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# Electrostatic bender and fringe fields

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**Abstract:** I describe the `COSY` code `GES` that we have used in the past 30 years to calculate maps to third order through electrostatic bend elements. It was always a mystery that `COSY`'s in-built procedures `ES`, `ESP` and `ECL` disagreed with our own code. Recent investigation finds the issue: the `COSY` routine does not include the fact of the orbit curvature changing in the fringe fields. This renders them incorrect as it is a fundamental source of second order aberration.

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# 1 Introduction

A long time ago (the 90s), I discovered two things concerning fringe fields of optical elements. The first is that the nonlinear effects of 'fringe fields' are intrinsic to the elements. That is, they are not reducible by modifying element boundaries. So it was a surprise to me that codes such as `GIOS`[1] and `COSY`[2] (and many others) allowed higher order (beyond linear) calculations with fringe fields shut off. Those codes still exist, and they still commit the same sins. Why call it a sin? Because when there is no fringe field, the fields that are used are discontinuous and therefore non-physical. In effect, the no-fringe-field options violate the Maxwell equations.

What is the cost? It is that non-specialists tasked with designing a beamline for example design without these effects at the start, thinking that they can deal with their complicating effects later, or not at all. Further, that the elements themselves are somehow at fault, and that there is a way of shaping elements to reduce these effects.

The second discovery was that the irreducible effect of the fringe field is independent of the shaping of the fringe field falloff. This is a general result and applies to all beam optical elements[3]. For solenoids, this effect is in first order so is correctly handled by all transport codes. But in dipoles, the effect is in second order and for quadrupoles in third order[4, 5].

But this makes it even less forgivable that the major transport codes do not include the effects. The default mode of operation should be to have the irreducible effect built in. Then the motion would always obey all conservation laws, even with the details of the fringe fields turned off. This is the way in which `TRANSOPTR` operates, but it is only a linear code, so these effects are used only to estimate the rms emittance growth, to be used in optimization calculations.

In `COSY`, *"bending magnets can be computed with their fringe fields or without, which is the default"*[6] (and the same is true for electrostatic bends), even though this violates Maxwell. The second order map of fringe fields is simple and uses negligible computing time as compared with Runge-Kutta integrating through an Enge function, and gives most of the effect needed for designing the optics of a transfer line matching section for example.

## 2 Theory

To find the transfer map of a fringe field, one must find a canonical transform that eliminates the terms in the expanded Hamiltonian that become singular in the limit of hard-edge fringe fields. I have done this for quadrupoles and in the particular case of electrostatic bends[7]. But the hard-edge limit is not given explicitly there so will be amplified here.

For the general toroidal bend, note from [7, eq. 17], there is only one term in the series expanded Hamiltonian that becomes signular in the limit and it is $h''x^3/6$. $h = h(s) = 1/\rho(s)$, $\rho$ being the radius of curvature of the reference trajectory. This results in a shift in $P_x'$ of $h''x^2/2$. Integrating by parts twice for entry into the field where $h$ goes from zero to $1/\rho$, we find

$$\Delta P_x = -hxx' = -\frac{xP_x}{\rho}. \tag{1}$$

But if $P_x$ is to change discontinuously, $x$ must also change. We can find this from the canonical transformation [7, eq. 20], since we know that while $x$ and $P_x$ suffer shifts in the hard edge limit, the transformed coordinates $X$ and $P_X$ do not. From outside where $h = h' = 0$ to inside the field where $h = 1/\rho$, $h' = 0$, we recover the change above in $P_x$, but also

$$\Delta x = \frac{x^2}{2\rho}. \tag{2}$$

These two equations (plus of course the first order identity matrix) are the second order map for entry into the bender field. For exit, the signs are reversed. Note that since the singular coefficient of the $x^3$ term, $h''$, depends only upon the electrode curvature in the bend plane, this second order shift applies to all toroidal cases.

## 3 `GES`: the `COSY` code

Writing a subroutine (aka 'procedure') for the electrostatic bender element is a straightforward application of the general element `GE`, in which the curvature $h$ and the electric potential $V$ are given on a grid of $s$ values. ($s$ is the independent variable, the path length along the reference trajectory.) Crucially, one must supply a function upon which the variation of $h$ and $V$ are based in the region of the fringe fields; in the body of the bender, both are constant. The approach is to simply specify $h(s)$ and derive $V(x, s)$ from it according to [7, eq. 3] as below:

$$V(x, s) = hx - \frac{1}{2}h(k+h)x^2 + \frac{1}{6}\left[2h\left(k^2 + kh + h^2\right) - h''\right]x^3 \tag{3}$$

It is natural to simply use the Enge function already available in `COSY`, but for the limiting cases discussed below, a simple polynomial (based on `COSY`'s `INTPOL`) is more stable.

This approach is 'kludgy' in the sense that it interpolates using a grid of data even though the analytic formulas for potential and curvature are known. Better would be to write it directly in the same way that magnetic bends are written. I am not sufficiently proficient in the `fox` language and the low level `COSY` code to do this.

`GES` has been written as a general toroidal electrostatic bend. Inputs are, in order, radius/m; angle/degrees; aperture/m (as usual, given as half the $D$ parameter in the Enge mode; kind, the bend-plane radius divided by the non-bend-plane radius of the ground surface e.g. 0 for Cylindrical, 1 for Spherical.

As a first test for the spherical bend, the code `GES` was run with fringe fields omitted. This gave a transfer map identical (in all 7 displayed digits) to that from using `COSY`'s routine `ESP`, with the setting `FR 0;`.

But with fringe fields, the two codes disagreed in second order. The case shown is where $\rho = 1$ metre, aperture radius 1 cm, bend angle $\pi/4$. For the `ESP` case, the fringe field option `FR 3;` was used.

```
GES
  0.1014558E-09-0.4112887E-09  0.000000      0.000000     -0.1964926E-09 000000
  0.7100503    -0.7014841      0.000000      0.000000     -0.7083172      100000
  0.7071161     0.7097656      0.000000      0.000000     -0.2928924      010000
   0.000000      0.000000     0.7071060    -0.7070988      0.000000       001000
   0.000000      0.000000     0.7071146     0.7071077      0.000000       000100
   0.000000      0.000000      0.000000      0.000000      1.000000       000010
  0.2928942     0.7081986      0.000000      0.000000     0.1180495       000001
 -0.3947180    -0.8518621      0.000000      0.000000     -0.7267119      200000
 -0.2069099    -0.7031506      0.000000      0.000000     -0.4576240      110000
 -0.4336389E-01-0.2038682      0.000000      0.000000     -0.3307033      020000
   0.000000      0.000000    -0.2041606     0.7109454      0.000000       101000
   0.000000      0.000000    -0.2076406     0.2922510      0.000000       011000
  0.1054571     0.1481579      0.000000      0.000000     -0.2295844      002000
   0.000000      0.000000      1.208664    -0.2935368      0.000000       100100
   0.000000      0.000000     0.5009565    -0.1055730E-02  0.000000       010100
  0.2065751     0.7072377      0.000000      0.000000     -0.4229213E-01 001100
  0.7874654      1.353561      0.000000      0.000000     0.4000934       100001
 -0.1468671     0.7059128      0.000000      0.000000     0.1653692       010001
   0.000000      0.000000     0.2075016     0.3533623      0.000000       001001
 -0.2495308    -0.5014689      0.000000      0.000000     -0.1250027      000200
   0.000000      0.000000    -0.1460474     0.2919105      0.000000       000101
 -0.2500816    -0.1482069      0.000000      0.000000     -0.5039539E-01 000002
 --------------------------------------------------------------------------------
```

```
symplectic error: 0.2589307634045153E-010

ESP
  0.7117938    -0.6972614      0.000000       0.000000     -0.7090489      100000
  0.7069680     0.7123671      0.000000       0.000000     -0.2928647      010000
   0.000000      0.000000     0.7118297     -0.6968199       0.000000      001000
   0.000000      0.000000     0.7074332      0.7123143       0.000000      000100
   0.000000      0.000000      0.000000       0.000000       1.000000      000010
  0.2928156     0.7093063      0.000000       0.000000      0.1179428      000001
 -0.4967818    -0.1054895E-01  0.000000       0.000000     -0.3707249      200000
   1.002711    -0.8830090E-02  0.000000       0.000000     -0.7489250      110000
  0.2071271    -0.7090427      0.000000       0.000000     -0.3290377      020000
   0.000000      0.000000     0.1776334E-01-0.9683271       0.000000      101000
   0.000000      0.000000     -1.414055     -0.4050090       0.000000      011000
  0.2140086    -0.6851324      0.000000       0.000000     -0.5779540      002000
   0.000000      0.000000     0.5606538E-03-0.9806706       0.000000      100100
   0.000000      0.000000    -0.1601973E-03  1.012666        0.000000      010100
  -1.006252     0.5536150E-02  0.000000       0.000000      0.2539799      001100
  0.9930520      1.050299      0.000000       0.000000      0.4025517      100001
  0.6111984E-01-0.1054575E-01  0.000000       0.000000      0.1664747      010001
   0.000000      0.000000    -0.9317907E-02 0.6428093       0.000000      001001
 -0.5003768     0.2548899E-02  0.000000       0.000000     -0.1225465      000200
   0.000000      0.000000     -0.3544161     0.9951060       0.000000      000101
 -0.2067942    -0.3589569      0.000000       0.000000     -0.4992080E-01 000002
 ------------------------------------------------------------------------------
symplectic error: 0.9751132434303467E-014


GIOS
       TRANSFER MATRIX OF THE SYSTEM AT PATH LENGTH L= 0.78540     LLU
       ------------------------------------------------------------
       X AND Y IN TLU, A AND B IN RAD, G AND D IN PARTS FROM M0 AND K0

     (X,X  )=0.71051         (A,X  )=-.70027         (T,X  )= 1.8042
     (X,A  )=0.70713         (A,A  )=0.71051         (T,A  )=0.74585
     (X,G  )= 0.0000         (A,G  )= 0.0000         (T,G  )=0.50000
     (X,D  )=0.29289         (A,D  )=0.70850         (T,D  )=-.30060

     (X,XX )=-.39269         (A,XX )=-.85070         (T,XX )= 1.8658
     (X,XA )=-.20751         (A,XA )=-.70335         (T,XA )= 1.1721
     (X,XG )= 0.0000         (A,XG )= 0.0000         (T,XG )=0.90210
     (X,XD )=0.78708         (A,XD )= 1.7034*        (T,XD )=-1.0169
     (X,AA )=-.42891E-01     (A,AA )=-.20397         (T,AA )=0.84572
     (X,AG )= 0.0000         (A,AG )= 0.0000         (T,AG )=0.37293
     (X,AD )=0.20711*        (A,AD )=0.70407         (T,AD )=-.45215E-01
     (X,GG )= 0.0000         (A,GG )= 0.0000         (T,GG )=-.12500
     (X,GD )= 0.0000         (A,GD )= 0.0000         (T,GD )=-.15031
```

```
(X,DD )=-.25000          (A,DD )=-.50342          (T,DD )=0.12896
(X,YY )=0.10525          (A,YY )=0.14985          (T,YY )=-.28221E-01
(X,YB )=0.20711          (A,YB )=0.71050          (T,YB )=-.39111
(X,BB )=-.25000          (A,BB )=-.49999          (T,BB )=0.18860


(Y,Y  )=0.70711          (B,Y  )=-.70711
(Y,B  )=0.70711          (B,B  )=0.70711

(Y,YX )=-.20810          (B,YX )=0.70850
(Y,YA )=-.20711          (B,YA )=0.29290
(Y,YG )= 0.0000          (B,YG )= 0.0000
(Y,YD )=0.20710          (B,YD )=0.70711*
(Y,BX )= 1.2095          (B,BX )=-.29288
(Y,BA )=0.50001          (B,BA )=-.70256E-16
(Y,BG )= 0.0000          (B,BG )= 0.0000
(Y,BD )=0.20711*         (B,BD )=0.29288
```

As one can see, `GES` and `ESP` are nothing alike in second order, but `GES` and `GIOS` are substantially in agreement[1]; the differences in the few % range can be explained by the default fringe field integrals used in `GIOS`.

To emphasize the point, a more clinical comparison has been made with vanishing (actually 0.1 mm) aperture. Only the relevant lines are shown, along with symplectic checks $g_2$ and $g_3$ from [8]. These are expected to be zero.

```
GES
  0.7071366    -0.7070500      0.000000      0.000000     -0.7071188     100000
  0.7071069     0.7071337      0.000000      0.000000     -0.2928931     010000
 -0.3976300    -0.8537546      0.000000      0.000000     -0.7278071     200000
 -0.2065952    -0.7056108      0.000000      0.000000     -0.4577152     110000
 -0.4337109E-01-0.2061276      0.000000      0.000000     -0.3320926     020000
 symplectic error: 0.1791276000985399E-011,
 g2= 0.1459810050619126E-010, g3= -.4534211894835494E-010

 ------------------------------------------------------------------------------
ESP
  0.7071544    -0.7070065      0.000000      0.000000     -0.7071262     100000
  0.7071069     0.7071594      0.000000      0.000000     -0.2928931     010000
 -0.4999687    -0.1075439E-03  0.000000      0.000000     -0.3749572     200000
   1.000028    -0.8968466E-04  0.000000      0.000000     -0.7499888     110000
  0.2071069    -0.7071261      0.000000      0.000000     -0.3320790     020000
 symplectic error: 0.2390266367238521E-013,
 g2= 0.2220446049250313E-015, g3= -.5014977148831701E-015
```

[1] It seems there are 4 errors in the `GIOS` calculation. These are marked with '*'. They may have been corrected later; I am using a 40-year-old version of `GIOS`.

This is for easy comparison with the Valetov paper[8, sections 4.1.1,4.1.5], which makes the claim that the disagreement between `GIOS` and the `ESP` routine in `COSY` is explained by the former being incorrect and the latter correct. In fact as we shall see, the opposite is the case. Note again, that `GES` values for second order agree within about 1% with the `GIOS` values above.

To investigate further, `GES` was run for the entry fringe field alone:

```
GES
  0.2783365E-09 0.9785234E-11  0.000000      0.000000      0.000000      000000
  0.9999786    -0.6774082E-02  0.000000      0.000000     -0.1093746E-01 100000
  0.1093732E-01 0.9999473      0.000000      0.000000     -0.5316821E-04 010000
   0.000000      0.000000     0.9999340    -0.1093709E-01  0.000000      001000
   0.000000      0.000000     0.1093735E-01 0.9999463      0.000000      000100
   0.000000      0.000000      0.000000      0.000000      1.000000      000010
  0.6645946E-04 0.1093725E-01  0.000000      0.000000      0.2734100E-02 000001
  0.4988888****-0.4231316E-02  0.000000      0.000000     -0.1053210E-01 200000
  0.1095947E-01-0.9979132****  0.000000      0.000000     -0.1217889E-03 110000
  0.6037104E-04-0.1093734E-01  0.000000      0.000000     -0.2734803E-02 020000
   0.000000      0.000000     0.2051418E-02 0.1284744E-01  0.000000      101000
   0.000000      0.000000    -0.2349555E-04-0.2029813E-02  0.000000      011000
  0.1085322E-02 0.6427298E-02  0.000000      0.000000     -0.2751460E-02 002000
   0.000000      0.000000     0.2185152E-01-0.2149924E-02  0.000000      100100
   0.000000      0.000000     0.1052144E-03 0.1427831E-06  0.000000      010100
 -0.2219241E-04-0.1910681E-02  0.000000      0.000000      0.5974595E-05 001100
  0.1754182E-03 0.1014972E-01  0.000000      0.000000      0.1093658E-01 100001
 -0.5467420E-02-0.2736212E-04  0.000000      0.000000      0.4651682E-04 010001
   0.000000      0.000000     0.6579551E-04 0.5479675E-02  0.000000      001001
 -0.6701825E-04-0.1093671E-01  0.000000      0.000000     -0.2734133E-02 000200
   0.000000      0.000000    -0.5467981E-02 0.5394849E-04  0.000000      000101
 -0.6645505E-04-0.5468580E-02  0.000000      0.000000     -0.2050348E-02 000002
 ------------------------------------------------------------------------------
symplectic error: 0.2318995512897258E-010
```

Note that this a nearly unity matrix; the exceptions are the two elements denoted by `****`. These correspond to $\Delta x/x^2$ and $\Delta P_x/(xP_x)$ and are in substantial agreement with the theoretical values (resp.) 2 and 1. They are expected to approach these values in the limit of zero aperture size, but the `COSY` Runge-Kutta integrater begins to break down at very small aperture (hence the $<$ 1% discrepancy). This confirms that `GES` is consistent with theory.

Further, it allows very fast calculation to second order, using the known entry and exit matrices. Thus, when user specifies a zero aperture in `GES`, the exact second order matrices are used and no integration through the fringe field occurs. However, the maps will only be good to second order and a warning message given if higher order is requested. As is well known, the third and higher orders diverge for zero aperture.

Usage of third and higher order is still possible maintaining symplecticity in a transport system with a second order electrostatic bender, using `COSY`'s handy symplectifier routine `SY`. From this we discover that eq. 1 expands as:

$$P_{x\mathrm{f}} = P_x\left(1 \mp \frac{x}{\rho} \pm \frac{x^2}{\rho^2} \mp \frac{x^3}{\rho^3} \pm ...\right) = \frac{P_x}{1 \pm x/\rho}. \tag{4}$$

The entrance and exit hard-edge fringe field maps `MAPENT` and `MAPEX` are coded as follows:

```
if lfr=0.;
write 6 'WARNING: zero fringe field length only valid for order up to 2';
   LOOP I 1 8;MAPENT(I) := XX(i);MAPEX(I) := XX(i);endloop;
MAPENT(1):=MAPENT(1)+DD(1)%(-1)/R;MAPENT(2):=MAPENT(2)-DD(2)%(-1)/R;sy MAPENT;
 MAPEX(1):= MAPEX(1)-DD(1)%(-1)/R; MAPEX(2):= MAPEX(2)-DD(2)%(-1)/R; sy MAPEX;
endif;
```

`lfr` is the length of the fringe field.

As a second test, we try the case of the bender as a thin lens. Then $h = 1/\rho$, let $c \equiv k/h$, (`kind` parameter in `GES` input). From [7, eq. 23], the nonlinear part of $P'_X = -\partial H/\partial X$ is integrated through the bend length $L = \rho\theta$ to give:

$$\Delta P_x \approx \frac{\theta}{\rho^2}\left(\left(-1 - \frac{3}{\gamma^2} + \left(2 + \frac{3}{2\gamma^2}\right)c - c^2\right)x^2 + \left(-\frac{c}{2\gamma^2} + c^2\right)y^2\right) \tag{5}$$

$$\Delta P_y \approx \frac{\theta}{\rho^2}\left(-\frac{c}{\gamma^2} + 2c^2\right)xy \tag{6}$$

Set the bend radius to $\rho = 1\,\mathrm{m}$ again, but angle to $\theta = 10\,\mathrm{mrad}$ so that the length is only 1 cm and $\theta/\rho^2 = 0.100$. The extreme non-relativistic ($\gamma = 1$) spherical ($c = 1$) case would be

$$\Delta P_x \approx 0.0100\left(\frac{-3}{2}x^2 + \frac{1}{2}y^2\right) \tag{7}$$

$$\Delta P_y \approx 0.0100\,(1\,xy) \tag{8}$$

The relevant `COSY` map rows are:

```
 -0.7499812E-04-0.1499925E-01  0.000000       0.000000     -0.1249942E-01 200000
   0.000000       0.000000    -0.4999708E-04 0.9999833E-02  0.000000      101000
  0.2499854E-04 0.4999417E-02  0.000000       0.000000     -0.2500083E-02 002000
```

For extreme relativistic, ($\gamma \gg 1$ not practical for electrostatic, but anyway...)

$$\Delta P_x \approx 0.0100 \left(0\,x^2 + 1\,y^2\right) \tag{9}$$
$$\Delta P_y \approx 0.0100\,(2\,xy) \tag{10}$$

The relevant COSY map rows are:

```
-0.1807110E-11-0.3614216E-09  0.000000      0.000000    -0.4999224E-02 200000
  0.000000      0.000000    -0.1204678E-11 0.1999967E-01  0.000000      101000
 0.4999833E-04 0.9999333E-02  0.000000      0.000000    -0.3333954E-06 002000
```

One more comparison non-relativistic cylindrical bend ($c = 0$),

$$\Delta P_x \approx 0.0100 \left(-4\,x^2 + 0\,y^2\right) \tag{11}$$
$$\Delta P_y \approx 0.0100\,(0\,xy) \tag{12}$$

The relevant COSY map row is:

```
-0.1999900E-03-0.3999600E-01  0.000000      0.000000    -0.1499867E-01 200000
```

(the other two relevant rows are missing since they are null).

These three examples all show good agreement with theory.

# 4 Conclusion

Current COSY version 10.2 does not include the variation of the orbit's curvature as a function of the independent variable $s$ in the fringe field of the electrostatic bends, and thus does not give correct higher order aberrations. This should be incorporated into the code; it is no different from the variation $\rho(s)$ that already correctly finds transfer maps for magnetic benders.

# References

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