Weakly Damped Anharmonic Oscillator

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Contents

Chapter 1

Damped Anharmonic Oscillator

1.1 Introduction

Let t, x and $\dot{x}$ be time, position and motion; and dots denote time derivatives ($dx/dt \equiv \dot{x}$). The damped anharmonic oscillator has the equation of motion (EOM):

$$\ddot{x} \; + \delta{\times}\dot{x} \; + \; \alpha{\times}x \; + \; \beta{\times}x^3 \; = \; 0\,. \tag{1.1}$$

Here $\delta \geq 0$ is responsible for damping. The physical unit of δ is Hz. The constants $\alpha \geq 0$ while β may be positive or negative or zero. The units of α and β are Hz^2. The case $\beta < 0$ is known as the soft (or softening) oscillator, while the case $\beta > 0$ is called the hard (or hardening) oscillator. Although the two cases differ only by the polarity of β, they are as alike as chalk and cheese. For the hard/soft case, an increase in amplitude is opposed/assisted by the increasing/decreasing stiffness. So there is an intrinsic self-stabilisation for the hard and de-stabilization for the soft oscillator.

The harmonically driven counterpart to Eq. (1.1), known as Duffing's equation[1, 2], has an extensive literature. In contrast, publications on the damped, free-oscillations (when $\beta \neq 0$) are very limited. There are two reasons for this disparity. Primarily, driven oscillators (i.e. resonators) are of engineering significance. Resonators convert small excitations into large responses, and can be beneficial or damaging. Intentional resonators such as musical instruments are an example of the former. Un-intended resonances in mechanical structures, for example beams and bridges, driven by vibrations (or unforeseen sources such as winds) can lead to damage or destruction of equipment. Contrastingly, undriven damped oscillations simply die away. The second reason is mathematical difficulty: driven oscillators succumb to simple Fourier series whereas free oscillations of (simple) non-linear oscillators demand the use of elliptic functions[3, 4].

1.1.1 Literature survey

The references herein contain a small but notable subset of the literature for the damped anharmonic oscillator. We begin with Ludeke and Wagner[5] who found approximations for the variation of frequency with amplitude and the (time-dependent) frequency depression from damping. They consider the case $0 < \delta \leq \sqrt{\alpha}$ and $|\beta\, x^2| \ll \alpha$. They assume simple exponential damping and employ harmonic balance of Fourier components. The frequency and damping have two adjustable parameters which are the solution of nonlinear simultaneous equations. They appear also to have started the fashion of calling the undriven anharmonic oscillator the Duffing oscillator - despite the century of work by A.M. Legendre, N.H. Abel and C.G.J. Jacobi prior to Duffing.

We continue with the works of Soudack and Barkham, who attempted to treat the general form $\ddot{x} + \alpha x + \beta x^3 + F(x, \dot{x}, t) = 0$ where F is a perturbation. To describe their approach we have to introduce a few simple results for the lossless hardening oscillator. The elliptic cosine $\mathrm{cn}(\phi, m) \times A$ with constant amplitude A has the properties:

$$x(0) = A, \dot{x}(0) = 0, \quad x(t) = A\,\mathrm{cn}(\omega t, m), \quad \dot{x} = -\omega A\,\mathrm{dn}(..)\,\mathrm{sn}(..), \quad \omega^2 = \alpha + \beta A^2, \quad m = \beta A^2/(2\omega^2)\ .$$

Initially[6, 7] they introduced solutions of the form $x = A(t)\,\mathrm{cn}[\omega(0)t + \Theta(t), m(A)]$. The Jacobi m-parameter keeps lock step with the amplitude, while the frequency ω is held constant at the initial value; i.e. $\dot{\omega} \times t$ is discarded in comparison to ω. Θ accounts for the residual phase variation. Although $m[A(t)]$ is time varying, they hold it constant when performing time derivatives. The functional forms for A, Θ are unknown at the outset. For compactness we suppress the arguments of the elliptic functions. The trial is substituted into the EOM, yielding:

$$\left[-A(\alpha + \beta A^2)\,\mathrm{cn}^2\,\dot{\Theta}(2\omega + \dot{\Theta})/\omega^2 + \ddot{A}\right]\mathrm{cn}\ - [2\dot{A}(\omega + \dot{\Theta}) + A\,\ddot{\Theta}]\mathrm{dn}\,\mathrm{sn}\ + F = 0\ . \tag{1.2}$$

$F(x, \dot{x}, t)$ is a known function. Balance of the elliptic functions is then used to generate simultaneous differential equations for A and Θ. Later[8] they realized the necessity of the time varying frequency, otherwise the phase is compromised. Hence they introduce the phase $\phi = \omega(t) \times t + \Theta(t)$ and set $\ddot{\omega} = 0$; and went on to apply the method to the damped[9] non-linear oscillator. (From these works later emerged the elliptic-Krylov–Bogoliubov method.) However, when forming derivatives with respect to A and Θ they fail to take the derivative with respect to the second argument of the elliptic cosine. This is a fundamental error, and no amount of tinkering can save the situation.

In a more recent review (2011) Yabuno[10] explores some global properties of the motion in phase space (phasen-raume) and Cveticanin[11] gives an erroneous solution of Eq. (1.1); the method fails because it does not recognise that amplitude and frequency are inextricably linked; the matter is discussed in Appendix A.

Johannessen[12, 13, 14] makes a significant advance in the use of elliptic functions, which the present author carries forward. He emphasizes the difficulty of accurately predicting the phase; and explicitly states that when dealing with time derivatives of $\mathrm{cn}[\phi(t), m(t)]$ we must take derivatives with respect to both arguments of the elliptic function. For the pendulum oscillator[12], he takes the trial function $x(t) = 2 \arcsin\left\{\sqrt{m(t)}\,\mathrm{sn}[\phi(t), m(t)]\right\}$. This is substituted in the equation of motion $\ddot{x} + \delta\dot{x} + \alpha \sin(x) = 0$, yielding simultaneous differential equations for m and ϕ; which are solved by inspection. For the hardening oscillator[13], he uses the trial form $x(t) = A(t)\,\mathrm{cn}[\phi(t), m(t)]$ where the functions A, m, ϕ are initially unknown. Fortunately, the instantaneous value of $\dot{\phi} = \omega(A)$ is known. He assumes $A(t)$ decays exponentially. The trial is substituted in the EOM 1.1, and the coefficients of $\dot{x}, x, x^3$ inspected to give simultaneous equations for the unknowns; which are solved by inspection. His approximate solution for (i) the pendulum is good up to amplitude $x_0 = \pi/2$; for (ii) the hardening oscillator is good up to amplitude $x_0 = 0.5$ when $\beta = \alpha = 1$; and for (iii) the softening oscillator[14] is good up to amplitude $\dot{x}_0 = 0.5$ when $-\beta = \alpha = 1$. The present authors results herein have a similar range of validity for the phase, corresponding to the Jacobi parameters $m = [\frac{1}{2}, \frac{1}{8}, \frac{1}{4}]$ respectively. And herein, the amplitude variation is predictable for the complete range of m; and the cubic oscillator is treated at large amplitude with $m = \frac{1}{2}$ and $x_0 = 2$.

Euler[15] *et al* find that a solution of Eq. 1.1 in terms of elliptic functions is possible under the special condition $2\delta^2 = 9\alpha$ and $\beta = 1$. The solution is not given explicitly, but they provide sufficient detail to construct the function. They also find a constant of motion $I = \exp(\frac{4}{3}\delta t)[(\dot{x}+x\delta/3)^2+\frac{1}{2}x^4]$. Panayotounkanos[16] *et al*, using the properties of Abel's equation of the second kind, prove that there is no exact solution of the damped pure cubic oscillator (PCO) in terms of standard functions - including elliptic functions. Panayotounakos[17] *et al* propose a methodology to construct exact solutions for the damped PCO ($\alpha = 0$), based on successive solutions of nonlinear transcendental equations. This is an impressive result; but unlike approximate results based on elliptic functions, does not build intuition as to oscillator behaviour.

1.1.2 Balance methods

Suppose there is an equation of the form $A(\dots) = B(\dots)$ that contains or generates periodic motion. In the method of *harmonic balance* (HB), the trial solution is a Fourier series; and the components either side of the equality must balance. HB has been used since the time of Duffing (1918) and earlier. In the method of *elliptic balance* (EB), the trial is an elliptic function; and the terms must balance. HB is suited to the harmonically-forced nonlinear oscillator. EB is suited to free oscillation of the nonlinear oscillator. The methods may be applied either to the EOM or the energy equation $\text{EE} = \int^t \text{EOM}[\dots]\dot{x}dt$. Working with the EOM is more straight forward. Working from the EE is more complicated but will uncover a constant of motion. If the trial is not an exact solution, there will be a residual. In this case, the EE is superior because it constrains the components to respect energy conservation.

1.1.3 Road map

There are two key properties of the damped anharmonic oscillator: (1) the frequency depends on amplitude, and (2) damping induces a frequency shift; and both will be addressed. We shall explore the motion governed by the EOM (1.1) by way of a series of progressively more challenging examples. In each case, the problem is addressed in three steps: (1) the conservative motion when $\delta = 0$; followed by (2) the damped motion; and finally (3) the frequency and phase advance. The approach is perturbative; and we restrict to the regime where the damping rate is much less than the oscillation frequency.

In Sec. 1.2 we introduce the elliptic functions and discuss their derivative with respect to the Jacobi-parameter. The Jacobi elliptic functions cn, dn, sn have two arguments: the phase ϕ and the elliptic modulus $k^2 = m$. k is related to the amplitude of oscillation, and m is called the (Jacobi) parameter. The $\text{cn}(\phi, m)$ and $\text{sn}(\phi, m)$ are analogues of the $\cos(\phi)$ and $\sin(\phi)$ trigonometric functions. We shall speak loosely of $\dot{\phi}$ as being the angular frequency. However, the nominal frequency is $\nu = \dot{\phi}/[4K(m)]$. Moreover, the elliptic functions have an infinite set of Fourier harmonics at multiples of ν; and the period of the fundamental changes with the amplitude. Further, when damping is present, both $\dot{\phi}$ and $m(t)$ are slowly time varying. We assume variations sufficiently slow that ν is effectively constant over at least an oscillation cycle. We shall find that most of the change in ν is due to $m(t)$, and that changes in $\dot{\phi}$ are "fine tuning".

Section 1.3, the linear oscillator, introduces the method of energy balance; and demonstrates that damping is associated with a frequency shift. This property is also shared by non-linear oscillators. However, for the latter, the damping rate and frequency shift are time dependent. Sec. 1.4, the pure cubic oscillator, is our first application of the elliptic functions; the working is simple because the Jacobi parameter is constant. Sec. 1.5, the pendulum oscillator, introduces an important side effect[12] when the parameter, $m(t)$, is time dependent. Sections 1.6 and 1.8 reprise established results for free oscillations of the lossless anharmonic oscillator; while Secs. 1.6.1 and 1.8.1 present new results for the damped motion. Section 1.7 reprises the solution of Euler, and comments on the same. An obvious approach[12, 13] is to make the Jacobi parameter (m) time-dependent; and occurred to the present author independently. This work differs from Johannessen in two respects. First, we derive the damping rate from the energy equation; while Johannessen works from the differential equation. Second, we use the known analytic expression[4] for the derivative $\partial/\partial m J(\phi, m)$ whereas Johannessen uses approximate forms. The EE allows to find the damping rate for large amplitudes without knowing the details of the motion.

We are guided by three principles. The energy equation guides the damping rate and expression for dm/dt. The quasi-static assumption provides the relation between amplitude r and m. The EOM determines the frequency $\dot{\phi}(t)$, which must satisfy the quasi-static value in the limits $t \to 0$ and $t \to \infty$. With damping, the frequency is shifted to a lower frequency; thereby ensuring mutual consistency of all three functions $m(t), r(t), \dot{\phi}(t)$. In this approach, unlike Johannessen, we avoid having to solve simultaneous differential equations. Typically, the trial solution does not equate the EOM exactly to zero; and the residual could (in principle) quantify the error.

In section 1.9, we find what range of values δ are consistent with the mathematical formalism; and in the final section 1.10, we make some observations and speculations on errors, improvements and extensions.

1.2 Elliptic functions and integrals

The solution of Eq. 1.1 (with $\delta = 0$) relies on the properties of Jacobian elliptic functions. There are four sets of 3 functions, giving a total of twelve. However, the properties of all may be derived from the principal set: cn, sn, dn. Two key identities are: $\mathrm{cn}^2 + \mathrm{sn}^2 = 1$ are $\mathrm{dn}^2 + m\,\mathrm{sn}^2 = 1$. Those needing a primer may turn to TRI-BN-24-04[20] and the extensive references therein.
Let $F(\phi, m)$ and $K(m) \equiv F(\pi/2, m)$ be the incomplete and complete elliptic integrals of the 1st kind, respectively. Let $E(\phi, m)$ and $E(m) \equiv E(\pi/2, m)$ be the incomplete and complete elliptic integrals of the 2nd kind, respectively. Let $A(\phi, m)$ be the Jacobi amplitude; such that $A[n \times K(m), m] = n \times \pi/2$ for integer n. Then $F[A(\phi, m), m] = \phi$.

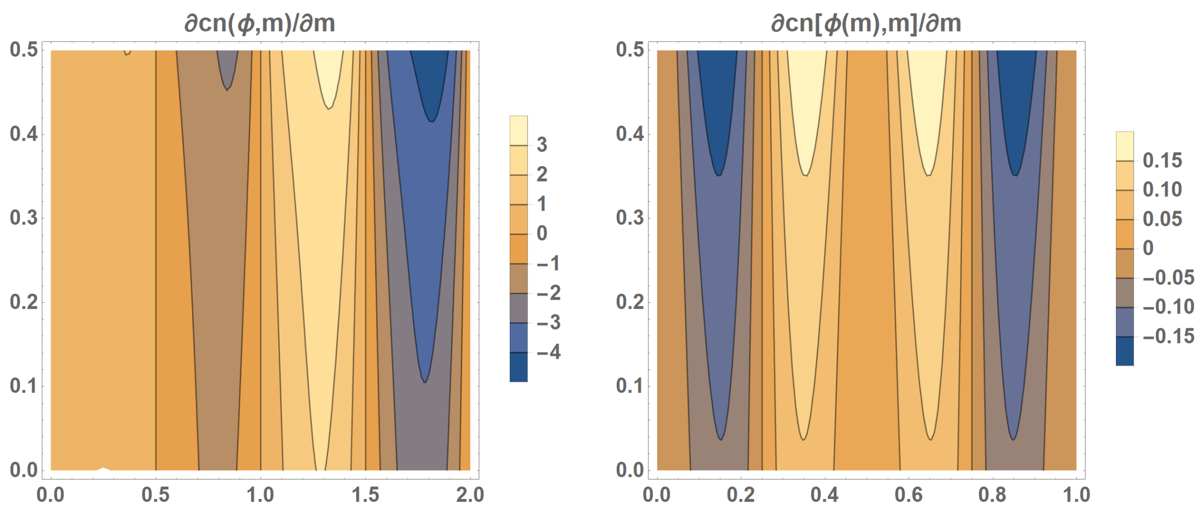

Figure 1.1: Left: contour plot of $\{(\partial/\partial m)\,\mathrm{cn}[\phi\, 4K(n), m]\}\, n \to m$. Abscissa ϕ and ordinate m. Right: contour plot of $(\partial/\partial m)\,\mathrm{cn}[\phi\, 4K(m, m]$.

1.2.1 Parameter derivative of elliptic function

In the following sections, we shall need to form the derivative of the elliptic functions with respect to the Jacobi parameter m. Formulae for the 1st and 2nd derivatives are given in Refs.[4, 19]. For brevity, we omit the arguments of the elliptic functions. Let $\mathrm{cd} \equiv \mathrm{cn}/\mathrm{dn}$. For example, the first derivative of $\mathrm{cn}(\phi, m)$ is:

$$2m(1-m) \times \frac{\partial}{\partial m}\mathrm{cn}(\phi, m) = \mathrm{dn}\,\mathrm{sn}\left[(m-1)\phi + E[A(\phi, m), m] - m\,\mathrm{cd}\,\mathrm{sn}\right] . \tag{1.3}$$

The second derivative is more complicated and contains the term $F[A(\phi, m), m]$, which is simply equal to ϕ. The first derivative is sketched in Fig. 1.1-Left as "height" versus increasing ϕ for a variety of m. Perhaps surprising, the function is not periodic, not small and it increases with ϕ. For the purpose of the next sentence only, let δm mean a small change in m. The explanation is that the period of ϕ varies with m. Hence when we construct $[\mathrm{cn}(\phi, m + \delta m) - \mathrm{cn}(\phi, m)]/\delta m$ there is a cumulative phase mismatch due to the differing periods. In contrast, if we make the phase track with the period, and form the derivative $(\partial/\partial m)\mathrm{cn}[\phi\, 4K(m), m]$ we find a function that is small and periodic; as shown in Fig 1.1-Right. To facilitate making plots with similar scales, for the left-side figure we formed the derivative Eq. (1.3) and then made the replacement $\phi \to \phi \times 4K(m)$.

The frequency is $\nu = \dot{\phi}/[4K(m)]$. The majority of the frequency change, as the oscillation decays, comes from the denominator $K(m)$. This variation, which does not appear in the first argument (ϕ) of the elliptic function, is hidden from us (inside the internal mechanics of the Jacobi function); but becomes exposed when we form the derivative of the elliptic function with respect to the second argument.

The $E[A(\phi, m), m]$ appearing in Eq. 1.3 is a sophisticated function not amenable to manipulation. However, $E[A]$ is a function linear in ϕ with a high-frequency ripple super-imposed. Therefore, in the analysis, we shall replace $E[A]$ by the excellent approximation

$$E[A(\phi, m), m] \;\approx\; \phi \times (1 - m/2)(1 - m^4)^{1/4} \; . \tag{1.4}$$

In fact over the range $m = [0, \frac{1}{2}]$ the relation $E[A(\phi, m), m] \approx \phi \times (1 - m/2)$ is perfectly adequate.

1.3 Simple Harmonic Oscillator

For the linear oscillator $\beta \equiv 0$. We start by considering the lossless case $\delta = 0$. We multiply Eq. (1.1) by $\dot{x}$ and integrate over time, leading to the energy equation:

$$\frac{1}{2}\dot{x}^2 + \frac{1}{2}\alpha x^2 = \alpha J \tag{1.5}$$

where J is a constant of motion. We substitute into Eq. (1.5) the trial solution $x(t) = \sqrt{r}\cos\phi$ where r is an adjustable parameter and $\phi(t)$ a simple function. The result is

$$r[\alpha - (\dot{\phi})^2](\cos\phi)^2 \;+\; r(\dot{\phi})^2 \;=\; 2\alpha J \; .$$

The method of harmonic balance for the coefficients of $\cos^2$ and $\cos^0$ leads to two simultaneous equations with solution $\dot{\phi} = \pm\sqrt{\alpha}$ and $J = r/2$ provided $\alpha > 0, r > 0$. Evidently, $\phi(t) = \phi_0 \pm \sqrt{\alpha}t$. We use the subscript notation X_0 to indicate the initial value $X(t\!=\!0)$. Depending on the initial phase ϕ_0 there are two *steady state* solutions.

$$\begin{array}{rlll} \phi_0 \to 0 & \text{implies} & x = \sqrt{r}\cos(\sqrt{\alpha}t) & \text{valid for } (x_0 \neq 0, \dot{x}_0 = 0) \; . \\ \phi_0 \to \pm\pi/2 & \text{implies} & x = \sqrt{r}\sin(\sqrt{\alpha}t) & \text{valid for } (x_0 = 0, \dot{x}_0 \neq 0) \; . \end{array}$$

General initial conditions are satisfied by a linear super-position of these principal solutions.

1.3.1 Damping

Restoring δ to a non-zero value will result in damped oscillations, and a changing energy value J. We multiply Eq. (1.1) by $\dot{x}$ and integrate over time, leading to the energy equation:

$$\alpha J(t) \equiv \frac{1}{2}\dot{x}^2 + \frac{1}{2}\alpha x^2 = -\delta \int^t (\dot{x})^2 ds \quad \text{or} \quad \alpha \dot{J} = -\delta(\dot{x})^2 \; . \tag{1.6}$$

Substituting the steady state solution into Eq. (1.6) yields $(\alpha/2)\dot{r} = -r\alpha\delta(\sin\sqrt{\alpha}t)^2$. Now, the damping rate is considered to be slowly varying compared with the oscillation frequency. Hence, by integrating over a single period we find $\langle \dot{r} \rangle = -\delta r$ where $\langle \ldots \rangle$ denotes cycle average. This has solution $r(t) = r_0 \exp(-\delta t)$ where r_0 is the initial value. The revised solution is

$$x = \exp(-\delta t/2)\sqrt{r_0}\cos\phi(t) \; , \quad \dot{x} = \exp(-\delta t/2)\sqrt{\alpha}\sqrt{r_0}\sin\phi(t) \; - (\delta/2)x \quad \text{and} \quad \phi(t) = \phi_0 + \sqrt{\alpha}t$$

with ϕ_0 equal zero or $\pi/2$.

1.3.2 Frequency shift and phase

The revised solution has two undesirable properties. First, substitution into the EOM $\ddot{x}+\delta\dot{x}+\alpha x = 0$ shows the equation to be violated; there is a residual $x\,\delta \neq 0$. Second, the new solutions cease to be principal functions. These properties are shared by damped non-linear oscillators. The residual can be made zero by allowing a frequency shift. We substitute $x = \exp(-\delta t/2)\sqrt{r_0}\cos[\phi(t)]$ into the EOM, leading to

$$\exp(-\delta t/2)\sqrt{r_0}\left\{[-4\alpha + \delta^2 + 4(\dot{\phi})^2]\cos\phi \; + 4\ddot{\phi} \times \sin\phi\right\} = 0\,. \tag{1.7}$$

Let $\alpha' \equiv \alpha - \delta^2/4$. Harmonic balance implies $\dot{\phi} = \pm\sqrt{\alpha'}$ and $\ddot{\phi} = 0$; with solution $\phi(t) = \phi_0 \pm t\sqrt{\alpha'}$. The frequency shift is $\epsilon = \sqrt{\alpha'} - \sqrt{\alpha} \approx -\delta^2/(8\sqrt{\alpha})$.

1.3.3 Principal functions

Linear combinations of the new solutions are the new principal functions. Let $x(t) = e^{-\delta t/2}\times (A\cos\phi + B\sin\phi)$ where $\phi_0 = 0$. Consider two cases. The condition $[x(0) = x_0,\, \dot{x}(0) = 0]$ has solution $A = x_0$, $B = \delta x_0/(2\dot{\phi})$. The condition $[x(0) = 0,\, \dot{x}(0) = \dot{x}_0]$ has solution $A = 0$, $B = \dot{x}_0/\dot{\phi}$. Forming the Wronskian determinant $W = x_1\dot{x}_2 - x_2\dot{x}_1$ of these two solutions gives $W = e^{-\delta t/2}x_0\dot{x}_0$. W is non-zero, confirming the functions are linear independent.

For the case of non-linear oscillators, linear combinations of the principal functions (such as cn,sd or sn,cd) do not satisfy the equation of motion because of cross-product terms arising in the cubic term βx^3. However, general initial conditions can be satisfied by choice of the phase ϕ_0. We demonstrate for the linear case ($\beta = 0$). We take the solution $x = \exp(-\delta t/2)\sqrt{r}\cos\phi$ and $\phi(t) = \phi_0 \pm t\sqrt{\alpha'}$ where r, ϕ_0 are to be determined. The condition $[x(0) = x_0,\, \dot{x}(0) = 0]$ has solution $r = x_0^2(\alpha/\alpha')$, $\phi_0 = \pm\arccos\sqrt{\alpha'/\alpha}$. The condition $[x(0) = 0,\, \dot{x}(0) = \dot{x}_0]$ has solution $r = \dot{x}_0^2/\alpha'$, $\phi_0 = \pm\pi/2$.

1.3.4 Critical damping

In the case of critical damping, the oscillator returns to the equilibrium position as quickly as possible, without oscillating, and passes it once at most. For the linear oscillator the critical condition is $\delta^2 = 4\alpha$. The sine and cosine solutions cease to be valid, and are replaced by $x = \exp(-\sqrt{\alpha}t)[1 + t\sqrt{\alpha}]x_0$ and $x = \exp(-\sqrt{\alpha}t)t\,\dot{x}_0$.

1.4 Pure Cubic Oscillator

The equation of motion (EOM) for the undamped cubic oscillator is $\ddot{x} + \beta x^3 = 0$, with $\beta > 0$. There is no linear restoring force, and so $\alpha = 0$. The free oscillations may be written in terms of the Jacobi $\mathrm{cn}(\phi, m)$ function. We multiply Eq. (1.1) by $\dot{x}$ and integrate over time, leading to the energy equation:

$$\frac{1}{2}\dot{x}^2 + \frac{1}{4}\beta x^4 = J \tag{1.8}$$

where J is a constant of motion. We substitute into Eq. (1.8) the trial solution $x(t) = \sqrt{r}\mathrm{cn}[\phi(t), m]$ and $\dot{x} = -\sqrt{r}\,\mathrm{dn}[\phi(t), m]\,\mathrm{sn}[\phi(t), m]\dot{\phi}$ where the adjustable parameters r, m and phase function $\phi(t)$ must be consistent with one another. In the absence of damping, the frequency $\dot{\phi}$ is constant. The energy equation becomes:

$$[r^2\beta - 2mr(\dot{\phi})^2][\mathrm{cn}(\phi, m)]^4 \; + \; 2r(2m-1)[\dot{\phi}\,\mathrm{cn}(\phi, m)]^2 \; - \; 2mr(\dot{\phi})^2 \; = \; 4J\,. \tag{1.9}$$

The principle of elliptic balance dictates that the coefficients of cn^0, cn^2, cn^4 must each be zero, leading to three simultaneous equations; with solution $\dot{\phi} = \pm\sqrt{\beta\, r}$ and $J = \beta r^2/4$ and $m = 1/2$.

Thus $\phi(t) = \phi_0 \pm t\sqrt{\beta r}$. Initial phase $\phi_0 = 0$ gives $x = \sqrt{r}\,\mathrm{cn}$; while $\phi_0 = \pm K[m]$ gives the other principal function $x = -\sqrt{r/2}\,\mathrm{sd}(\phi, \frac{1}{2})$. Note, the Jacobi m-parameter is a defined constant. Thus the effect of damping will show as variation of the amplitude $r \to r(t)$ and a corresponding change of the frequency $\dot{\phi} \to \sqrt{\beta\, r(t)}$.

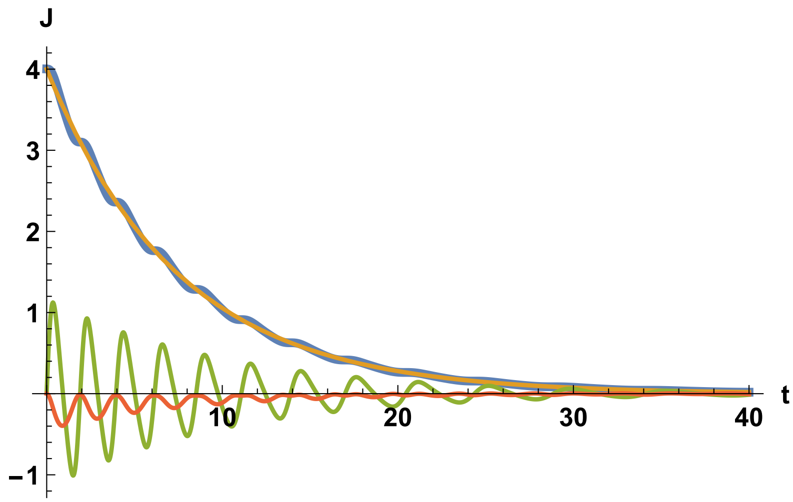

Figure 1.2: Energy of damped cubic oscillator versus time. Blue: numerical solution J_n. Gold: analytic variation J_r. Olive: the magnified difference $10 \times (J_n - J_r)$. Coral: the magnified difference $100 \times (J_n - J_a)$ where J_a is calculated from the analytic solution for $(x, \dot{x})$. Here $x_0 = 2, \dot{x}_0 = -\frac{1}{15}$.

1.4.1 Damping & variation r(t)

Restoring δ to a non-zero value will result in damped oscillations, and a changing energy value $J(t) = -\delta \int^t (\dot{x})^2 ds$ or $\dot{J} = -\delta(\dot{x})^2$. Substituting the free oscillation, and writing J in terms of r, leads to

$$\frac{1}{4}\beta\frac{d}{dt}r(t)^2 = \frac{1}{2}\beta r\dot{r} \;=\; -\delta\beta r^2[\mathrm{dn}(\phi, 1/2)\,\mathrm{sn}(\phi, 1/2)]^2\,. \tag{1.10}$$

We integrate over one period to find the average rate of change; the cycle average denoted $\langle \ldots \rangle$. We assume that $\dot{\phi}$ is constant during one period. The period is $4K(\frac{1}{2})$ and the integral $\int_0^{4K} (\mathrm{dn} \times \mathrm{sn})^2 dt = (4/3)K(\frac{1}{2})$. Hence $\frac{1}{2}\langle\dot{r}\rangle = -\frac{1}{3}\delta\, r$, with solution $r(t) = r_0 \exp[-\frac{2}{3}\delta t]$ where r_0 is the initial value.

These results are confirmed in Fig. 1.2 which compares the energy J_n calculated by numerical solution of the EOM for $(x, \dot{x})$ and substitution in Eq. 1.8 versus $J_r = \beta r^2/4 = \exp(-4\delta t/3)\beta\, r_0^2/4$. The quantity J_a is the energy calculated from the approximate analytic expressions for $(x, \dot{x})$. The values chosen are $\beta = 1, \delta = \frac{1}{10}$ and the large initial amplitude $r_0 = 4, x(0) = 2, \dot{x}(0) = \dot{r}(0)/(2\sqrt{r_0}) = -\frac{1}{15}$. The values J_n, J_r are in good agreement; and the values J_n, J_a in excellent agreement.

1.4.2 Differential equation

Suppose that instead of the energy equation, we chose to work directly with the differential equation. Substituting the trial $x(t) = \sqrt{r(t)}\mathrm{cn}[\phi(t), \frac{1}{2}]$ into the EOM leads to the condition:

$$r^2[\beta r - (\dot{\phi})^2]\,\mathrm{cn}^3 \;- r[\dot{\phi}\,\dot{r} + r(\dot{\phi}\,\delta + \ddot{\phi})]\,\mathrm{dn}\,\mathrm{sn} \;+ (1/4)[-(\dot{r})^2 + 2r(\dot{r}\delta + \ddot{r})]\,\mathrm{cn} \;= 0\,. \tag{1.11}$$

For brevity we have omitted the arguments of the elliptic functions. The EB principle implies three simultaneous equations, one each for the coefficients of cn, cn^3 and dn×sn. There is no exact solution, because the trial is not exactly the correct function. However, there are *two* inexact solutions. In no particular order, the first[1] solution is: $r(t) = \exp(-2t\delta)r_0$ and $\dot\phi = \sqrt{\beta\, r(t)}$; and the second[2] solution is $r(t) = \exp(-2t\delta/3)r_0$ and $\dot\phi = \sqrt{\beta\, r(t)}$. It is not straight forward to determine which one is the more correct. In each case, the EOM has a small residual. The first has residual $2\delta\sqrt{\beta}r(t)\,\mathrm{dn}\,\mathrm{sn} \approx -2\delta\dot{x}(t)$. The second has residual $-(2/9)\delta^2\, x(t)$. Although the second is order δ smaller, $x_0 = \sqrt{r_0}$ while $\dot{x}_0 = 0$; so it is no clear which is actually the smaller quantity. However, we have the benefit of the energy equation which tells us the correct variation is $r(t) = \exp(-2t\delta/3)r_0$.

1.4.3 Frequency shift and phase

We may now calculate the frequency and phase advance:

$$\dot\phi = \pm e^{-\delta t/3}\sqrt{\beta\, r_0} \quad \text{and} \quad \phi(t) = \phi_0 \pm \sqrt{\beta}\int_0^t \sqrt{r(u)}du = 3[1-\exp(-\delta t/3)]\sqrt{\beta\, r_0}/\delta\ . \tag{1.12}$$

In the limit $\delta \to 0$, we recover the phase advance at constant frequency: $\phi(t) = \pm\sqrt{\beta\, r_0}\, t$.

The trial solution has become $x(t) = \sqrt{r(t)}\,\mathrm{cn}[\phi(t), \frac{1}{2}]$ where r and ϕ are known functions. We may now test whether this trial satisfies the EOM. Substituting $x, \dot{x}, \ddot{x}$ and $r, \dot{r}, \ddot{r}$ and $\dot\phi, \ddot\phi$ into the EOM, we find the residual $-\frac{2}{9}\delta^2\sqrt{r_0}\exp(-\delta t/3)\,\mathrm{cn}[\phi(t), \frac{1}{2}] \neq 0$. Although non-zero, the residual is a small quantity (of order δ^2) and decays exponentially.

If we substitute into the EOM, the found functions $x, \dot{x}, \ddot{x}$ and $\dot{r}, \ddot{r}$ but hold $\dot\phi$ and $\ddot\phi$ as unknown free variables, the residual is

$$\sqrt{r(t)}\left\{[\beta\, r(t) - (\dot\phi)^2]\,\mathrm{cn}^3 \ - \ (2/9)\delta^2\,\mathrm{cn} \ - \ (1/3)(\dot\phi\delta + 3\ddot\phi)\,\mathrm{dn}\,\mathrm{sn}\right\} = 0\ .$$

Equating the coefficients of cn^3 and $\mathrm{dn}\times\mathrm{sn}$ each to zero, leads to two simultaneous equations for $\dot\phi$; and which both have the solution Eq. 1.12. However, the coefficient of cn cannot be made zero unless $\delta = 0$. This implies the exponentially damped frequency is a good approximation, but not an exact behaviour. Let X be the exact solution of the EOM and y the error such that $X = x + y$. A crude estimate is y of order $(4/27)\delta^2\, X(t)/(\beta\sqrt{r(t)}) \simeq (4/27)\delta^2\sqrt{r(t)}/\beta$. The comparison of numerical and analytic solutions in Fig. 1.3 confirms that the error is of order δ^2 smaller than X. The PCO parameters are $\beta = 1$ and $\delta = \frac{1}{10}$.

1.4.4 Principal functions

The effect of damping is that $x = \sqrt{r(t)}\,\mathrm{cn}[\phi(t), \frac{1}{2}]$ with $\phi_0 = 0$ and $\phi_0 = \pm K(\frac{1}{2})$ are nolonger orthogonal functions. The properties of the principal functions can be restored by adjusting ϕ_0. The time derivative is $\dot{x} = -\sqrt{r(t)}\mathrm{dn}(\phi, \frac{1}{2})\mathrm{sn}(\phi, \frac{1}{2})\dot\phi + \frac{1}{2}(\dot{r}/\sqrt{r})\,\mathrm{cn}(\phi, \frac{1}{2})$.

The condition $[x(0) = x_0, \dot{x}(0) = 0]$ leads to simultaneous equations

$$\sqrt{r_0}\,\mathrm{cn}(\phi_0, 1/2) = x_0 \quad \text{and} \quad \sqrt{r_0}\,\delta\,\mathrm{cn}(\phi_0, 1/2) + 3r_0\sqrt{\beta}\,\mathrm{dn}(\phi_0, 1/2), \mathrm{sn}(\phi_0, 1/2) = 0\ ,$$

to be solved for r_0 and ϕ_0. Substituting $r_0 = (x_0/\mathrm{cn})^2$ into the second condition gives a transcendental equation for ϕ_0: $\delta + 3\sqrt{\beta}x_0\,\mathrm{dn}(\phi_0, \frac{1}{2})\,\mathrm{sn}(\phi_0, \frac{1}{2})/[\,\mathrm{cn}(\phi_0, \frac{1}{2})]^2 = 0$. This is solved numerically for ϕ_0, and then ϕ_0 inserted into $r_0 = (x_0/\mathrm{cn})^2$.

The condition $[x(0) = 0, \dot{x}(0) = \dot{x}_0]$ leads to simultaneous equations

$$\sqrt{r_0}\,\mathrm{cn}(\phi_0, 1/2) = 0 \quad \text{and} \quad \sqrt{r_0}\,\delta\,\mathrm{cn}(\phi_0, 1/2) + 3r_0\sqrt{\beta}\,\mathrm{dn}(\phi_0, 1/2)\,\mathrm{sn}(\phi_0, 1/2) + 3\dot{x}_0 = 0\ .$$

These have the solution $\phi_0 = \pm K(\frac{1}{2})$ and $r_0 = |\dot{x}_0|\sqrt{2/\beta}$.

[1] The coefficient of cn is zeroed first, followed by the coefficient of cn^3; and the residual is generated by the coefficient of $\mathrm{dn}\times\mathrm{sn}$.

[2] The coefficient of cn^3 is zeroed first, followed by the coefficient of $\mathrm{dn}\times\mathrm{sn}$; and the residual is generated by the coefficient of cn.

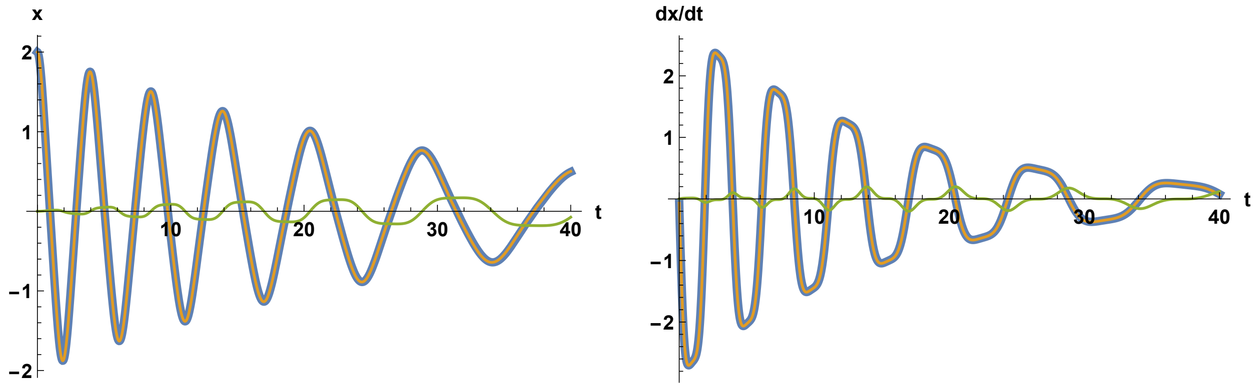

Figure 1.3: Cubic oscillator with $x_0 = 2, \dot{x}_0 = -\frac{1}{15}$. Left: overlay of approximate analytic x_a (gold) on the exact numerical $x_n(t)$ (blue); and the magnified error $10 \times (x_n - x_a)$. Right: overlay of approximate $\dot{x}_a$ (gold) on the exact $\dot{x}_n$ (blue); and the magnified error $10 \times (\dot{x}_n - \dot{x}_a)$.

1.5 Pendulum Oscillator

The damped rigid pendulum has the equation of motion

$$\ddot{x} \;+\; \delta \dot{x} \;+\; \alpha \sin(x) \;=\; 0\,. \tag{1.13}$$

This is an archetype nonlinear oscillator. However, its principal use as a clock (since the time of Christiaan Huygens, 1656) relies on small amplitude oscillations of a few degrees. Arguably, because of the infinite series $\sin x \;=\; x - x^3/3 + x^5/5 + \ldots,$ the pendulum is more challenging than the anharmonic oscillator. However, the pendulum has only two parameters (α, δ) and its "frequency" has a very simple formula; and so we treat it before the anharmonic oscillator.

We begin with the lossless motion, $\delta = 0$. We multiply the EOM by $\dot{x}$ and integrate over time, leading to the energy equation:

$$\frac{1}{2}\dot{x}^2 \;-\; \alpha \cos[x(t)] = (J-1)\alpha\;, \tag{1.14}$$

where J is a constant of motion. The couplet $(J-1)$ ensures that the energy J is always positive. We substitute into Eq. (1.14) the trial solution $x(t) \;=\; 2\,\mathrm{arcsin}[\sqrt{m}\,\mathrm{sn}[\phi(t), m]$ and $\dot{x} \;=\; 2\sqrt{m}], \mathrm{cn}[\phi(t), m]\,\dot{\phi}$ where the m parameter and phase function ϕ must be consistent with one another. In the absence of damping, the frequency $\dot{\phi}$ is constant. The energy equation becomes:

$$2m[\alpha - (\dot{\phi})^2]\,\mathrm{sn}^2 \;+\; 2m(\dot{\phi})^2 \;=\; \alpha J \tag{1.15}$$

We equate the coefficients of sn^2 and sn^0 each to zero, leading to the conditions $\dot{\phi} \;=\; \pm\sqrt{\alpha}$ and $J = 2m$ and $\alpha > 0$. The period of motion is $\tau = 4K(m)/\sqrt{\alpha}$.

1.5.1 Damping & variation m(t)

Restoring δ to a non-zero value results in a changing energy value $\alpha \dot{J} \;=\; -\delta(\dot{x})^2$. Under the assumption of weak damping, we substitute $J(m)$ and $\dot{x}$ for the lossless case, leading to

$$\alpha \frac{d}{dt} 2m \;=\; -4m\alpha\delta[\,\mathrm{cn}(\phi, m)]^2\;. \tag{1.16}$$

The average damping rate is found by integrating over one period of the oscillation. The integral

$$2m \int_0^{2K} \mathrm{cn}^2 dt = 2\,\{E[A(2K, m), m] + 2(m-1)K(m)\} = 4[E(m) + (m-1)K(m)]\;.$$

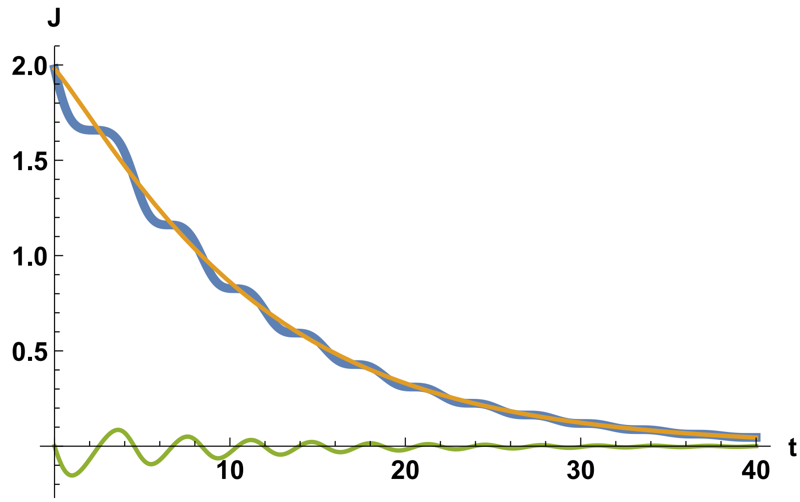

Figure 1.4: Energy of damped pendulum oscillator versus time. Blue: numerical solution J_n. Gold: approximate solution J_r. Olive: the difference $(J_n - J_r)$. Here $m_0 = 0.99$.

Hence the damping law:

$$\langle \dot{m} \rangle \; = \; -2\delta \left[(m-1) \; + \; E(m)/K(m) \right] \; . \tag{1.17}$$

This result is confirmed in Fig. 1.2 which compares the energy J_n calculated by numerical solution of the EOM and substitution in Eq. 1.14 versus $J_r = 2m(t)$ where m is computed by numerical integration of Eq. 1.17. The values chosen are $\alpha = 1, \delta = \frac{1}{10}$ and the large initial amplitude $m_0 = 0.99, x(0) = 0, \dot{x}(0) = 2\sqrt{\alpha m_0} = 1.99$. The values are in good agreement for $J < 1$ (or $m < \frac{1}{2}$); but there is an oscillator mismatch prior to that. The damping law is sketched in Fig. 1.5 Left.

Approximate m(t)

The function Eq. 1.17 goes to zero at $m = 0$ because the amplitude is zero; and also at $m = 1$ because the period becomes infinite due to the pendulum being inverted $(x = \pi)$. Over the range $m = [0, \frac{1}{2}]$ the approximation $\dot{m}/\delta = -m$ is perfectly adequate; leading to $m(t) = m_0 \exp(-\delta t)$ where m_0 is the initial value. In the range $m = [0, 1]$, the function is well approximated by $\dot{m}/\delta \approx -m \times (1 - m^2)^{1/5}$. Unfortunately, this does not have a closed-form integral. The poor approximation $\dot{m}/\delta \approx -m \times (1 - m^2)^{1/2}$ and initial value m_0, however does have an integral:

$$m(t) = \sin\left\{ 2 \,\mathrm{arccot} \left[e^{t\delta} \cot(u_0/2) \right] \right\} \; = \frac{e^{\delta t} m_0 (\sqrt{1 - m_0^2} - 1)}{(\sqrt{1 - m_0^2} - 1) \; + m_0^2 (1 - e^{2\delta t})/2} \; . \tag{1.18}$$

Here $\sin(u_0) = m_0$. The derivation[3] of Eq. 1.18 begins with the substitution $m(t) \to \sin[u(t)]$; and ends with the replacement $\cot(\theta/2) \to \sin\theta/(1 - \cos\theta)$. We call this solution $m_1(t)$.

A better approximation is $\dot{m}/\delta \approx -m \times (1 - m^2)^{1/4}$ which has the approximate integral: $F[u(t)] = t\delta + F[u_0]$ where $F(X) = \sqrt{(\cos X)^3/(2 \sin X)}$ and $m = \sin u$. Hence $F(m) = (\sqrt{1 - m^2})^{3/4}/\sqrt{(2m)}$. This leads to a sextic algebraic equation for $m(t)$, which is soluble. The formal solution is $m(t) = F^{-1}[t\delta + F(m_0)]$ where F^{-1} denotes inverse. We call this solution $m_2(t)$. Fig. 1.5 Right compares the approximate solutions (m_1, m_2) against numerical integration of Eq. 1.17, all starting from the initial value $m(0) = 0.99$. The agreement is satisfactory for m_2.

[3]The substitution $m(t) \to \tanh[u(t)]$ leads to the same result.

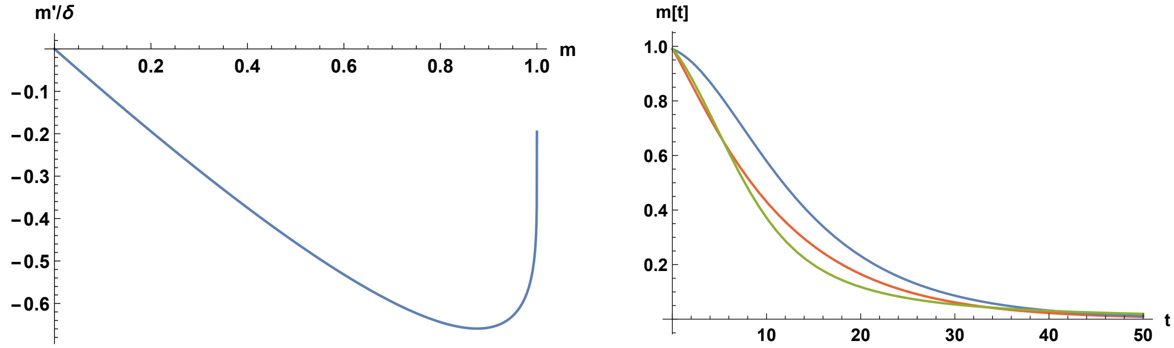

Figure 1.5: Left: Pendulum damping form factor. Right: Jacobi parameter m versus time for pendulum; Blue = $m_1(t)$; Olive = $m_2(t)$; Coral = Exact variation; $m_0 = 0.99$

1.5.2 Frequency shift and phase

We have a successful trial solution $x(t) = 2\arcsin[\sqrt{m(t)}\,\mathrm{sn}[\phi(t), m(t)]$ with known dependencies for $\dot{m}$ and $\dot{\phi}$. Unless it is the exact solution, substitution into the EOM (1.13) will leave a residual error. For the linear oscillator, we found that the error could be reduced (in fact zeroed) by introducing a frequency shift related to the damping rate δ. We shall attempt a similar reduction for the pendulum oscillator. We substitute the trial into the EOM taking full account that the time dependency of m implies we must take derivatives of the elliptic functions with respect to both arguments. The resulting expression is very lengthy, so we shall restrict attention to the dominant terms which are large and/or growing. The missing terms are oscillatory and will integrate to zero effect. To further simplify the expression, we restrict to $m < \frac{1}{2}$ such that $\dot{m} \approx -\delta m$ and $\ddot{m} \approx \delta^2 m$, and $E[A(\phi, m), m] \approx \phi \times (1 - m/2)$. For brevity we omit the arguments ϕ, m of the elliptic functions and write $m(t)$ as simply m. Our approximate EOM has a common factor $\sqrt{m}$ multiplying terms in cn and sn; elliptic balance implies each must be zero.

$$\mathrm{dn}\,\mathrm{sn}\left[-\frac{1}{8}\frac{(\delta\,\phi\,m)^2}{(1-m)^2} - \frac{\delta\,\phi\,m\,\dot{\phi}}{(1-m)} + 2(\alpha - \dot{\phi})^2\right] = 0 \tag{1.19}$$

$$2\,\mathrm{cn} \times \ddot{\phi} - \mathrm{cn} \times \frac{1}{2}\frac{\delta^2\,\phi\,m\,\mathrm{dn}^2}{(1-m)^2} = 0 \tag{1.20}$$

We solve the quadratic Eq. (1.19) for $\dot{\phi}$

$$\dot{\phi} = \pm\sqrt{\alpha} - \delta \times \phi(t) m(t)/[1 - m(t)]/4\ . \tag{1.21}$$

We substitute the explicit $m(t) = m_0 e^{-\delta t}$. The resulting equation has an exact and complicated solution in terms of the ${}_2F_1$ hypergeometric function and powers of decaying exponentials. However, our approach is perturbative and leads to a simple, compact expression that is an excellent approximation. We start from $\dot{\phi} = \sqrt{\alpha}$, and $\ddot{\phi} = 0$, and substitute the original $\phi(t) = \pm\sqrt{\alpha}\,t$.

$$\pm\dot{\phi} = \sqrt{\alpha} - \delta\,t\,\sqrt{\alpha}\,m_0/(e^{\delta t} - m_0)/4\ . \tag{1.22}$$

Notably $\dot{\phi} = \pm\sqrt{\alpha}$ when $t \to 0$ and as $t \to \infty$. Generally, the damping shifts $\dot{\phi}$ to lower frequency. The depressed frequency is crucial to obtaining a match between values of $(x, \dot{x})$ computed by numerical solution of the EOM and those calculated from the approximate analytic expressions.

The phase is the integral of Eq. (1.22). The first term has integral $\pm\sqrt{\alpha}t$. The second term has an exact integral; the precise form depends on the sign of δ and whether $m(t)/m(0)$ is greater or smaller than unity. Here $\delta > 0$ and $m(t)/m(0) < 1$. Hence the phase:

$$\phi(t) \approx \sqrt{\alpha}\,t + \Phi(t) \quad \text{where} \quad \Phi(t) \times (4/\sqrt{\alpha}) \equiv -t\ln[1 - m(t)] + \mathrm{Li}_2[m(t)] - \mathrm{Li}_2[m(0)] \tag{1.23}$$

and where $\mathrm{Li}_2[\dots]$ is the polylogarithm $\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} z^k/k^n$.

We must verify if the solution (1.21) also satisfies Eq. (1.20). We substitute $\ddot{\phi}$ the time derivative of Eq. (1.21), and $\dot{m} = -\delta \times m$, and replace dn^2 by its average value $\mathrm{dn}^2 \approx 1 - m/2$, to find the residual

$$\delta \times m[-4\sqrt{\alpha} \ + \ (4\sqrt{\alpha} + 3\,\delta\,\phi)m]\,\mathrm{cn}(\phi, m)/(1-m)^2\,. \tag{1.24}$$

At $t = 0$ this quantity is small, but not zero. With $\phi \approx \sqrt{\alpha}t$ and to second order in $m(t)$, the residual is $\mathrm{cn} \times [-4\sqrt{\alpha}\,\delta\,m(1+m) + 3\,t\sqrt{\alpha}\,\delta^2 m^2]$. The maximum value occurs at $t \approx 1/(2\delta)$, and then decays as $m = m_0 e^{-2\delta t}$. This implies the trial $x(t)$ will deviate from the exact solution of the EOM; but the error is bounded because it mostly accrues during the interval $t = [0, \frac{1}{\delta}]$. The comparison of numerical and analytic solutions in Fig. 1.6 confirms that the error is small and peaks around $t \approx 1/\delta$. The pendulum parameters are $\alpha = 1$ and $\delta = \frac{1}{10}$.

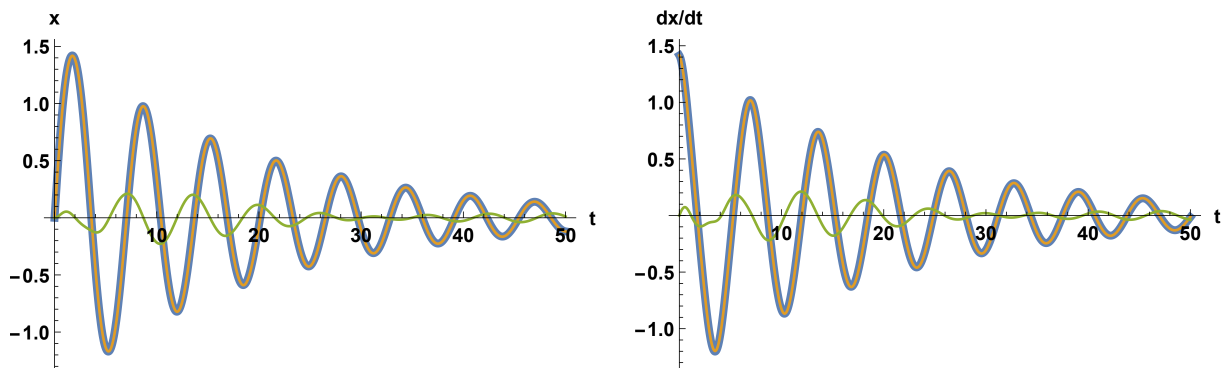

Figure 1.6: Pendulum oscillator with $x_0 = 0, \dot{x}_0 = \sqrt{2}$. Left: overlay of approximate analytic x_a (gold) on the exact numerical $x_n(t)$ (blue); and the magnified error $10 \times (x_n - x_a)$. Right: overlay of approximate $\dot{x}_a$ (gold) on the exact $\dot{x}_n$ (blue); and the magnified error $10 \times (\dot{x}_n - \dot{x}_a)$. Here $m_0 = \frac{1}{2}$.

The approximate expression for the EOM, after the trial solution has been inserted, and highly oscillatory terms dismissed, contains the powers $\phi\delta$, $\phi\delta^2$ and $(\phi\delta)^2$ where $\delta \ll 1$ and $\phi \gg 1$. In the absence of damping, ϕ increases linearly with time. It might be supposed, therefore, that the term $(\phi\delta)^2$ will dominate. But this is not the case. The phase always appears in combination with $m(t)$ which is exponentially damped, preventing the product from reaching large values; and hence the terms containing $\phi\delta^2$ and $(\phi\delta)^2$ are comparable.

1.6 Hard Anharmonic Oscillator

After introducing the preliminaries above, we now focus on the anharmonic EOM Eq. (1.1). We shall find a solution in terms of an elliptic function with all the adjustable parameters $r, m, \dot{\phi}$ varying in lock step to follow the decaying oscillation. In the limit $\delta \to 0$ the system will approach the quasi-static limit: a dynamic equilibrium in which the EOM is identically equal zero. For finite values of δ, there will be a small residual to the EOM despite the coupled variation of the oscillator parameters; and this must be reduced by introducing a frequency shift.

We begin with the lossless motion, $\delta = 0$. Many authors work directly with the differential equation, Eq. 1.1, but it is more elegant to work with the integral of motion. We multiply the EOM by $\dot{x}$ and integrate with respect to time, leading to the Hamiltonian:

$$\frac{1}{2}(\dot{x})^2 \ + \ \frac{1}{2}\alpha\,x^2 \ + \ \frac{1}{4}\beta\,x^4 \ = \ J\alpha\,. \tag{1.25}$$

The constant of integration J is chosen to satisfy the initial conditions, and is dimensionless. The next step is to substitute a trial function

$$x(t) = \sqrt{r}\,\mathrm{cn}(\phi, m) \quad \text{and} \quad \dot{x} = -\dot{\phi}\sqrt{r}\,\mathrm{dn}(\phi, m)\,\mathrm{sn}(\phi, m) \tag{1.26}$$

with adjustable constants $(\dot{\phi}, r, m, J)$ into Eq. 1.25, leading to

$$2r(1-m)(\dot{\phi})^2 \; + 2r[\alpha + (2m-1)(\dot{\phi})^2]\,\mathrm{cn}(\phi, m)^2 \; + \; r[r\beta - 2m(\dot{\phi})^2]\,\mathrm{cn}(at, m)^4 \; = \; 4\alpha J\,. \tag{1.27}$$

The coefficients of the time-varying functions must all be zero. This leads to three simultaneous non-linear algebraic equations for the adjustable constants.

$$2\alpha J + r(m-1)(\dot{\phi})^2 = 0 \qquad \alpha + (2m-1)(\dot{\phi})^2 = 0 \qquad r\beta - 2m(\dot{\phi})^2 = 0\,. \tag{1.28}$$

There are 3 equations and 4 adjustable constants. Therefore, we may choose one quantity as the control parameter; and find the remainder constants in terms of that parameter. If we treat $r > 0$ as the parameter, we find the solution:

$$\left[\dot{\phi} = \pm\sqrt{\alpha + r\beta}, \quad J = r(2\alpha + r\beta)/(4\alpha), \quad m = r\beta/[2(\alpha + r\beta)]\right] \quad \text{with} \quad \beta > 0\,. \tag{1.29}$$

If we treat m as the parameter, we find an alternative form (for the same solution):

$$\left[\dot{\phi} = \pm\frac{\sqrt{\alpha}}{\sqrt{1-2m}}, \quad J = \frac{(1-m)m\alpha}{(1-2m)^2\beta}, \quad r = \frac{2m\alpha}{\beta(1-2m)}\right] \quad \text{with} \quad 0 < m \leq 1/2\,. \tag{1.30}$$

We shall refer to these conditions with m replaced by $m(t)$ as *quasi-static*. By inspection, the adjustable constants diverge as $m \to \frac{1}{2}$. This is the limit of very large amplitude such that $\beta x^2 \gg \alpha$. In this case, the linear restoring force (αx) becomes insignificant, and the motion tends to that of the pure cubic oscillator. The period of oscillation $\tau(m)$ is given by the condition $\dot{\phi}\tau = 4K[m]$.

The quadrature solution is obtained by making the Jacobi argument substitution $[\phi, m] \to [\phi + K(m), m]$ in Eq. 1.26, leading to

$$x(t) = -\sqrt{r}\sqrt{1-m}\,\mathrm{sd}(\phi, m) \quad \text{and} \quad \dot{x} = -\dot{\phi}\sqrt{r}\sqrt{1-m}\,\mathrm{cd}(\phi, m)\,\mathrm{nd}(\phi, m)\,. \tag{1.31}$$

1.6.1 Damping & variation m(t)

We consider the case of weak damping, wherein the energy loss per cycle of the oscillation is a small fraction of the total energy. In this case, we may adopt a perturbative approach. We multiply the EOM, Eq. 1.1, by $\dot{x}$ and integrate over time, leading to

$$\alpha J = -\delta \times \int^t (\dot{x})^2 du \quad \text{or} \quad \alpha\frac{d}{dt}J(t) = -(\dot{x})^2 \times \delta\,. \tag{1.32}$$

If the damping constant δ is sufficiently small, then we may insert the unperturbed motion in the right hand side of Eq. 1.32. This will give the instantaneous damping rate. If we then form the time average over an oscillation cycle, we shall find the right side is related to J; and so we shall have an evolution equation for $J(t)$.

It is expedient to write J in terms of the Jacobi m parameter. The left side of Eq. 1.32 is

$$\frac{\alpha^2}{\beta}\frac{d}{dt}\left[\frac{(1-m)m}{(1-2m)^2}\right] \; = \; \frac{\alpha^2}{\beta}\frac{\dot{m}}{(1-2m)^3} \quad \text{where} \quad m = m(t)\,.$$

The right side of Eq. 1.32 is $-2m\alpha^2\delta\,\mathrm{dn}(\phi, m)^2\,\mathrm{sn}(\phi, m)^2/(1-2m)^2/\beta$. Hence the rate equation

$$\dot{m} \; = \; -\delta\, 2m(1-2m)\,\mathrm{dn}(\phi, m)^2\,\mathrm{sn}(\phi, m)^2\,. \tag{1.33}$$

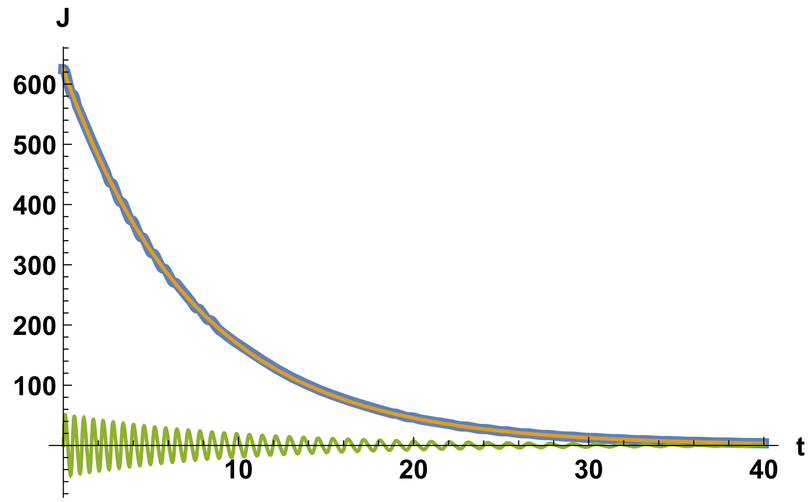

Figure 1.7: Energy of damped hardening oscillator versus time. Blue: numerical solution J_n. Gold: approximate solution J_r. Olive: the magnified difference $10 \times (J_n - J_r)$. Here $m_0 = 0.490$.

The cycle-average of Eq. 1.33 is: $\langle \dot{m} \rangle = -\delta \times 2m(1-2m) \times F(m)$ where

$$F(m) \equiv 2\int_0^{2K(m)} \mathrm{dn}^2(z,m)\mathrm{sn}^2(z,m)dz/4K(m) \;=\; \frac{(1-m)}{3m} \;+\; \frac{(2m-1)E(m)}{3mK(m)} \,. \tag{1.34}$$

The product function on the right side of $\langle \dot{m} \rangle$ is plotted in Fig. 1.8; and simplifies to

$$\begin{aligned} \langle \dot{m} \rangle/\delta \;&=\; -\frac{2}{3}(1-3m+m^2) \;+\; \frac{2}{3}(1-2m)^2\frac{E(m)}{K(m)} \\ &\approx\; -\frac{2}{3}(1-3m+m^2) \;+\; \frac{2}{3}(1-2m)^2(1-m/2) = -m + \frac{8m^2}{3} - \frac{4m^3}{3} \,. \end{aligned} \tag{1.35}$$

This result is confirmed in Fig. 1.7 which compares the energy J_n calculated by numerical solution of the EOM and substitution in Eq. 1.25 versus $J_r = (1-m)m]\alpha/[(1-2m)^2\beta]$ with $m(t)$ the numerical solution of Eq. 1.35. The values chosen are $\alpha = 1, \beta = 1, \delta = \frac{1}{10}$ and the large initial amplitude $m_0 = 0.49$, $r_0 = 49$, $x(0) = 7$, $\dot{x}(0) = -0.235862$. The values are in good agreement.

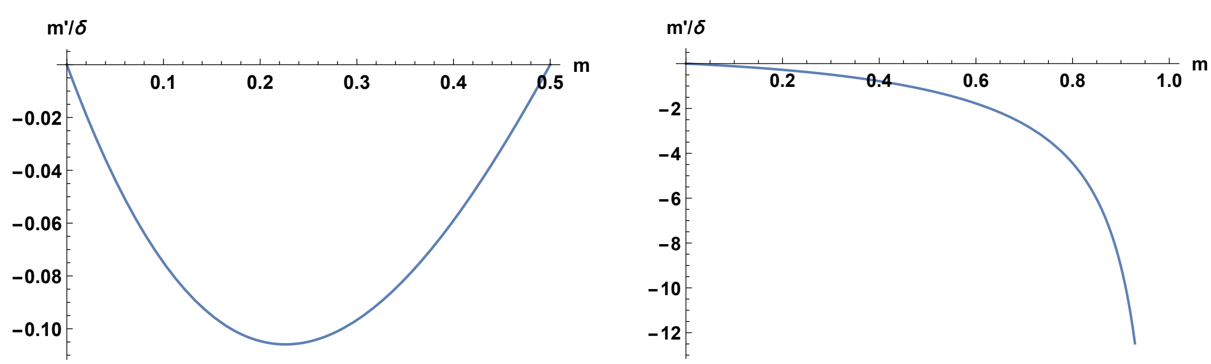

Figure 1.8: Damping form factors for the hard (left) and soft (right) Duffing oscillator.

Approximate m(t)

The first term in Eq. 1.35 is precisely $-m$, with implication of exponential damping $m(t) = m(0)\exp(-t\delta)$. The cubic form for $\langle \dot{m} \rangle$ is an excellent approximation, but unfortunately does not lead to an integral. In its place, a reasonable approximation is $-A\sin(2m\pi)$, where the constant $A = 0.105912$ is the value at the minimum $m = 0.22615$. The equation $\dot{m} = -A\delta\sin(2m\pi)$ has the exact solution $\tan[\pi\, m(t)] = \exp(-bt)\tan[\pi\, m(0)$ where $b = 2\pi A\delta$ and $m(0)$ is the initial value. Fig. 1.10 compares this approximate solution for $m(0) = 0.49$ against numerical integration of the exact evolution equation; the agreement is good.

1.6.2 Frequency shift and phase

When $\delta = 0$, we have a successful trial solution $x(t) = \sqrt{r}\,\mathrm{cn}[\phi, m]$ with known dependencies for $\dot{\phi}$ and r given by Eq. 1.30. When $0 < \delta \ll \sqrt{\alpha}$ is non-zero, we know that $\langle\dot{m}\rangle$ depends only on m; see Eq. 1.35. In the limit of very slow variation, the solution takes on the quasi-static form $x(t) = \sqrt{r(t)}\,\mathrm{cn}[\phi(t), m(t)]$ with $r(t)$ following the variation of $m(t)$ through Eq. 1.30. However, the phase variation will depart from that given in Eq. 1.30. Substitution of the quasi-static form into the EOM (1.1) will leave a residual error; and this may be used to improve the estimate of the frequency $\dot{\phi}$.

We substitute the quasi-static form $x = \sqrt{r(t)}\,\mathrm{cn}[\phi(t), m(t)]$ into the EOM taking full account that the time dependency of m implies we must take derivatives of the elliptic functions with respect to both arguments. The resulting expression is very lengthy, so we shall restrict attention to the dominant terms which are large and/or growing. We use the quasi-static form $r(m)$ Eq. 1.30, and the relation between $\dot{r}$ and $\dot{m}$ to achieve some cancellations and simplification. We leave $\dot{\phi}$ as a free variable. For the hardening oscillator, $0 \le m \le \frac{1}{2}$ for which $E[A(\phi, m), m] \approx \phi \times (1 - m/2)$. However, to further simplify the expression we must restrict $\langle\dot{m}\rangle$ to the near-linear range $m < \frac{1}{4}$ such that $\dot{m} \approx -\delta m$ and $\ddot{m} \approx \delta^2 m$. For brevity we omit the arguments ϕ, m of the elliptic functions and write $m(t)$ as simply m. The EOM contains terms in cn and sn; elliptic balance implies the coefficients of both must each be zero. Hence two simultaneous equations:

$$\sqrt{r}\left\{\frac{2\alpha\, m}{(1-2m)}\,\mathrm{cn}^3 + \mathrm{cn}\left[\frac{\phi(t)\dot{\phi}\,\delta\,\Lambda\, m}{2(1-m)} + \frac{\phi^2\delta^2\Lambda\, m^2}{4^2(1-m)^2} + (\alpha + \Lambda\times\dot{\phi}^2)\right]\right\} = 0 \tag{1.36}$$

$$\frac{\mathrm{dn}\,\mathrm{sn}}{\sqrt{r}}\left\{\frac{\alpha\,\delta^2\,\phi\, m^2[\Lambda - 5m + 4m^2 + 4m\,\mathrm{dn}^2]}{2\beta(1-2m)^2(1-m)^2} + \frac{2\alpha\, m\,\ddot{\phi}}{\beta(1-2m)}\right\} = 0 \tag{1.37}$$

where $\Lambda(t) \equiv 1 - 2\,\mathrm{dn}^2$. It is worth stating that had we not relied on the quasi-static forms and the guiding hand of the energy equation, but left $r, \dot{\phi}, m$ all as free functions, the equations would have contained dozens of terms; a bewildering expression with no hope of solution.

The Eq. (1.36) is quadratic in $\dot{\phi}$, and can be solved[4] exactly:

$$\dot{\phi} = \pm\frac{\sqrt{\alpha}}{\sqrt{1-2m(t)}} - \delta\times\frac{\phi(t)m(t)}{4[1-m(t)]}\ . \tag{1.38}$$

We substitute the explicit $m(t) = m_0\exp(-\delta t)$. The first term has an exact integral. We substitute $\phi \approx \sqrt{\alpha}t$ into the second term, and find its integral to be $\Phi(t)$. Hence the phase

$$\phi(t) \approx \sqrt{\alpha}\, t + 2(\sqrt{\alpha}/\delta)\ln\left[\frac{1+\sqrt{1-2m(t)}}{1+\sqrt{1-2m(0)}}\right] + \Phi(t)\ . \tag{1.39}$$

The terms excluding Φ have the first order approximation $\sqrt{\alpha}t + (\sqrt{\alpha}/\delta)[m(0) - m(t)]$.

For consistency, the coefficient of dn×sn must be zero or very small. Certainly it is small: the first term is order δ^2, and in the absence of damping $\ddot{\phi} = 0$. We now quantify the assertion. The solution Eq. 1.38, and its derivative $\ddot{\phi}$ is substituted into Eq. 1.37. We substitute $\dot{m} = -\delta\, m$, and

[4] The elliptic functions cancel between denominator and numerator leaving this remarkably simple form.

replace dn^2 by its (approximate) average value $1 - m/2$. Simplifications are achieved, with the residual EOM equal to

$$\frac{\sqrt{r}\,\mathrm{dn}\,\mathrm{sn}}{4(1-m)(1-2m)}\left[\frac{\sqrt{\alpha}\,\delta\, m(5-6m)}{\sqrt{1-2m}} + \frac{\phi\,\delta^2\, m^2(7-6m)}{4(1-m)}\right] . \tag{1.40}$$

The second term, in $\phi\,\delta^2$ is of order δ times smaller than the first and can be neglected. Thus the residual has magnitude of order $\sqrt{r}\sqrt{\alpha}\,m\,\delta$; and is bounded because both m and r decay exponentially. It is an effort to find even a crude estimate for the deviation of the trial function $x(t)$ from the exact solution $X(t)$ of the EOM. Let $X = x + y$ where y is the error. It may be inferred that y is of order $\delta\, m(t)\sqrt{r/\alpha}$. The comparison of numerical and analytic solutions in Fig. 1.9 confirms that the error is of order $\delta \times m$ smaller than X. The oscillator parameters are $\alpha = \beta = 1$ and $\delta = \frac{1}{10}$; and $m_0 = \frac{1}{8}$.

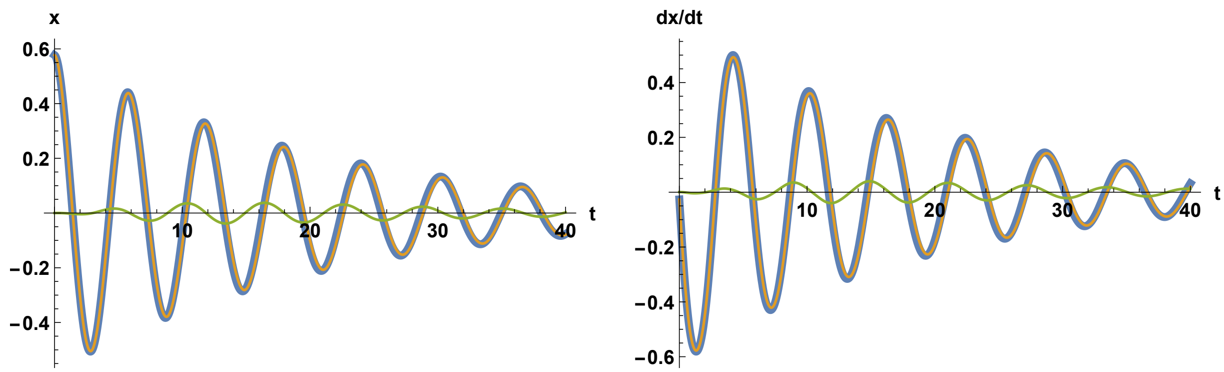

Figure 1.9: Hardening oscillator with $x_0 = 1/\sqrt{3} \approx 0.5774, \dot{x}_0 = -0.0266$. Left: overlay of approximate analytic x_a (gold) on the exact numerical $x_n(t)$ (blue); and the error $(x_n - x_a)$. Right: overlay of approximate $\dot{x}_a$ (gold) on the exact $\dot{x}_n$ (blue); and the error $(\dot{x}_n - \dot{x}_a)$. Here $m_0 = \frac{1}{8}$.

1.7 Euler's solution for hardening oscillator

The goal of Euler[15] *et al* was to find conditions under which the EOM 1.1 has a solution of the form $x(t) = v(t)\,\mathrm{cn}[\phi(t), \frac{1}{2}]$ with m a constant. The restriction $\beta = 1$ is unnecessary. We outline their solution, and comment on critical damping. Euler begins from the trial function $x = v(t)u[\phi(t)]$ where u, v, ϕ are initially unknown. Substitution into the EOM yields:

$$u \times [\alpha\, v + \dot{v}\delta + \ddot{v}] \;+\; \dot{u} \times [2\dot{\phi}\dot{v} + v(\dot{\phi}\delta + \ddot{\phi})] \;+\; v(\dot{\phi})^2\ddot{u} + \beta \times (uv)^3 = 0\,. \tag{1.41}$$

The coefficients of u and $\dot{u}$ are equated to zero (see below), leaving $v[(\dot{\phi})^2\ddot{u} + \beta v^2 u^3] = 0$. The latter is suggestive of the pure cubic oscillator. They insert the trial $\dot{\phi} = v(t)/\sqrt{k}$ and find the condition for u to be of the form $u(z) = \sqrt{r}\,\mathrm{cn}[\omega z, \frac{1}{2}]$ where $z \equiv \phi(t)$. The condition is $\omega = \sqrt{k\, r\, \beta}$.

The coefficient of u becomes zero when $v(t) = c_1 \exp(t\rho_1) + c_2 \exp(t\rho_2)$ where the roots $2\rho_1 = -\delta - \Delta$ and $2\rho_2 = -\delta + \Delta$ where $\Delta = \sqrt{\delta^2 - 4\alpha}$, and c_1, c_2 are adjustable constants. Next, we must find the conditions under which substitution of v into the coefficient of $\dot{u}$ makes that coefficient become zero. There are two such conditions: (1) that $\delta = -3\sqrt{\alpha/2}$ and $c_2 = 0$; and (2) that $\delta = +3\sqrt{\alpha/2}$ and $c_1 = 0$. Because $\alpha > 0$, the former is a growing solution and is discarded. Hence $v(t) = c_2 \exp[-t\sqrt{\alpha/2}]$. With no loss of generality, c_2 may be set equal to one. Now $\dot{\phi} = v(t)/\sqrt{k}$ may be integrated to find $\phi(t)$; and the first argument of the elliptic function is $\omega\,\phi$. Hence Euler's solution:

$$x(t) = \exp[-t\sqrt{\alpha/2}]\sqrt{r}\,\mathrm{cn}\left\{[1 - \exp(-t\sqrt{\alpha/2})]\sqrt{2\, r\beta/\alpha}, 1/2\right\} . \tag{1.42}$$

The quadrature solution is found by adding $\pm K(\frac{1}{2})$ to the argument of the elliptic cosine.

1.7.1 Critical damping

The condition $\delta^2 = 9\alpha/2$ is suggestive of the critical damping for the simple harmonic oscillator. However, for the nonlinear case the oscillation frequency depends on amplitude; and there will be one precise amplitude at which critical damping occurs. The condition that there be no more than one zero crossing during the oscillation is $\sqrt{2\,r\beta/\alpha} \leq K[\frac{1}{2}]$. The equality determines the critical amplitude $r \approx 1.7188\,\alpha/\beta$.

1.8 Soft Anharmonic Oscillator

For the softening oscillator $\beta < 0$. This has implications for large amplitude oscillations. Let us write the EOM as $A + B = 0$ where $A = \alpha x + \beta x^3$ and $B = \delta\dot{x} + \ddot{x}$. At a certain amplitude $x_0 = \pm\sqrt{\alpha/|\beta|}$ the quantity $A = 0$ and the equation $B = 0$ has the momentary solution $x(t) = x_0 \exp(-\delta t)$. In this regime of amplitude, the term $\delta \times \dot{x}$ cannot be considered a small quantity. The analysis below is perturbative in δ; and therefore will break down as the limit $|x_0| = \sqrt{\alpha/|\beta|}$ is approached. This coincides with the limit of Jacobi parameter $m \to 1$.

The first step is to find the steady state solution, in the absence of damping. We substitute the trial function

$$x(t) = \sqrt{r}\,\mathrm{sn}(\phi, m) \quad \text{and} \quad \dot{x} = +\dot{\phi}\,\sqrt{r}\,\mathrm{cn}(\phi, m)\,\mathrm{dn}(\phi, m) \tag{1.43}$$

with adjustable constants $(\dot{\phi}, r, m, J)$, into the energy equation 1.25; leading to

$$4J\alpha = 2(\dot{\phi})^2 r \; + \; 2r[\alpha - (1+m)(\dot{\phi})^2]\,\mathrm{sn}^2(\phi, m) \; + \; r[2m(\dot{\phi})^2 + r\beta]\,\mathrm{sn}^4(\phi, m)\;. \tag{1.44}$$

The coefficients of the time-varying functions must all be zero. This leads to three simultaneous non-linear algebraic equations for the four adjustable constants. If we treat $r > 0$ as the parameter, we find the solution:

$$\left[\dot{\phi} = \pm\sqrt{\Gamma/2}, \quad J = r\Gamma/(4\alpha), \quad m = -r\beta/\Gamma\right] \quad \text{with} \quad \Gamma = (2\alpha + r\beta) \quad \text{and} \quad 0 < r \leq -\alpha/\beta\;. \tag{1.45}$$

If we treat m as the parameter, we find the equivalent form:

$$\left[\dot{\phi} = \pm\frac{\sqrt{\alpha}}{\sqrt{1+m}}, \quad J = \frac{-m\alpha}{(1+m)^2\beta}, \quad r = \frac{-2m\alpha}{(1+m)\beta}\right] \quad \text{where} \quad 0 < m \leq 1\;. \tag{1.46}$$

We shall refer to these conditions with m replaced by $m(t)$ as *quasi-static*. The period of oscillation $\tau(r)$ is given by the condition $\dot{\phi}\tau = 4K[m]$. The quadrature solution is obtained by the Jacobi argument substitution $[\phi, m] \to [\phi + K(m), m]$ in Eq. 1.43, leading to

$$x(t) = \sqrt{r}\,\mathrm{cd}(\phi, m) \quad \text{and} \quad \dot{x} = a\sqrt{r}(m-1)\,\mathrm{sd}(\phi, m)\,\mathrm{nd}(\phi, m)\;. \tag{1.47}$$

1.8.1 Damping & variation m(t)

The starting point is Eq. 1.32. We substitute $\dot{x}$ by the expression in Eq. 1.43 and form the average over one oscillation cycle. As before, working is simplified if we choose to use the Jacobi m parameter. The left side of Eq. 1.32 becomes

$$-\frac{\alpha^2}{\beta}\frac{d}{dt}\left[\frac{m}{(1+m)^2}\right] \; = \; \frac{\alpha^2}{\beta}\frac{(m-1)\dot{m}}{(1+m)^3} \quad \text{where} \quad m = m(t)\;.$$

The right side of Eq. 1.32 is $2m\alpha^2\,\delta\,\mathrm{cn}(\phi, m)^2\,\mathrm{dn}(\phi, m)^2/(1+m)^2/\beta$. Hence the rate equation

$$\dot{m} = -\delta\, 2m(1+m)\,\mathrm{cn}(\phi, m)^2\,\mathrm{dn}(\phi, m)^2/(1-m)\;. \tag{1.48}$$

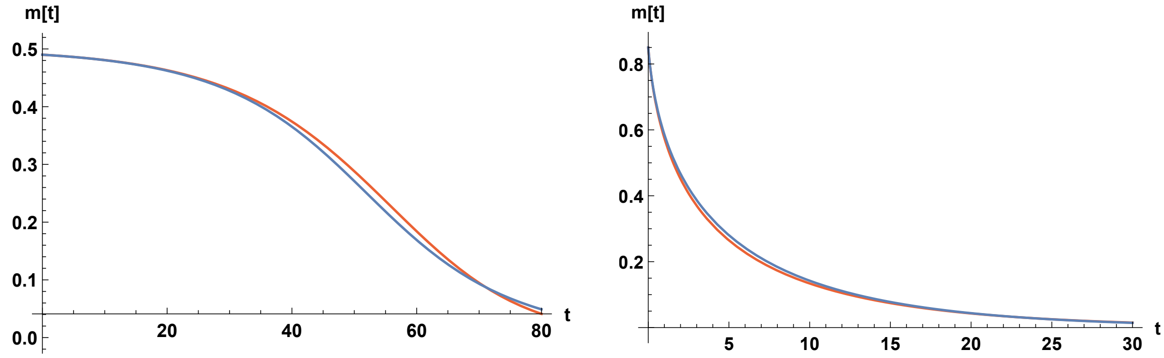

Figure 1.10: Jacobi amplitude m versus time for the hard (left) and soft (right) Duffing oscillator. The damping parameter $\delta = 0.1$ in both cases. The blue curve is the approximate analytic solution, whereas the coral coloured curve is that from numerical integration. Note that the time scale is markedly different between the two plots, as are the initial values $m_0 = 0.49$ and $m_0 = 0.85$.

The cycle-average of Eq. 1.48 is: $\langle \dot{m} \rangle = -\delta \times [2m(1+m)/(1-m)] \times F(m)$ where

$$F(m) \equiv 2\int_0^{2K(m)} \mathrm{dn}^2(z,m)\,\mathrm{cn}^2(z,m)dx/[4K(m)] \;=\; \frac{m-1}{3m} + \frac{(1+m)E(m)}{3mK(m)}\,. \tag{1.49}$$

The product function on the right side of $\langle \dot{m} \rangle$ is plotted in Fig. 1.8; and simplifies to

$$\langle \dot{m} \rangle/\delta \;=\; \frac{2}{3}(1+m) - \frac{2}{3}\frac{(1+m)^2E(m)}{3(1-m)K(m)} \;\approx\; -\frac{m}{(1-m)}\,. \tag{1.50}$$

This result is confirmed in Fig. 1.11 which compares the energy J_n calculated by numerical solution of the EOM and substitution in Eq. 1.25 versus $J_r = -m\alpha/[(1+m)^2\beta]$ with $m(t)$ the numerical solution of Eq. 1.35. The values chosen are $\alpha = 1, \beta = 1, \delta = \frac{1}{10}$ and the large initial amplitude $m_0 = 0.99$, $r_0 = 0.995$, $x(0) = 0$, $\dot{x}(0) = 0.7071$. The values are in good agreement.

Approximate m(t)

The first term in the Taylor expansion of $\dot{m}$ (Eq. 1.50) about $m = 0$ is precisely $-m$, with implication of exponential damping $m(t) = m(0)\exp(-t\delta)$. For the purpose of the next few sentences only, let A be an adjustable constant. The approximate equation $\dot{m} = \delta \times Am/(m-1)$ has the exact solution $m(t) = -\mathrm{ProdLog}\{-m(0)\exp[-m(0) \;-\; A\,t\,\delta]\}$ where the product $\mathrm{ProdLog}\{z\}$ is the solution for w of $z = w\exp(w)$. Over the range $m = [0,1]$ the best fit to $\dot{m}$ is $A = 1$. Over the reduced range $m = [0, 0.85]$ the best fit to $\dot{m}$ is $A = 1.08$; but we must accept the derivative is not quite correct at $m = 0$. Fig. 1.10 compares this approximate solution against numerical integration of the exact evolution equation starting from the initial value $m(0) = 0.85$. The agreement is good: the relative fractional error is no more than a few percent over the entire time interval.

1.8.2 Frequency shift and phase

When $\delta = 0$, we have a successful trial solution $x(t) = \sqrt{r}\,\mathrm{sn}[\phi, m]$ with known dependencies for $\dot{\phi}$ and r given by Eq. 1.46. When $0 < \delta \ll \sqrt{\alpha}$ is non-zero, we know that $\langle \dot{m} \rangle$ depends only on m; see Eq. 1.50. In the limit of very slow variation, the solution takes on the quasi-static form $x(t) = \sqrt{r(t)}\,\mathrm{sn}[\phi(t), m(t)]$ with $r(t)$ following the variation of $m(t)$ through Eq. 1.46. However, the phase variation will depart from that given in Eq. 1.46. Substitution of the quasi-static form into the EOM (1.1) will leave a residual error; and this may be used to improve the estimate of the frequency $\dot{\phi}$. We take full account that the time dependency of m implies we must take derivatives

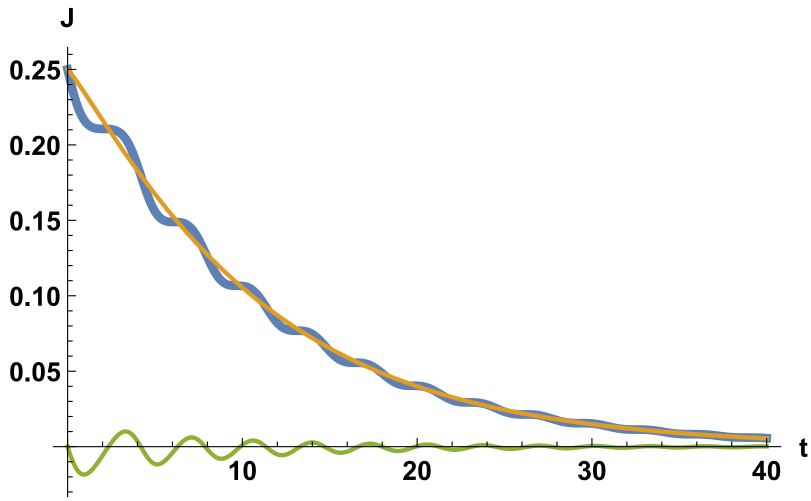

Figure 1.11: Energy of damped softening oscillator versus time. Blue: numerical solution J_n. Gold: approximate solution J_r. Olive: the difference $(J_n - J_r)$. Here $m_0 = 0.99$.

of the elliptic functions with respect to both arguments. The resulting expression is very lengthy, so we shall restrict attention to the dominant terms such as $\phi\delta, \phi\delta^2, (\phi\delta)^2$. We use the quasi-static form $r(m)$ Eq. 1.46, and the relation between $\dot{r}$ and $\dot{m}$ to achieve some simplification. We leave $\dot{\phi}$ as a free variable. For the softening oscillator, $0 \leq m \leq 1$. However, we restrict the range to $0 \leq m \leq \frac{1}{2}$ for which $E[A(\phi, m), m] \approx \phi \times (1 - m/2)$. In the near-linear range $m < \frac{1}{2}$ the cycle-average $\langle \dot{m} \rangle$ behaves such that $\dot{m} \approx -\delta m$ and $\ddot{m} \approx \delta^2 m$. For brevity we omit the arguments ϕ, m of the elliptic functions and write $m(t)$ as simply m. The approximate EOM residual contains terms in cn and sn; elliptic balance implies their coefficients must each be zero. Hence two simultaneous equations:

$$\sqrt{r}\left\{-\frac{2\alpha\, m}{1+m}\mathrm{sn}^3 \;+\; \mathrm{sn}\left[\frac{\phi^2\,\delta^2\, m(\Lambda - m)}{4^2(1-m)^2} + \frac{\phi\,\delta\, m(\Lambda - m)\dot{\phi}}{2(1-m)} + \alpha + (\Lambda - m)(\dot{\phi})^2\right]\right\} \;=\; 0 \quad (1.51)$$

$$-\frac{\mathrm{dn}\,\mathrm{cn}}{\sqrt{r}}\left\{\frac{2\alpha\, m}{\beta(1+m)}\ddot{\phi} \;+\; \frac{\alpha\,\delta^2\phi\, m^2(-1+2m+m^2)}{2\beta(1-m)^2(1+m)^2}\right\} \;=\; 0 \quad (1.52)$$

where $\Lambda(t) \equiv 1 - 2\,\mathrm{dn}^2$. The Eq. (1.51) is quadratic in $\dot{\phi}$, and can be solved exactly:

$$\dot{\phi} = \pm\frac{\sqrt{\alpha}}{\sqrt{1+m}} \;-\; \frac{\phi\,\delta\, m}{4(1-m)}\,. \tag{1.53}$$

We substitute $m(t) = m_0 \exp(-\delta t)$. The first term on the right side of Eq. 1.53 has an exact integral, and the second term has the approximate integral $\Phi(t)$. Hence the phase:

$$\phi(t) = \sqrt{\alpha}t \;+ 2(\sqrt{\alpha}/\delta)\ln\left[\frac{1+\sqrt{1+m(t)}}{1+\sqrt{m(0)}}\right] \;+\; \Phi(t)\,. \tag{1.54}$$

To first order in m, the first term approximates to $\sqrt{\alpha}t + \sqrt{\alpha}/(2\delta)[m(t) - m(0)]$. This may be substituted in Eq. 1.53 to give an improved version of Φ; however, the expressions become lengthy.

For consistency, the coefficient of dn×cn must be zero or very small. We now quantify the value. The solution Eq. 1.53, and its derivative $\ddot{\phi}$ is substituted into Eq. 1.52. The residual EOM simplifies to

$$\frac{m\,\mathrm{cn}\,\mathrm{dn}}{2\sqrt{2}\,(1-m)}\left[\frac{\sqrt{\alpha}\,\delta\,(1-3m)\sqrt{r}}{(1+m)^{3/2}} \;-\; \frac{\phi\,\alpha\, m^2\,\delta^2(13+5m)}{4\beta(1-m)\sqrt{r}}\right]\,. \tag{1.55}$$

The two terms are roughly equal in magnitude. Let X be the exact solution of the EOM and y the error such that $X = x + y$. It is not straight forward to infer what this residual implies about the error $y(t)$, other than it is bounded and decays exponentially.

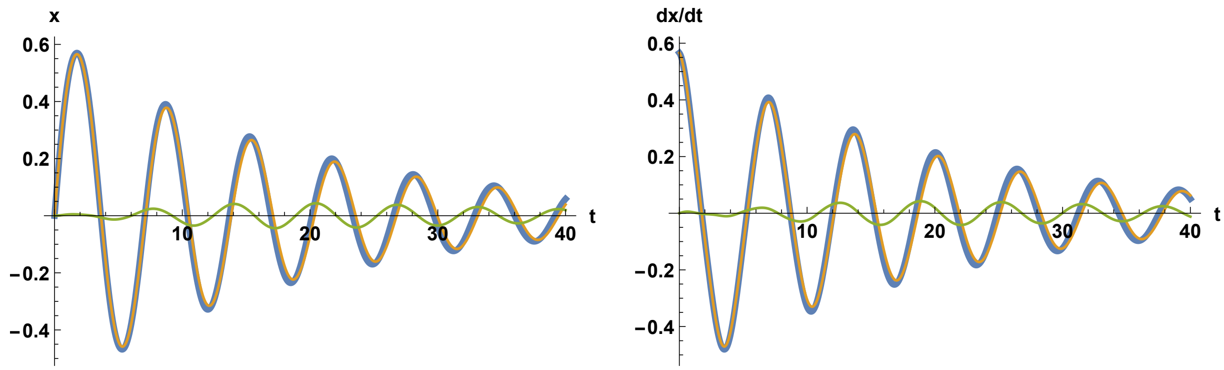

Figure 1.12: Softening oscillator with $x_0 = 0, \dot{x}_0 = 2\sqrt{2}/5 \approx 0.5657$. Left: overlay of approximate analytic x_a (gold) on the exact numerical $x_n(t)$ (blue); and the error $(x_n - x_a)$. Right: overlay of approximate $\dot{x}_a$ (gold) on the exact $\dot{x}_n$ (blue); and the error $(\dot{x}_n - \dot{x}_a)$. Here $m_0 = \frac{1}{4}$.

1.8.3 Energy variation, hard and soft

The article would be incomplete without showing the energy variation $J(t)$ corresponding to the motions $(x, \dot{x})$ shown in Figs. 1.9 and 1.12. Figure 1.13 is the counterpart to Figs. 1.7 and 1.11 but for smaller initial value of Jacobi-m. For the hardening oscillator $m_0 = \frac{1}{8}$, and for the softening oscillator $m_0 = \frac{1}{4}$. As elsewhere $\alpha = 1, |\beta| = 1, \delta = \frac{1}{10}$. The quantity J_n is the energy evaluated from $(x, \dot{x})$ calculated by numerical solution of the EOM; and can be treated as exact. The quantity J_a is evaluated from the approximate analytic expressions for $(x, \dot{x})$. The quantity $J_r[m(t)]$ is evaluated from $m(t)$ calculated by numerical solution of the damping law (obtained from the energy equation). The error $(J_n - J_a)$ is at the percent level. J_a faithfully reproduces the ripple in J_n due to the envelope oscillations.

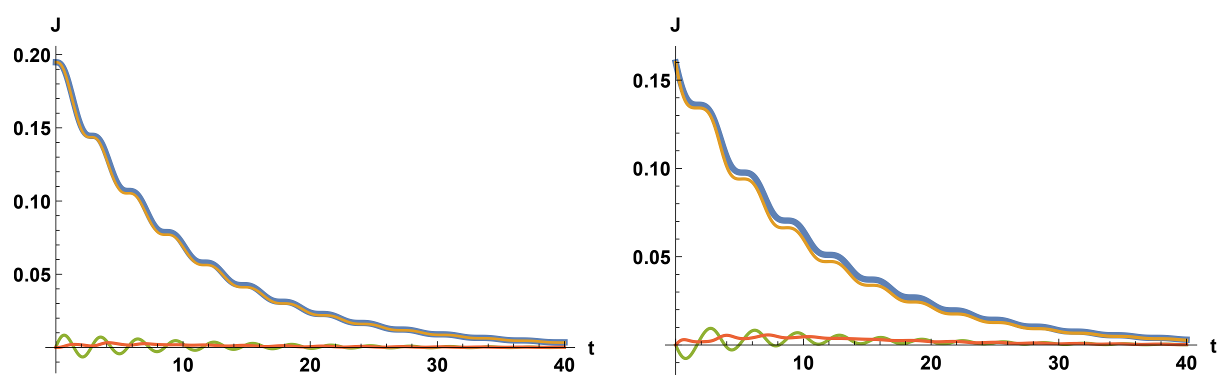

Figure 1.13: Energy versus time. Left: hardening oscillator. Right: softening oscillator. Blue: numerical solution J_n. Gold: approximate solution J_a. Olive: $J_n - J_r$. Coral: $(J_n - J_a)$.

1.9 Valid range of damping constant

Ludek[5] and Barkham[9] both claim their results are for *large damping rate* δ. Caution must be exercised when making claims of this sort. Ludek asserts the restriction $\delta^2 < 4\alpha$, as for the critically damped linear oscillator. However, it is not straight forward in the case of nonlinear oscillators; and careful study is required. In our reliance on the energy equation, it is explicit that δ must be sufficiently small that parameters are (at worst) slowly varying or (at best) almost constant over one period of the (undamped) oscillation. It might be thought that working directly from the EOM releases that restriction, but this is an illusion. Inevitably, there will be time-averaging over non-linear terms; and similar restrictions will arise.

There is no physical impediment to achieving the condition $\Delta J/J \to -1$ in a single oscillation; all that is required is a large damping constant δ in excess of $2\sqrt{\alpha}$. However, in such an extreme, frequency, phase and damping rate become ill-defined; as do kinematic integrals that depend on parameters being slowly varying or constant. We shall examine the condition $|\Delta J/J| \leq 1$ in the context of performing integrals and applying them. From this will emerge criteria for the valid range of δ consistent with time averaging of the non-linear term $(\dot{x})^2$.

We rely on the energy equation and time averaging to find $\langle \dot{J} \rangle$, thus we shall study the variation $\Delta J \propto -\delta \times \int_{-\tau/2}^{+\tau/2} (\dot{x})^2 dt$ per oscillation period τ. There are three considerations. (1) There is a difference between the zero and first order perturbation theory in the rate of phase advance that depends on δ. (2) We treat the energy J as a continuous variable. This appears to imply that the relative fractional change $\Delta J/J$ per oscillation cycle shall be small, say $|\Delta J/J| \ll 1$; however, that is incorrect. Although the change $\Delta J(m)$ is calculated from a window of width 2τ, the quantity $\langle \dot{J}[m(t)] \rangle$ we employ (in our differential equations) acts point-like, changes continuously with time, and at any instant contributes an infinitesimal effect. Large step-like values of ΔJ never accumulate because we use differential equations to determine $m(t)$ and $\dot{\phi}(t)$. This being so, the correct condition is $|\Delta J/J| < 1$. We take care of (relatively) fast parameter changes by continuously moving from one integral to another. To be more precise, moving from one value of the integral, as parametrised by m, to another - following the continuous variation of $m(t)$. One might say that fast variation of $m(t)$ within the integrand is mimicked by the changing value of the integral parametrised by $m(t)$. (3) Previously, when performing the ΔJ integrals, we held m constant. In the following, we shall find the effect of allowing $m(t)$ to vary during the integration over time. We shall derive results for the pendulum oscillator in detail. Results for the hardening and softening anharmonic oscillators are similar to the pendulum, and are presented briefly. In all three cases, it is concluded that $\delta/\sqrt{\alpha} \ll 1$ is a sufficient condition for validity of the perturbative formalism. For the cubic oscillator, substitute β for α.

1.9.1 Pendulum oscillator

The integrand for ΔJ is

$$-\delta \times (\dot{x})^2/\alpha = -4m\,\delta \times (\mathrm{cn}[\phi(t), m])^2 (\dot{\phi})^2/\alpha \,.$$

Previously, the integral was evaluated with the zero order approximation $\dot{\phi} = \sqrt{\alpha}$ and $\phi(t) = \phi_0 + \sqrt{\alpha}\, t$. However, we discovered the first order perturbation $\dot{\phi} = \sqrt{\alpha} - \phi\xi$. If m is constant and $\phi_0 = 0$, this has solution

$$\phi(t) = [1 - \exp(-\xi t)]\sqrt{\alpha}/\xi \approx \sqrt{\alpha} t(1 - \xi t/2) \quad \text{and} \quad \dot{\phi} \approx \sqrt{\alpha}(1 - \xi t) \quad \text{with} \quad \xi = \frac{m\delta}{4(1-m)} \,.$$

Evidently these two versions of the integrand will differ little if $\xi \times t$ evaluated at $\pm\tau/2 = \pm 2K(m)/\sqrt{\alpha}$ is small compared with unity. This leads to the condition $\delta/\sqrt{\alpha} \ll 4(1-m)/[mK(m)]$. In the near-linear regime, $m \ll 1$, the righthand side is typically much greater than one. The right-hand side equals 2.157 when $m = \frac{1}{2}$, and is greater than one for $m < 0.6639$; and only becomes restrictive as $m \to 1$.

Evaluating the integral for ΔJ, we may form the fractional change $|\Delta J/J| < 1$ per cycle and hence a condition on δ:

$$\frac{\delta}{\sqrt{\alpha}} \; < \; \lambda(m) \equiv \frac{m}{8[E(m) + (m-1)K(m)]} \quad \text{where} \quad \frac{1}{8} \le \lambda \le \frac{1}{2\pi} \; . \tag{1.56}$$

This condition is plotted in Fig. 1.14 as a function of m; and for $m < 0.9223$ is much more restrictive than the condition immediately above. Nevertheless, the condition is slowly varying function of m.

It is worth noting that (i) the potential and the oscillation is symmetric about $x = 0$; and (ii) the integrand has half the period of the oscillation and repeats itself. Consequently, the value of $\langle \dot{J} \rangle$ is the same whether the average is over a whole or half period. This being so, we could from the beginning have taken a half-size window.

Previously, when performing the ΔJ integrals, we held m constant. Now we let m vary during the integration. We make the substitution $m \to m + \Delta m$ and Taylor expand the integrand to first order in Δm. This necessitates taking the derivative of the Jacobi functions with respect to the second argument. If $\phi(t) = \phi_0 + \dot{\phi} t$ and $\Delta m(t) = \dot{m} t$ are both locally linear in time t, and the integral is constructed symmetrically about its midpoint $t = 0$, then there is no change in its value when $m(t)$ varies. This transpires because the variation introduces additional terms into the integrand that are either the product of t and a symmetric function, or the product of t^2 and an anti-symmetric function; and both cancel to zero when integrated. The insensitivity of ΔJ with respect to variation $m(t)$ during the integration is reassuring; and common to all three oscillator types.

1.9.2 Hardening oscillator

As above, we investigate the integrand for ΔJ:

$$-\delta \times (\dot{x})^2/\alpha = -2m\,\delta \times (\mathrm{dn}[\phi(t), m])^2 (\mathrm{sn}[\phi(t), m])^2 (\dot{\phi})^2 / [\beta(1 - 2m)] \; .$$

Here $\beta > 0$. Previously, we evaluated the integral using the unperturbed ($\delta = 0$) phase advance $\dot{\phi}_0 \equiv \sqrt{\alpha}/\sqrt{1-2m}$. However, the perturbed value is $\dot{\phi} = \dot{\phi}_0 - \phi\xi$; and this has solution $\phi(t) = [1 - \exp(-\xi t)]\dot{\phi}_0/\xi$. The two versions of the integrand will differ little if $|\xi \times t| \ll 1$ when evaluated at $\tau = \pm 2K(m)/\dot{\phi}_0$. This leads to the condition $\delta \ll 4(1-m)/[mK(m)]\dot{\phi}_0$. The smallest value of the righthand side is 7.533 and occurs at $m = 0.3909$. The condition does not constitute a practical restriction.

Evaluating the integral for ΔJ, we may form the fractional change $|\Delta J/J| < 1$ per cycle and hence a condition on δ:

$$\frac{\delta}{\sqrt{\alpha}} \; < \; \lambda(m) \equiv \frac{3m(1-m)}{8\sqrt{1-2m}[(2m-1)E(m) + (1-m)K(m)]} \; . \tag{1.57}$$

This condition is plotted in Fig. 1.15 as a function of m. In the near-linear regime $m < \frac{1}{4}$, $\lambda(m)$ is a slowly varying function. $\lambda(0) = 1/(2\pi)$ is the smallest value. $\lambda(1/4) = 0.1874$.

1.9.3 Softening oscillator

As before, we investigate the integrand for ΔJ:

$$-\delta \times (\dot{x})^2/\alpha = 2m\,\delta \times (\mathrm{dn}[\phi(t), m])^2 (\mathrm{cn}[\phi(t), m])^2 (\dot{\phi})^2 / [\beta(1 + m)] \; .$$

Here $\beta < 0$. Previously, we evaluated the integral using the unperturbed ($\delta = 0$) phase advance $\dot{\phi}_0 \equiv \sqrt{\alpha}/\sqrt{1+m}$. However, the perturbed value is $\dot{\phi} = \dot{\phi}_0 - \phi\xi$; and this has solution $\phi(t) = [1 - \exp(-\xi t)]\dot{\phi}_0/\xi$. The two versions of the integrand will differ little if $|\xi \times t| \ll 1$ when evaluated at $\tau = \pm 2K(m)/\dot{\phi}_0$. This leads to the condition $\delta \ll 4(1-m)/[mK(m)]\dot{\phi}_0$. In the near-linear regime, $m \ll 1$, the righthand side is typically much greater than one. The righthand side equals 1.7615 when $m = \frac{1}{2}$, and is greater than one for $m < 0.6154$; and only becomes restrictive as $m \to 1$.

Evaluating the integral for ΔJ, we may form the fractional change $|\Delta J/J| < 1$ per cycle and hence a condition on δ:

$$\frac{\delta}{\sqrt{\alpha}} < \lambda(m) \equiv \frac{3m}{8\sqrt{1+m}[(1+m)E(m)+(m-1)K(m)]} \quad \text{where} \quad \frac{3}{16\sqrt{2}} \leq \lambda \leq \frac{1}{2\pi} . \tag{1.58}$$

This condition is plotted in Fig. 1.14 as a function of m; and for $m < 0.8955$ is much more restrictive than the condition immediately above. $\lambda(m)$ is a slowly varying function over the entire range.

1.9.4 Cubic oscillator

For the pure cubic oscillator, the role of α is taken on by β. We investigate the integrand for ΔJ:

$$-\delta \times (\dot{x})^2 = -r\,\delta \times (\mathrm{dn}[\phi(t), 1/2])^2 (\mathrm{sn}[\phi(t), 1/2])^2 (\dot{\phi})^2 .$$

The phase advance is $\dot{\phi} = \sqrt{\beta\, r(t)}$ and is non-perturbative. The action is $J = r^2\beta/4$. Performing the integral, we may form the fractional change $\Delta J/J = -\delta\, 16K(1/2)/[3\sqrt{r\beta}]$. The condition $|\Delta J/J| < 1$ is solved for δ, leading to $\delta/\sqrt{\beta} < \lambda \equiv 3\sqrt{r}/[16K(1/2)]$. This condition is plotted in Fig. 1.15 as a function of r. Evidently, the restriction on $\delta/\sqrt{\beta}$ is weak except for small oscillations in the vicinity of $r < 1$.

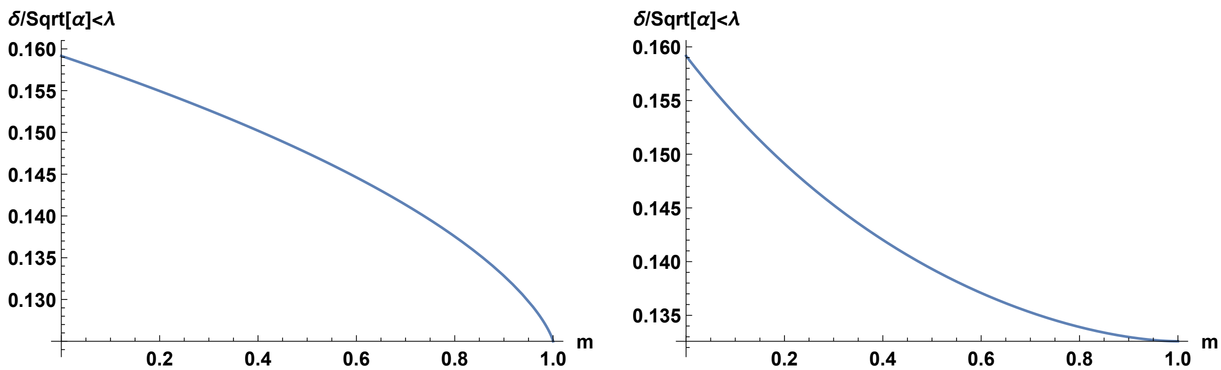

Figure 1.14: Permissible values of δ. Left: simple pendulum. Right: softening oscillator.

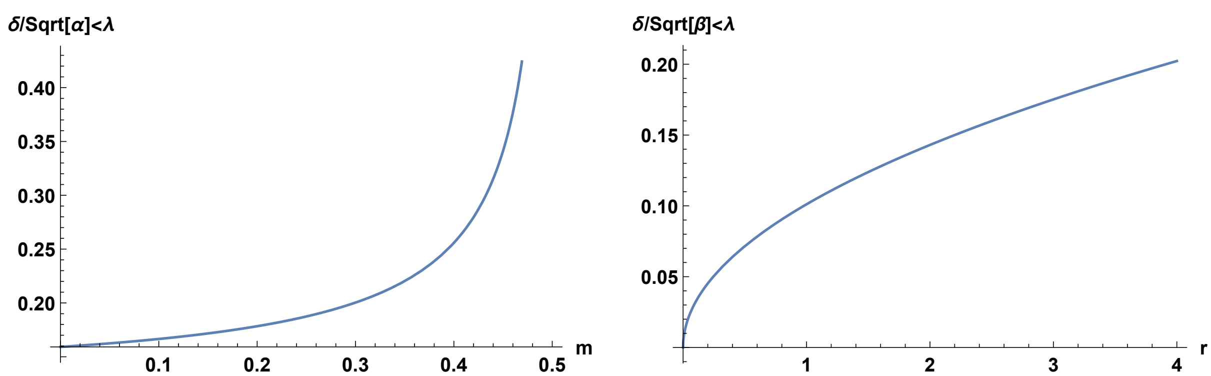

Figure 1.15: Permissible values of δ. Left: hardening oscillator. Right cubic oscillator.

If the conditions on δ are respected, the relative fractional errors in $x(t), \dot{x}(t), J(t)$ calculated from the approximate analytic formula are 1-2% when $\delta/\sqrt{\alpha} \approx \frac{1}{10}$; and less for smaller values of δ. Perhaps remarkable, if $\delta/\sqrt{\alpha} \approx \frac{1}{2}$, graphs of $x(t), \dot{x}(t), J(t)$ calculated from the approximate expressions still resemble plots calculated by exact numerical solution of the EOMs. However, due to mis-match of the phase advance, the relative fractional errors contain ripple that rises to 20%.

1.10 Further observations

1.10.1 Second order expansion

We have expressions for $\langle \dot{m} \rangle$ valid for the entire range of Jacobi-m. We have presented the formalism as if the only mathematically tractable variation was $\dot{m} = -\delta m$. However, the second order expansion $\dot{m} = -\delta m + \epsilon m^2$ with $m(0) = m_0$ also has an exact solution:

$$m(t) = \frac{m_0 \delta}{m_0 \epsilon + e^{t\delta}(\delta - m_0 \epsilon)} \ . \tag{1.59}$$

This expression may be used directly for the pendulum amplitude, or when forming $r[m(t)]$ for the hard and soft anharmonic oscillators. Appropriate parameters are given in the table immediately below. The form Eq. 1.59 should not be substituted into the first-order phase advance ϕ which assumes $m(t) = m_0 e^{-t\delta}$.

Oscillator type	Maclaurin expansion $\dot{m}/\delta =$	Best fit $\dot{m}/\delta =$	Fit range $m =$	Epsilon $\epsilon =$	Linear Regime $m =$
Pendulum	$-m + m^2/8 + \ldots$	$-m + m^2/6 + \ldots$	$[0, 1/2]$	$(1/6)\delta$	$[0, 1/3]$
Hardening	$-m + 21m^2/8 + \ldots$	$-m + 19m^2/8 + \ldots$	$[0, 1/4]$	$(19/8)\delta$	$[0, 1/16]$
Softening	$-m - 13m^2/8 + \ldots$	$-m - 19m^2/8 + \ldots$	$[0, 1/2]$	$(-19/8)\delta$	$[0, 1/8]$

By "linear regime" we mean the error is very small; and by "near-linear" we mean that the error incurred is small when $\langle \dot{m} \rangle$ is treated as locally linear in m.

1.10.2 Residual and errors

Substitution of an inexact trial solution into the EOM leaves a residual R. Unfortunately, this cannot be used to estimate easily the error; as is demonstrated below. Suppose the exact solution, the trial solution, and the error are X, x, y respectively. Evidently, $X = x + y$. Substitution of X alone into the EOM leaves zero. Substitution of x alone into the EOM leaves R. Thus, substitution of $x + y$ into the EOM results in the equation

$$\ddot{y} + \delta\dot{y} + (\alpha + 3\beta\, x^2)y + \beta(3xy^2 + y^3) + R(t) = 0 \ . \tag{1.60}$$

Unfortunately, this cannot successfully be linearized in y. The terms are finely balanced against one another. If we assume y is initially small, and discard the terms in xy^2 and y^3 then we find that y grows quickly - in contradiction to the initial assumption. Only if all terms are retained, does the error estimate y remain small and valid. Hence, to find y we must solve a driven non-linear differential equation; and to resort to numerical methods.

1.10.3 Taylor versus Maclaurin expansion

We have given approximate solutions in the near-linear ranges: $m < \frac{1}{2}$ for the pendulum oscillator, $m < \frac{1}{8}$ for the hardening, and $m < \frac{1}{4}$ for the softening oscillator. However, the formalism is not limited to those ranges.

Because all oscillations finally terminate at $m = 0$, a Maclaurin series expansion of $m(t)$ about 0 is an expansion appropriate to late (i.e. large) time. To access times in the distant past, at which the initial m was large, we need either to (i) retain many power of m in the expansion, or (ii) move the the expansion point to some value $m_b \neq 0$ that is closer to $m = 1$ (or $m = 1/2$).

We have expressions for $\langle \dot{m} \rangle$ valid for the entire range of Jacobi-m. Previously, we made Maclaurin expansions in m about zero, thus $\dot{m} = c_1 m + c_2 m^2 + \dots$ where the c_n are coefficients. However, we could have made Taylor expansions about a non-zero value $m = m_b$; thus $\dot{m} = F(m_b) + b_1 \Delta m + b_2 \Delta m^2 + \dots$ where $\Delta m = (m - m_b)$. F is the local value, b_1 is the local gradient, and b_2 the local 2nd derivative, and so on. All expressions for $J, r, \dot{\phi}$, etc, have to be re-written in terms of the adjustable constant m_b and variable Δm. The evolution equations quickly become complicated and daunting, but the procedure could be continued to completion. Inevitably, we would truncate expressions to linear in Δm and valid for short time scales.

1.10.4 Conclusion

We have presented a literature survey for methods relating to the damped anharmonic oscillator ranging from the introduction of elliptic balance in 1969 to the most recent work of Johannessen in 2017. The damping has been modeled by a time-varying Jacobi-m parameter, and its influence on the oscillation frequency has been fully accounted for. The present author has introduced the quasi-static limit and the energy equation to guide and simplify the problem. We have given a systematic and uniform treatment of four classic non-linear oscillators: the pendulum and the anharmonic oscillator class. And have illustrated the methods with examples from the near-linear regime, and compared them against pure numerical integration of the equation of motion. In all cases, the agreement is found to be at the percent level when the ratio of damping and oscillation constants $\delta/\sqrt{\alpha}$ is $\simeq \frac{1}{10}$. The underlying strategy is perturbative, and so the accuracy improves as $\delta/\sqrt{\alpha}$ is progressively reduced.

Appendix A

Small error, larger consequences

Cveticanin's Review[11] is an authoritative account of methods used to solve Duffing's equation. However, in Article 4.3.2 "The Duffing equation with damping", it is stated[1] that solutions of Eq. (1.1) can be written as the product of a decaying exponential and a Jacobi elliptic function with time-independent frequency and modulus. We demonstrate this to be incorrect. Following Ref.[11] we write the equation of motion (EOM) as $\ddot{x}+2\delta\dot{x}+\alpha x+\beta x^3=0$, and derive the properties by balance of powers of the elliptic functions.

For the <u>hard</u> anharmonic oscillator ($\beta > 0$) take the trial form $x(t) = x_0 \exp(-st)\,\mathrm{cn}[\phi(t), m]$ where s, m are adjustable constants. For brevity, we omit the arguments of the elliptic function. Substitution into the EOM yields:

$$x_0 \left\{ \mathrm{cn}^3[x_0^2\beta - 2e^{2st}m \times (\dot{\phi})^2] + e^{2st}\,\mathrm{cn}[\alpha + s^2 - 2s\delta + (2m-1)(\dot{\phi})^2] + e^{2st}\,\mathrm{dn}\,\mathrm{sn}[2(s-\delta)\dot{\phi} - \ddot{\phi}] \right\} = 0\,.$$

Equating the (correct) coefficients of the powers cn^3, cn, dn×sn each to zero leads to three simultaneous equations, which are incompatible and have no solution for $\phi(t)$. In a simple algebraic error, Cveticanin gives the coefficient of cn^3 as $[x_0^2\beta - 2m \times (\dot{\phi})^2]$; omitting the exponential term e^{2st} fundamentally alters the nature of the equations thereby admitting a solution with constant values for $m, s, \dot{\phi}$ - albeit incorrect.

For the <u>pure cubic oscillator</u> ($\alpha = 0$) we take again the trial form $x(t) = x_0 e^{-st}\,\mathrm{cn}[\phi, m]$, and substitute into the EOM. The coefficients of cn^3 and dn×sn are unchanged. α is set to zero in the coefficient of cn. Hence, the same conclusion applies.

Cveticanin does not treat the softening oscillator, but for completeness we do. For the <u>soft</u> anharmonic oscillator ($\beta < 0$) we insert the trial form $x(t) = x_0 \exp(-st)\,\mathrm{sn}[\phi(t), m]$ into the EOM, giving:

$$x_0 \left\{ \mathrm{sn}^3[x_0^2\beta + 2e^{2st}m \times (\dot{\phi})^2] + e^{2st}\,\mathrm{sn}[\alpha + s^2 - 2s\delta - (1+m)(\dot{\phi})^2] - e^{2st}\,\mathrm{dn}\,\mathrm{cn}[2(s-\delta)\dot{\phi} - \ddot{\phi}] \right\} = 0\,.$$

Equating the coefficients of the powers sn^3, sn, dn×cn each to zero leads to three simultaneous equations, which are incompatible and have no solution for $\phi(t)$. The problem is the time-dependence e^{2st} appearing in the coefficient of sn^3: $[x_0^2\beta + 2e^{2st}m \times (\dot{\phi})^2] = 0$. This condition implies that either (or both) m and $\dot{\phi}$ must be time-dependent.

[1] The statement is not attributed. It may be that Cveticanin reproduces the mistake of a previous author.

Bibliography

[1] Ivana Kovacic and Michael J. Brennan: The Duffing Equation - Nonlinear Oscillators and their Behaviour, John Wiley & Sons (2011).

[2] Livija Cveticanin: Ninety years of Duffing's equation, Theoret.Appl.Mech. TEOPM7,Vol.40, No.1, pp. 49-63, Belgrade 2013.

[3] C.G.J.Jacobi: "New Foundations of the Theory of Elliptic Functions", Konigsberg, Borntraeger 1829, Reprinted by Cambridge University Press, 2012 (in Latin). The title is often abbreviated to"Fundamenta Nova".

[4] P.F. Byrd and M.D. Friedmann: "Handbook of Elliptic Integrals for Engineers and Scientists", Springer-Verlag Berlin, Heidelberg & New York. 1st Edition 1954; 2nd Edition 1971.

[5] C. A. Ludeke and W.S. Wagner: The Generalized Duffing Equation with Large Damping, Intternationla Journal of Non-Linear Mechanics, Vol.3 pp.383-395 (1968).

[6] Peter George Douglas Barkham: Transient analysis of nonlinear non-autonomous second order systems using Jacobian elliptic functions, University of British Columbia, Thesis, 1969.

[7] P.G.D. Barkham, A.C. Soudack: An extension to the method of Krylov and Bogolubov. International Journal of Control, 10, pp.377–392, 1969.

Abstract: An extension of the method of Kryloff and Bogoliuboff is presented which uses Jacobian elliptic functions to approximate the solutions to a class of second-order differential equations exhibiting a gross cubic non-linearity. The results obtained from an analysis of the error incurred by the approximation show that, provided certain conditions of smallness are satisfied, excellent solution accuracy can be obtained. Two examples are used to demonstrate application of the extension to systems of practical interest.

[8] A.C. Soudack, P.G.D. Barkham: Further results on Approximate solutions of non-linear, non-autonomous second-order differential equations, International Journal of Control, 12, pp.763–767, 1970.

Abstract: A method is presented for determining more accurate approximate solutions to a class of differential equations characterized by ... where resonance effects may be neglected. The results are an extension of work presented in a previous paper (Barkham and Soudack 1969). The extension involves the relaxation of the restriction that the fundamental oscillation frequency is amplitude invariant. Since, in general, the frequency is amplitude dependent for the system under consideration, relaxation of the aforementioned restriction leads to markedly improved approximate solutions.

[9] A.C. Soudack and P.G.D. Barkham: On the transient solution of the unforced duffing equation with large damping, International Journal of Control, Vol. 13, Issue 4 (1971) pp.767-770.

Abstract: The refined elliptic-function method previously developed (Soudack and Barkham 1970) is applied to the unforced Duffing equation with large damping. The theoretical development for this case, and an example, are presented. The Kryloff-Bogoliuboff (K-B) solution is shown for comparison with the present result.The advantage of this approach, well illustrated by the example, is in its ability to predict the solution phase. The approach is therefore superior to quasi-linear methods, such as the K-B method, which rapidly fails to give a good approximation due to phase errors.

[10] Hiroshi Yabuno: Free Vibration of a Duffing Oscillator with Viscous Damping, Chap. 3 in Ref.[1].

[11] Livija Cveticanin: Analysis techniques for the various forms of the Duffing equation, Chap. 4 in Ref.[1]

[12] Kim Johannessen: An analytical solution to the equation of motion for the damped nonlinear pendulum. European Journal of Physics, Volume 35, Number X (2014)

[13] Kim Johannessen: The Duffing oscillator with damping. European Journal of Physics, Volume 36, Number 6. (2015)

[14] Kim Johannessen: The Duffing Oscillator with Damping for a Softening Potential. Int. J. Appl. Comput. Math (2017) 3:3805–3816.

[15] N. Euler, W-H Steeb and K. Cyrus: On Exact Solutions for Damped Anharmonic Oscillators, J. Phys. A: Math. Gen. 22 (1989) L195-L199.

[16] D.E. Panayotounkanos, N.D. Panayotounakou, A.V. Vakakis: On the Solution of the Unforced Damped Duffing Oscillator with No Linear Stiffness Term, Nonlinear Dynamics **28**: p.1–16, 2002.

[17] D.E. Panayotounakos, E.E. Theotokoglou, M.P. Markakis: Exact analytic solutions for the damped Duffing nonlinear oscillator, Comptes Rendus Mecanique 334 (2006) pp. 311-316.

[18] N. Kryloff and N. Bogoliuboff: Introduction to Nonliear Mechanics (1937).

[19] https://functions.wolfram.com/EllipticFunctions/JacobiCN/20/01/02/

https://functions.wolfram.com/EllipticFunctions/JacobiSN/20/01/02/

[20] Shane Koscielniak: “Motion in Dual-Harmonic Potential Well”, TRIUMF Laboratory Internal Report TRI-BN-24-04R.