening $q < 0$ and $0 < p < 2q/(2q-1)$. The second solution is inconsistent with constraints. The third is an accidental $r(p)$ solution that duplicates the libration in the hanging valley.

Treat m as parameter

There are seven trivial solutions (such as $r = 0$) for the adjustable constants; these are discarded. There are six more distinct solutions; none of them are valid for $q > 0$. Moreover, only the second is valid for $q < 0$ (i.e. bunch lengthening). This duplicates the $r = 1$ condition, and is a rotation solution (across the main potential well) written in terms of the shifted origin.

1.7.6 Bunch lengthening mode

Libration

We provide two numerical examples for the case $q < 0$. The first has $q = -1/4$ and a potential well with a single minimum. The second has $q = -3/4$ and a potential well with two local minima; the motion escapes the local minima but is trapped by the encompassing well.

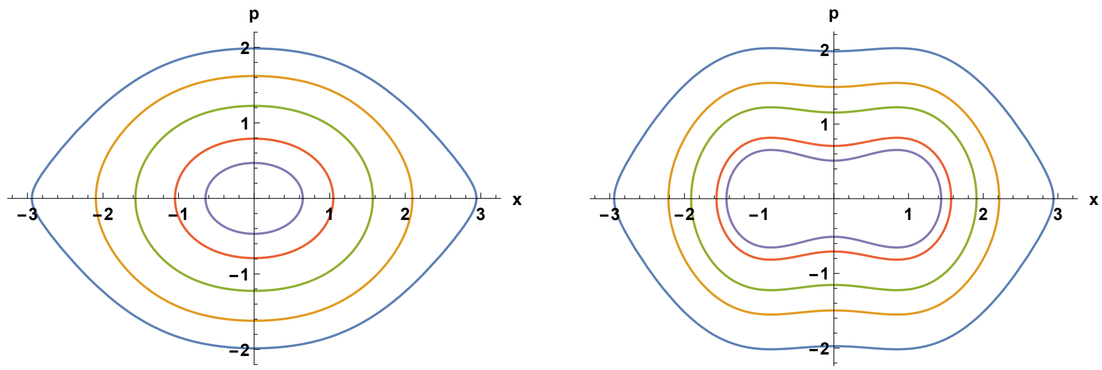

Figure 1.8: Contours of constant Hamiltonian for libration in bunch-lengthening mode. Left: $q = -1/4$. Right $q = -3/4$.

Rotation

We show the untrapped motion that streams over the periodic well for the cases $q = -1/4$ and $q = -3/4$.

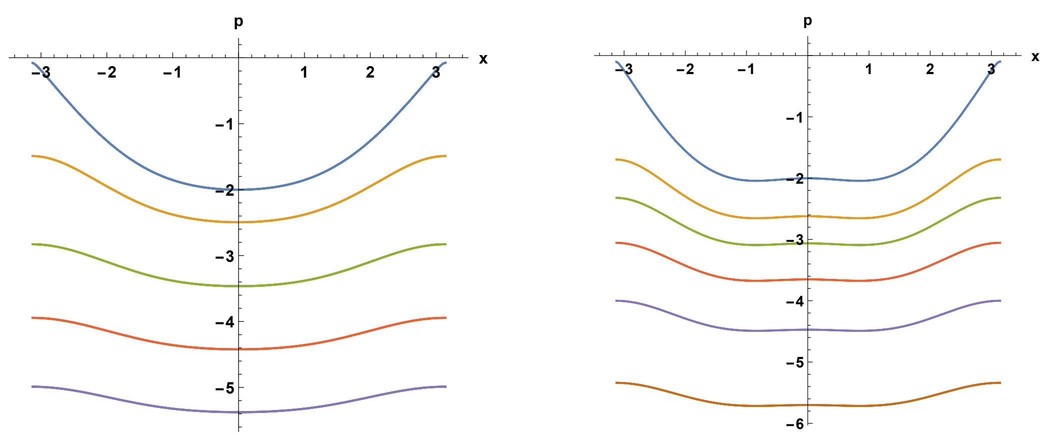

Figure 1.9: Contours of constant Hamiltonian for rotation in bunch-lengthening mode. Left: $q = -1/4$. Right $q = -3/4$.

1.7.7 Bunch shortening mode

We provide two numerical examples of *libration* for the case $q > 0$. The first has $q = +1/4$ and a potential well with a single minimum. The second has $q = +3/4$ and a main potential well with two local minima above; there are separate librations in the lower and upper wells. Concerning *rotation*, we show the motion corresponding to $q = 1/4$.

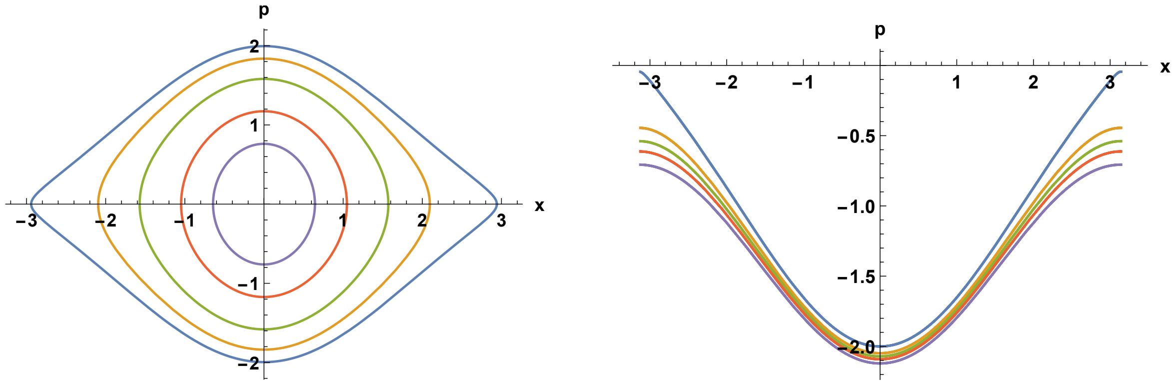

Figure 1.10: Contours of constant Hamiltonian for bunch shortening mode $q = 1/4$. Left = Libration. Right = Rotation.

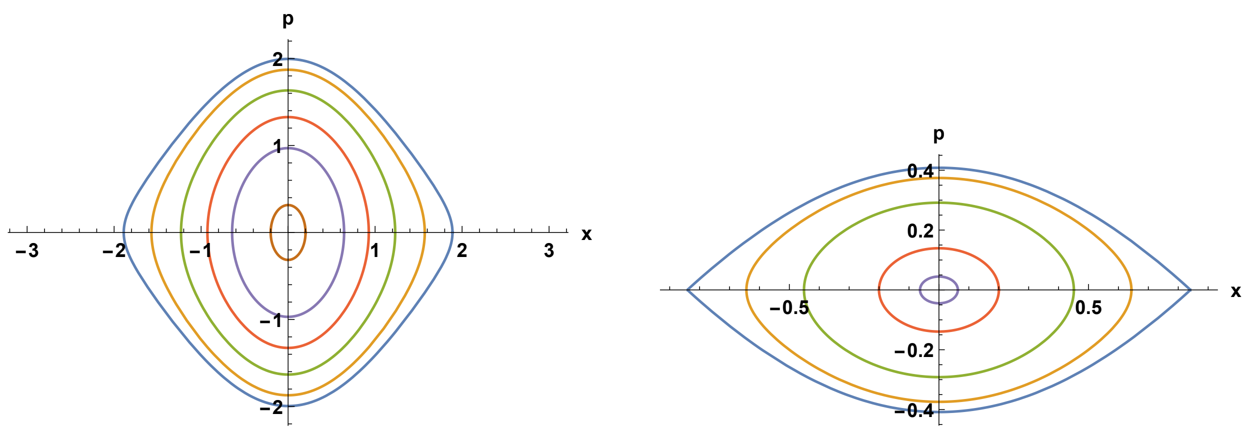

Figure 1.11: Contours of constant Hamiltonian in bunch shortening mode $q = 3/4$. Left: libration in the main valley. Right: libration in the hanging valley. The momenta are measured with respect to the local minimum of the potential.

No rotation solution, $q > 1/2$

Concerning rotation, we have not found formulae for the rotary motion when the amplitude ratio exceeds $q = 1/2$. A challenge to the reader is to find a valid trial function for motion that traverses the bunch-shortening potential well for $q > +1/2$.

1.8 Acceptance Area

The phase space (combinations of position and momenta) that is confined within the potential well is called the longitudinal acceptance, also known as the RF-bucket area. In all cases, this area may be computed numerically from the formulae in Eq. 1.33 for all harmonics n and amplitude ratio q. For particular cases the integrals may be obtained analytically. We adopt the notation $Y = \sin(x/2)$ and $\sqrt{m} = \sin(X/2)$ where $0 < X \leq \pi$ is the extreme of motion. It follows that $dx/dY = 2/\sqrt{1-Y^2}$.

1.8.1 First harmonic

In the case $q = 0$, the particle motion is governed only by the fundamental component of the waveform. The potential is $U = 1 - \cos x$ which may be re-written in terms of the half angle $x/2$. The Hamiltonian is $(1/2)p^2 + 2\omega^2[\sin(x/2)]^2 = 2\omega^2[\sin(X/2)]^2$. The momentum may be written $p = 2\omega\sqrt{m - Y^2}$. The area is $4\int_0^X p(x)dx = 4\int_0^{\sqrt{m}} p(Y)(dx/dY)dY$. The case $m = 1$ is trivial. The integrand becomes 4ω and the area $A = 16\omega$. In the general case, the integrand is $4\omega\sqrt{m-Y^2}/\sqrt{1-Y^2}$ and the area $A(m) = 16\omega[E(m) + (m-1)K(m)]$. Note that $E(1) = 1$. The period is $4\int_0^X dx/p(x) = 4\int_0^{\sqrt{m}}(dx/dY)dY/p(Y)$ equal to $4K(m)/\omega$.

1.8.2 Second harmonic

In the case $n = 2$, we recover the dual harmonic system. The potential is $U = (1-\cos x) + (q/2)(1-\cos 2x)$ which may be written $U = 2Y^2[1 + 2q(1-Y^2)]$. The Hamiltonian value $2J\omega$ is found by substituting $x = X$ and $p = 0$, leading to $(1/2)p^2 - 2\omega^2(m - Y^2)[1 - 2mq + 2q(1-Y^2)] = 0$, which may solved for the momentum $p(Y)$.

The case $m = 1$ is straight forward. The integrand becomes $p(Y)(dx/dY) = 4\omega\sqrt{1-2qY^2}$. For the *bunch shortening mode*, $q > 0$. The range of integration is $Y = [0, 1]$ for $q \leq 1/2$, and $Y = [0, 1/\sqrt{2q}]$ for $q \geq 1/2$. The acceptance area is

$$A(q \leq 1/2) = 4\omega\left[2\sqrt{1-2q} + \sqrt{2}\arcsin[\sqrt{2q}]/\sqrt{q}\right] \quad \text{and} \quad A(q \geq 1/2) = 2\pi\sqrt{2}\,\omega/\sqrt{q}\,. \tag{1.44}$$

This area falls as q rises, because the well becomes progressively narrower. In particular $q = 1/2$ gives $A = 4\pi\omega$, which is smaller than for the fundamental alone.

For the *bunch lengthening mode*, $q < 0$, and $m = 1$, the range of integration is $Y = [0, 1]$. The acceptance area is

$$A(q < 0) = 4\omega\left[2\sqrt{1-2q} \; + \; \sqrt{2}\,\mathrm{arcsinh}[\sqrt{-2\mathrm{q}}]/\sqrt{-\mathrm{q}}\right] \tag{1.45}$$

This area increases as $|q|$ rises. In the specific case $q = -1/2$, gives $A = 8\omega[\sqrt{2} + \mathrm{arcsinh}(1)] \approx 18.3647\omega$, which is larger than for the fundamental alone. For this case, $q = -1/2$, the period may also be found as function of m; thus $\tau = 2\sqrt{2}K[(1+m)/2]/(\sqrt{m}\,\omega)$. Wolfram *Mathematica* also finds the area $A(m)$; it is composed of hypergeometric functions, but well approximated by a quadratic $A(m) \approx 16\,\omega \times [0.218537m \; + \; 0.89138m^2]$.

1.8.3 Third harmonic

In the case $n = 3$, the potential well is $U = (1 - \cos x) + (q/3)(1 - \cos 3x)$. Starting from the equivalence $\cos 3z = (\cos z)^3 - 3\cos z(\sin z)^2$ and repeated use of the half-angle formulae, the well is re-written $U(Y) = (2/3)Y^2(3 + 9q - 24Y^2 + 16qY^4)$ with $Y = \sin(x/2)$. The Hamiltonian value is found by setting $p = 0$ and $x = X$, leading to $(1/2)p^2 + \omega^2 U(Y) = \omega^2 U(\sqrt{m})$; and which is solved for $p(Y)$. The general case is intractable, but we may set $m = 1$ leading to simplification: $p(Y) = 2\omega\sqrt{1-Y^2}\sqrt{1 + (q/3)(1-4Y^2)^2}$. The integrand for the area integration with respect to

Y is $p(Y)(dx/dY) = 4\omega\sqrt{1+(q/3)(1-4Y^2)^2}$. For bunch lengthening with zero restoring force at the centre $x = 0$, $q = -1/3$ and the integral is

$$A = (16/19)\sqrt{2}\,\omega\,[E(-2) + 3K(-2)] \approx 14.3274\omega \text{ where } K(-2) = K(2/3)/\sqrt{3},\;\; E(-2) = E(2/3)\sqrt{3}\,.$$

$$K(-m) = K[m/(m+1)]/\sqrt{m+1} \quad \text{and} \quad E(-m) = E[m/(m+1)\sqrt{m+1}\,.$$

The corresponding integral for bunch shortening with $q = +1/3$ is computed numerically equal to $A \approx 17.2357\omega$.

1.8.4 Fourth harmonic

In the case $n = 4$, the potential well is $U = (1-\cos x) + (q/4)(1-\cos 4x)$. Starting from the equivalence $\cos 4z = (\cos z)^4 - 6(\cos z)^2(\sin z)^2 + (\sin z)^4$ and repeated use of the half-angle formulae, the well is re-written $U(Y) = 2Y^2[1 + 4q(1 - 5Y^2 + 8Y^4 - 4Y^6)]$ with $Y = \sin(x/2)$. The Hamiltonian is $(1/2)p^2 + \omega^2 U(Y) = \omega^2 U(\sqrt{m})$. We set $m = 1$ and solve for the momentum $p = 2\omega\sqrt{1-Y^2}\sqrt{1-4qY^2(1-2Y^2)^2}$. The integrand for the area integration is $p(Y)(dx/dY)$. For $q < 0$ and $0 < q \leq 1/4$ the integration range is $Y = [0, 1]$. For bunch shortening with $q = 1/4$ the acceptance is $A \approx 14.9937\,\omega$; and for bunch lengthening with $q = -1/4$, the acceptance is $A \approx 16.7683\,\omega$.

1.9 Sources for elliptic functions and their applications

1.9.1 Historical

Many of the great names in mathematics have worked on the elliptic functions and integrals; for example Wallis (1655), Newton, Euler and Gauss[13] (1797). However, the fundamental investigations are credited (in historical sequence) to A.M. Legendre[14] (before 1825), N.H. Abel[15] (1823) and C.G. Jacobi[16] (1829 and onwards). Kovacic[26] presents a thumbnail history of the naissance of Jacobi elliptic functions; but does not place adequate credit with Abel. It was Abel [7] who introduced the idea to define elliptic functions as the inversion of elliptic integrals; and it was Abel that showed they are doubly periodic functions. Kovacic also lists a number of modern applications in physics and engineering. The extension of the function argument to complex values was already known to Jacobi, but the consequences for the function definition as the inverse of a contour integral in the complex plane was not sorted out until the work of Neville[17] (circa 1943). The functions, properties and applications are still an active area of research; see Prasolov and Solovyev[31].

Ultimately, although interesting and rewarding in their own right, none of the historical sources consider motion in multi-harmonic potential wells; perhaps because at the end of the 19th century such a problem may have appeared as contrived as a "rotating catenary" seems to the modern-day reader. Instead, one must look to treatises on the properties of elliptic functions and integrals.

1.9.2 Mathematical properties

The standard mathematical sources for concise properties of elliptic functions and integrals are Abramowitz and Stegun[18] or the NIST-DLMF[19], Gradshetyn and Rhyzik[20], and other works such as Whittaker and Watson[21]. However, for a deeper exposition (told with clarity), I recommend the treatise by Neville[17]. A large variety of elliptic and related integrals is given in the handbook by Byrd and Friedmann[22]. The properties of elliptic functions as encoded in the Wolfram *Mathematica* program are listed exhaustively in

`https://functions.wolfram.com/EllipticFunctions/JacobiAmplitude/introductions/JacobiPQs/ShowAll.html`

[7] Niels Henrik Abel died prematurely in 1829 at the age of 26.

1.9.3 Applications in dynamics

We mention, in historical order, some texts on the application of elliptic functions and integrals. None of these were known to the author until after the derivation in Secs. 1.3-1.5 was formulated.

Greenhill

Published in 1892, Greenhill's book[23] was intended to extend the introduction of elliptic functions and integrals into the curriculum of the Cambridge Mathematical Tripos by exhibiting their practical importance in Applied Mathematics.

Chap. 1, *The Elliptic Functions*, uses the pendulum to introduce the Jacobi elliptic functions.

Chap. 3, *Geometrical and mechanical illustrations of the elliptic functions*, covers the following: shape of revolving chain (under gravity), catenary curves, internal stresses of a swinging body, motion of Watt's Governor[8] with $\ddot{X} \propto (a^2-X^2)(X^2-b^2)$, shape of elastic[9] beams under compression of their end points, ellipse rolling over a periodic curved surface.

Chap. 4, *Addition theorems for the elliptic functions*, gives an example in spherical trigonometry.

Chap. 6 *Elliptic integrals of the 2nd and 3rd kind*, gives two examples: confocal ellipses and hyperbolae, and Euler's pendulum (like a rocking cradle).

Chap. 7: *Elliptic integrals in general and their applications*, covers the following: shape of revolving chain, shape of elastic wire, spherical pendulum, spinning top, gyrostat[10], trajectory of projectile with cubic resistance law.

Chap. 8: *The double periodicity of the elliptic functions*, covers confocal orthogonal Cartesian ovals, confocal quadratic surfaces, and the electric potential between confocal ellipsoids.

Whittaker

Initially published in 1904, Whittaker's encyclopaedic book[24] is a thorough treatment of analytical dynamics, covering topics in Newtonian and Hamiltonian mechanics and celestial mechanics and in particular the three-body problem. Whittaker enumerates the complete class of one-body classical mechanical problems that are solvable in terms of elliptic functions and integrals.

Chap. 4, *The soluble problems of particle dynamics*, covers the following: simple pendulum, problem of motion under central forces, motion governed by $(\dot{Y})^2 = 4(Y-e_1)(Y-e_2)(Y-e_3)$, spherical pendulum (motion on the surface of a sphere), motion under two centres of force (e.g. gravitational).

Chap. 6, *The soluble problems of rigid bodies* covers the following: pendulum with a moment of inertia, motion of a rod on a rotating frame, motion of a disc with various constraints, motion of systems with two degrees of freedom, one cylinder rotating on another, hoop and ring, motion of a body about a fixed point under no forces, motion of a top, motion of gyrostat.

Nonlinear waves

In recent times, there is a resurgence of interest in Jacobian elliptic functions - for a particular application: the solution of non-linear wave equations. Kovacic[26] provides an enticing review that represents the "tip of the iceberg". An earlier and more lengthy review is given by Kivshar[27]. Other applications include the nonlinear Schrodinger and Klein-Gordon and Korteweg-de Vries equations. Entering the phrase "Jacobi elliptic function method for finding periodic wave solutions" into the search engine of *Academia.edu* yields over three hundred (relevant) results. The introduction of the paper by Mustafa[28] sets the stage quite well.

[8] This rotating mechanical device regulates the motion of a stationary steam engine, and is based on the counterbalance of centripetal and gravitational forces.

[9] This problem known as the *Elastica* is also that of the shape of a capillary surface and Bernoulli's Lintearia.

[10] A modified gyroscope consisting of a rotating wheel pivoted within a rigid case.

> In recent years, the exact solutions of nonlinear PDEs have been investigated by many workers who are interested in nonlinear physical phenomena. Many powerful methods have been presented, such as the homogeneous balance method [1], the hyperbolic tan-gent expansion method [2], the trial function method [3], the tanh method [4], the non-linear transformation method [5], inverse scattering transformation [6], Bäcklund trans-formation [7], Hirota's bilinear method [8], the generalized Riccati equation method [9],the Weierstrass elliptic function method [10], the theta function method [11], sine-cosine method [12] and the Jacobi elliptic function expansion method [13, 14] and so on.

The citations are to references in their paper, not to the bibliography of this work. The three references I have chosen, are simply those publications[28, 29, 30] I have actually read; they happen (mostly by chance) to have some relation to the method I have formulated in this note.

1.9.4 Applications to curves

Greenhill provides examples of catenary curves and lemniscates[11]. But there exists a more expansive landscape. The introduction[12] to the 1992 book by Prasolov and Soloviev[31] sets the scene:

> The theory of elliptic function and its geometric twin - the *theory of elliptic curves* - occupies one of the central places in mathematics having unified several of its branches. In spite of its senior age, the theory ... remains an alive and rapidly developing domain of mathematics ... In the past decade, elliptic functions and curves became the subject of close attention by experts in such nonclassical fields as algebraic topology and quantum field theory; quite recently, with the help of the elliptic curve theory, Fermat's Last Theorem was finally proved[13].

Requiring less mathematical stamina is the small compendium of historical results assembled by Snape[32].

[11] In algebraic geometry, a lemniscate is any of several figure-eight shaped curves.

[12] The introduction was updated for the 1997 English translation published by the American Mathematical Society.

[13] Andrew John Wiles' proof of Fermat's Last Theorem is a special case of the modularity theorem for elliptic curves. The proof was delivered in final form in 1995.

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[1] Roger Gram and Phil Morton: "Advantages of a Set of Second-Harmonic RF Cavities", SLAC-TN-67-30.

[2] P. Bramham, S. Hansen, A. Hofmann, E. Peschardt: "Active Landau Cavity on the 4th Harmonic of the RF Frequency", 1977, ISR Peformnace Report ISR-RF-TH-AH-cr. The notation TH-AH means Theoretical - Albert Hofmann.

`https://cds.cern.ch/record/1131181`

[3] Albert Hofmann and Steve Myers: "Beam Dynamics in a Double RF System", CERN-ISR-TH-RF/80-26. Presented at 11th Int. Conf. on High Energy Accelerators, Geneva, 1980. https://cds.cern.ch/record/879237

[4] Yoshikazu Miyahara: "Double RF System for Landau Damping and Robinson Instability", Proc. 1989 Particle Accelerator Conf. p. 921.

[5] Albert Hofmann: "Kinetic Theory", CERN 89-01. See in particular sec. 3.2 "Stationary distribution in double RF system".

Brief discussion of the short-bunch limit and the bunch lengthening condition $q = -1/n$ leads to the approximate equation of motion $\ddot{x} + \omega^2(1/6)(n^2 - 1)x^3 = 0$ and corresponding Hamiltonian.

[6] Elena Shaposhnikova: "Bunched beam transfer matrices in single and double rf systems", CERN-SL-94-19 (RF).

[7] T. Bohl, T. Linnecar, E. Shaposhnikova: "Beam Transfer Functions and Beam Stabilisation in Double RF System", CERN-AB-2005-026 or Proc. 2005 Particle Accelerator Conference, Knoxville, Tennessee, p. 2300.

[8] T. Argyropoulos and E. Shaposhnikova: "Loss of Landau Damping for Inductive Impedance in a Double RF System", Proc IPAC2013, Shanghai, China, p.1796 (article TUPWA040).

[9] E. Shaposhnikova and J. Esteban Muller: "Possible Beam Parameters In Double RF Operation of the CERN LHC", Proc. of IPAC2016, Busan, Korea, p. 2430 (article WEPMW008).

[10] Ivan Karpov, Theodoros Argyropoulos, and Elena Shaposhnikova: "Thresholds for loss of Landau damping in longitudinal plane", Physical Review Accelerators and Beams **24**, 011002 (2021).

[11] Kenneth R. Myer: "Jacobi Elliptic Functions from a Dynamical Systems Point of View" The American Mathematical Monthly , Oct., 2001, Vol. 108, No. 8 (Oct., 2001), pp. 729-737.

[12] S.Y. Lee: "Accelerator Physics", World Scientific Publishing, 1999.

[13] Carl Friedrich Gauss: "Disquisitiones Arithmeticae" (Arithmetic Studies), 1801.

[14] A.-M.Legendre: “Theory on Elliptic Functions”, Impremiere de Houzard Courcier, Paris, Book in three volumes: Vol.1,1825; Vol.2,1826; Vol.3,1830(in French).

[15] N.H. Abel: Research on elliptic functions, Journal on Theoretical and Applied Mathematics 3 (1828)160–190.(in French).

[16] C.G.J.Jacobi: “New Foundations of the Theory of Elliptic Functions”, Konigsberg, Borntraeger 1829, Reprinted by Cambridge University Press, 2012 (in Latin). The title is often abbreviated to“Fundamenta Nova”.

[17] Eric Harold Neville: “Jacobian Elliptic Functions”, Oxford Clarendon Press, 1944.

[18] Milton Abramowitz and Irene Stegun:“ Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables”, published in 1964 by the National Bureau of Standards.

[19] The NIST Digital Library of Mathematical Functions (https://dlmf.nist.gov) may be considered a digital version (https://dlmf.nist.gov/about/) of Abramowitz and Stegun, Chap. 22 presents the Jacobian Elliptic Functions (https://dlmf.nist.gov/22). Sec. 22.19 lists some physical applications (https://dlmf.nist.gov/22.19); principaly the variants of motion in the quartic potential.

[20] I. S. Gradshteyn and I. M. Ryzhik: “Table of Integrals, Series, and Products”. Originally published in 1943 (in Russian), it is now in its 8th edition of the English translation. https://doi.org/10.1016/C2010-0-64839-5

The 7th edition:

`https://archive.org/details/GradshteinI.S.RyzhikI.M.TablesOfIntegralsSeriesAndProducts`

[21] E. T. Whittaker and G. N. Watson: “A Course of Modern Analysis”, Cambridge University Press (1902).

[22] P.F. Byrd and M.D. Friedmann: “Handbook of Elliptic Integrals for Engineers and Scientists”, Springer-Verlag Berlin, Heidelberg & New York. 1st Edition 1954; 2nd Edition 1971.

An world-wide-web version is available from University of Georgia ScholarWorksUGA at https://esploro.libs.uga.edu/

[23] Alfred George Greenhill: “The Applications of Elliptic Functions”, MacMillan and Co. London and New York, 1892.

[24] Edmund Taylor Whittaker: “A Treatise on the Analytical Dynamics of Particles and Rigid Bodies”, 4th edition (1964), Cambridge University Press, Cambridge U.K.

[25] Alain J. Brizard: “A primer on elliptic functions with applications in classical mechanics”, arXiv:0711.4064v1

Brizard covers the following: Motion in quartic potential, simple pendulum, force-free asymmetric top. He also provides examples of the use of the Weierstrass elliptic functions: Motion in cubic potential, spherical pendulum, heavy symmetric top, and Korteweg-de Vries equation (soliton solutions).

[26] Ivana Kovacic, Livija Cveticanin, Miodrag Zukovic, Zvonko Rakaric: “Jacobi elliptic functions: A review of nonlinear oscillatory application problems” Journal of Sound and Vibration Vol 380 (2016) 1-36.

Kovacic *et al* provide the following examples: Hard and soft Duffing-type conservative oscillators $\ddot{x} + c_x + c_3x^3 = 0$; bistable Duffing oscillator; quadratic oscillator $\ddot{x} + c_1x + c_2x^2 = 0$; and driven non-linear oscillators (cubic and quadratic).

[27] Yuri Kivshar and Boris Malomed: "Dynamics of solitons in nearly integrable systems", Reviews of Modern Physics, Vol. 61, No. 4, (1989)

[28] Mustafa Inc and Mahmut Ergüt: "Periodic wave solutions for the generalized shallow water wave equation by the improved Jacobi elliptic function method" Applied Mathematics E-Notes, 5(2005), 89-96. Available free at mirror sites of http://www.math.nthu.edu.tw/-amen

`https://www.researchgate.net/publication/266310159`

Like Meyer and the present author, they make use of the Jacobi Eqs. 1.5.

[29] E.J. Parkes, B.R. Duffy, P.C. Abbott: "The Jacobi elliptic-function method for finding periodic-wave solutions to nonlinear evolution equations", Physics Letters A 295 (2002) 280–286.

`https://www.academia.edu/26167601`

See also the large number of related papers associated with the hyperlink.

[30] Engui Fan: "Multiple traveling wave solutions of nonlinear evolution equations using a unified algebraic method", J. Phys. A: Math. Gen. 35 (2002) 6853–6872

[31] Viktor Prasolov and Yuri Solovyev: "Elliptic functions and Elliptic Integrals" Translations of Mathematical Monographs Vol 170, American Mathematical Society, 1991.

The theory of elliptic curves is exemplified by the following: Geometry of cubic curves, arcs of curves, construction of polygons[14]; cubic, quartic and quintic algebraic equations, etc.

[32] Jamie Snape: "Applications of Elliptic Functions In Classical and Algebraic Geometry" Dissertation submitted for the degree of Master of Mathematics at the University of Durham.

Snape provides a variety of geometric examples: Halphen's circles and Poncolet's polygons, Fagnano's ellipses, Bernoulli's lemniscate[15], Seiffert's spherical spiral, and the nine circles theorem. He also gives two examples drawn from dynamics: Simple pendulum, spherical pendulum

[33] J. V. Armitage: "Elliptic Functions" Department of Mathematical Sciences University of Durham and the late W. F. Eberlein, Cambridge University Press (2006).

[14] Related to Galois theory.

[15] A special case of Casinni's ovals; introduced by astronomer Giovanni Domenico Cassini to describe the orbits of the planets more precisely than with ellipses.