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Alpha magnet in TRANSOPTR

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Abstract: This report describes the implementation of an alpha magnet in the TRANSOPTR code.

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1 Equation of motion in Cartesian frame

An alpha magnet is a special kind of achromatic bending magnet with the magnetic field of the form $B_z = Gx_c$, where $G = \frac{\mathrm{d}B_z}{\mathrm{d}x_c}$ is the constant field gradient, while x_c is the x coordinate in the Cartesian plane. Particles of any energy entering the magnet at an angle of 40.7° leave the magnet with zero dispersion.

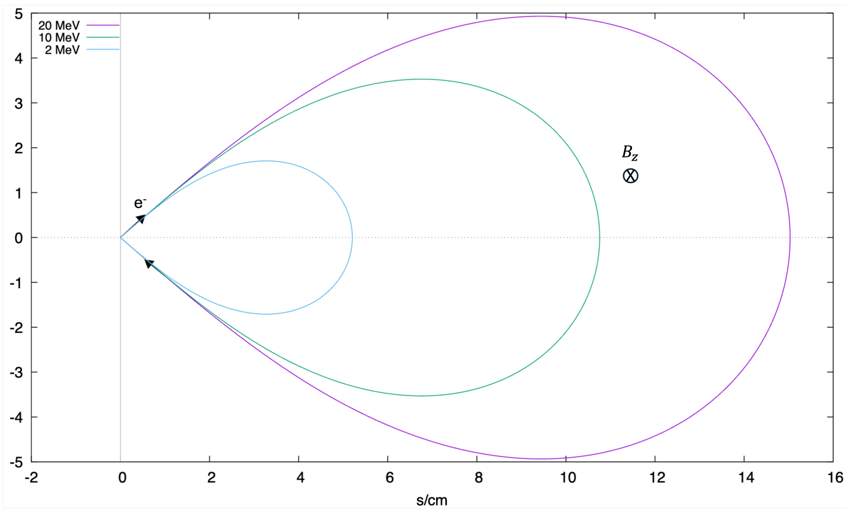

Figure 1: Trajectories of electrons at 2 (blue), 10 (green) and 20 MeV (purple) in an alpha magnet with $G = -0.1\,\mathrm{T/cm}$.

Assuming the entrance angle of the reference particle 40.7°, the equations of motion of the reference charged particle in an alpha magnet were given in [1]. Using the relationship of $\frac{\mathrm{d}x_c}{\mathrm{d}t} = \frac{\mathrm{d}x_c}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = v\frac{\mathrm{d}x_c}{\mathrm{d}s}$, we can rewrite them as follows:

$$\begin{aligned}\frac{\mathrm{d}^2 x_c}{\mathrm{d}s^2} &= \frac{qGx_c}{\gamma m v}\frac{\mathrm{d}y_c}{\mathrm{d}s}\\ \frac{\mathrm{d}^2 y_c}{\mathrm{d}s^2} &= -\frac{qGx_c}{\gamma m v}\frac{\mathrm{d}x_c}{\mathrm{d}s}\\ \frac{\mathrm{d}^2 z_c}{\mathrm{d}s^2} &= 0 \end{aligned} \tag{1}$$

where x_c, y_c and z_c are the coordinate in the Cartesian plane. In principle, eqn. 1 have to be solved numerically, but using the results from [2], the total length travelled can be estimated by

$$L = 1.91655\,\mathrm{cm}\sqrt{\frac{\beta\gamma}{G[\mathrm{T/cm}]}} \tag{2}$$

2 Equation of motion (Hamiltonian) in Frenet-Serret (FS) Frame

The Hamiltonian of the alpha magnet in FS frame expanded to second order is [3]

$$H_s = \frac{m^2c^2}{P^2} + \frac{P_x^2}{2P} + \frac{P_y^2}{2P} + \frac{P_z^2}{2P\gamma^2} - \frac{Gqx^2Q(s)}{2P} + \frac{Gqy^2Q(s)}{2P} - P_z x\Omega(s) + \frac{Gqx^2x_c(s)\Omega(s)}{2P} \tag{3}$$

where $\Omega(s) = \dfrac{Gqx_c(s)}{\gamma mv}$ is the curvature; $Q = \sin\alpha - \dfrac{Gqx_c(s)^2}{2\gamma mv}$; P is the momentum of the reference particle. The equation of motion are as follows:

$$\begin{aligned}
x' &= \frac{\partial H_s}{\partial P_x} = \frac{P_x}{P} \\
P_x' &= -\frac{\partial H_s}{\partial x} = \frac{Gqx\left(Q(s) - \Omega(s)x_c(s)\right)}{P} \\
y &= \frac{\partial H_s}{\partial P_y} = \frac{P_y}{P} \\
P_y' &= -\frac{\partial H_s}{\partial y} = -\frac{GqyQ(s)}{P} \\
z' &= \frac{\partial H_s}{\partial P_z} = \frac{P_z}{P\gamma^2} - x\Omega(s) \\
P_z' &= -\frac{\partial H_s}{\partial z} = 0
\end{aligned} \tag{4}$$

Note that when $(x, P_x, y, P_y, z, P_z) = 0$, all the first order terms in eq. 4 are zero. This is the main properties of the FS coordinate.

3 Infinitesimal Matrix

The infinitesimal transfer matrix $F(s)$ is defined as

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}s} = F\mathbf{X} \quad \text{where} \quad \mathbf{X} = \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ p_z \end{pmatrix}$$

Note that all p_x, p_y and p_z are the normalized momentum in TRANSOPTR, i.e. $p_x = \dfrac{P_x}{P}$ etc. (the Hamiltonian in eq. 3 is scaled by P). The corresponding F matrix can be obtained by taking $F = S\mathcal{H}$, where $\mathcal{H}$ is the Hessian Matrix (second order derivative of Hamiltonian),

$$\mathcal{H} = \begin{pmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x \partial P_x} & \cdots & \frac{\partial^2 H}{\partial x \partial P_z} \\ \frac{\partial^2 H}{\partial P_x \partial x} & \frac{\partial^2 H}{\partial P_x^2} & \cdots & \frac{\partial^2 H}{\partial P_x \partial P_z} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 H}{\partial P_z \partial x} & \frac{\partial^2 H}{\partial P_z \partial P_x} & \cdots & \frac{\partial^2 H}{\partial P_z^2} \end{pmatrix}$$

and S is the fundamental symplectic matrix,

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

Solving eqn 3, the F matrix is given as

$$F = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{Gq(Q(s) - x_c(s)\Omega(s))}{\mathrm{P}} & 0 & 0 & 0 & 0 & \Omega(s) \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{GqQ(s)}{\mathrm{P}} & 0 & 0 & 0 \\ -\Omega(s) & 0 & 0 & 0 & 0 & \frac{1}{\gamma^2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{5}$$

The evolution of the σ matrix along the reference orbit can be found by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}s} = F\sigma + \sigma F^T$$

The transfer matrix can then obtained by solving the integral of $M = I + F\,\mathrm{d}s$ (I is the identity matrix).

4 Implementation into TRANSOPTR

In TRANSOPTR, the subroutines for alpha magnet can be found in `TALPHA.f` in `transoptr/src`. The F matrix is as given in 10. The `SX` and `DSX` matrix are used to solve eqn. 1. The initial conditions are given in the subroutine `ABENDINTs` as:

```
!initial condition
sx(13,1) = 0.0 !xc0
sx(13,3) = 0.0 !yc0
```

```
      sx(13,2) = COS(alpha) !dxc0/ds
      sx(13,4) = SIN(alpha) !dyc0/ds
```

The solutions of the `SX` matrix (or x_c, y_c, x_c', y_c') are given in the `FS-x`, `FS-y`, `FS-x'`, and `FS-y'` columns respectively in the `fort.envelope`. Similar outputs are also produced in the `traj.dat` file, with an extra first column of the total path length travelled. The subroutine can be called by `ALPHABENDs(G,qlab)`, where G is the field gradient in T/cm, and qlab is the labelling in string. An example of its usage in `sy.f` and `data.dat` are given as follows:

data.dat :

```
10.0 0.0 0.0 0.510999 -1.0 0.0e-12
-1 5 1e-3 1e-4
0 0.0 1.0 0.0
0.1 0.01 0.1 0.01 0.1 0.05
1.0 1.0 1.0 1.0 1.0 1.0 1.0 1.0
0
0
0.001 20
10 0.0 0.95 20
```

sy.f :

```
      SUBROUTINE TSYSTEM
      COMMON/SCPARM/QSC,ISC,CMPS
      COMMON/MOM/P,BRHO,pMASS,ENERGK,GSQ,ENERGKi,charge,current
      COMMON/PRINT/IPRINT
      COMMON/SS/SX(13,6)
c
      CMPS=0.1 ! Number of cm per step, for plotting only
      wo=1.0 ! Weight aberration from optical elements
      G = -0.1 !T/cm
c
      call drift(0.0,".")
      call ALPHABENDs(G,"alpha")
      call drift(0.0,".")
c
      call print_transfer_matrix
      return
      end
```

Note that when the charge is negative, G should also be negative for a charge particle travelling clockwise.

5 Figure-of-merit

A sample calculation was performed using the ALPHABENDs subroutine in TRANSOPTR. The particle was taken to be an electron with energy of 2 MeV. The field gradient was assumed to be -0.01 T/cm (negative G for negative charge). The result is given below: The beam envelope in y is initially focused, defocused then focused again upon leaving the

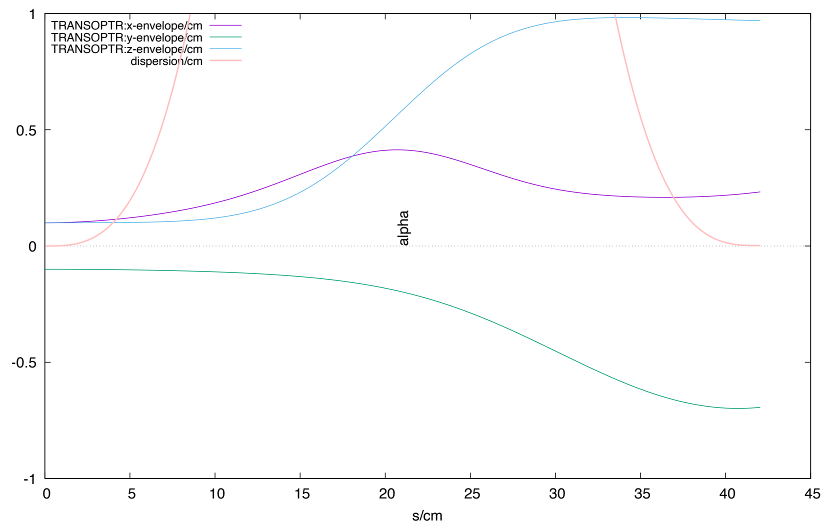

Figure 2: Beam envelope of a 2 MeV electron beam simulated by TRANSOPTR for $G = -0.01\,\mathrm{T/cm}$. Note that space charge is omitted here.

alpha magnet. This is consistent with the results shown in [4].

The corresponding transfer matrix is given as follows:

$$M = \begin{pmatrix} -0.9999831 & -21.02228 & 0 & 0 & 0 & 0.001802935 \\ 0.0002616404 & -0.9945130 & 0 & 0 & 0 & -5.84297 \times 10^{-6} \\ 0 & 0 & -0.7355179 & 68.99863 & 0 & 0 \\ 0 & 0 & -0.00684461 & -0.7174889 & 0 & 0 \\ 6.1728 \times 10^{-6} & 0.00188042 & 0 & 0 & 1 & -19.27766 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{6}$$

It was compared with the results generated by ELEGANT in [2] (transverse coordinate in m)

$$M_{ele} = \begin{pmatrix} -1.0 & -0.21 & 0 & 0 & 0 & 0 \\ 0 & -1.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.74 & 0.69 & 0 & 0 \\ 0 & 0 & -0.66 & -0.74 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.21 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{7}$$

as well as the transfer matrix generated using T. Planche's FFA alpha (electron with positive G)

$$M_{TP} =$$
$$\begin{pmatrix} -1 & -21.02646 & 0 & 0 & 0 & 0.19044 \times 10^{-2} \\ -0.622438 \times 10^{-4} & -1.001302 & 0 & 0 & 0 & -0.24114 \times 10^{-5} \\ 0 & 0 & -0.735459 & 68.99606 & 0 & 0 \\ 0 & 0 & -0.66093 \times 10^{-2} & -0.739658 & 0 & 0 \\ -0.258074 \times 10^{-5} & -0.195812 \times 10^{-2} & 0 & 0 & 1 & -19.27754 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The symplecticity is also checked as

$$M^T S M - S =$$
$$\begin{pmatrix} 0. & -3.52952 \times 10^{-6} & 0. & 0. & 0. & 0.000011544 \\ 3.52952 \times 10^{-6} & 0. & 0. & 0. & 0. & 0.00379629 \\ 0. & 0. & 0. & -5.42711 \times 10^{-6} & 0. & 0. \\ 0. & 0. & 5.42711 \times 10^{-6} & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. \\ -0.000011544 & -0.00379629 & 0. & 0. & 0. & 0. \end{pmatrix}$$

Overall the match are good except for the M_{56} term. The sign of the M_{56} term is flipped, as TRANSOPTR uses $z = -\beta_0 c\Delta t$ (so that particles with $\Delta t > 0$ has a greater z) as the fifth coordinate. Besides, the fifth coordinate in `elegant` is not Δt but the equivalent distance travelled, $s = \beta ct$. Tracking starts from $t_0 = 0$ through the accelerator.

$$
\begin{aligned}
s &= (\beta_0 + \Delta\beta)ct \\
P &= \beta\gamma mc \\
\frac{\mathrm{d}P}{\mathrm{d}\beta} &= \gamma mc + \beta\frac{\mathrm{d}\gamma}{\mathrm{d}\beta}mc \\
&= \gamma mc + \beta^2\gamma^3 mc \\
&= \gamma^3 mc \\
\therefore \Delta\beta &\approx \frac{\Delta P}{\gamma_0^3 mc} \\
\Delta\beta ct &= \frac{\Delta\beta}{\beta_0}L \\
&= \frac{\Delta P}{\beta_0\gamma_0^3 mc}L \\
&= \frac{\Delta P}{P_0\gamma_0^2}L \\
\therefore s &= \beta_0 ct + \frac{\Delta P}{P_0\gamma_0^2}L
\end{aligned}
$$

In order to account for the difference, M_{56} in TRANSOPTR can be scaled by L/γ^2. For instance, in this case, $-19.27 - \frac{42.04}{4.91} = -21.01$ cm, which is consistent with $M_{56} = 0.21\,\mathrm{m}$ from `elegant`.

5.1 Space charge effect

If space charge is included, for instance at Q=0.2, 20, and 200 pC, the beam envelope travelled through the alpha magnet with $G = -0.01\,\mathrm{T/cm}$ and 20 cm of drift are given in Fig. 3. Note that in this case the beam dispersion is not zero except for 0.2 pC. This means that the space charge from the beam mess up the achromatic property of the alpha magnet significantly.

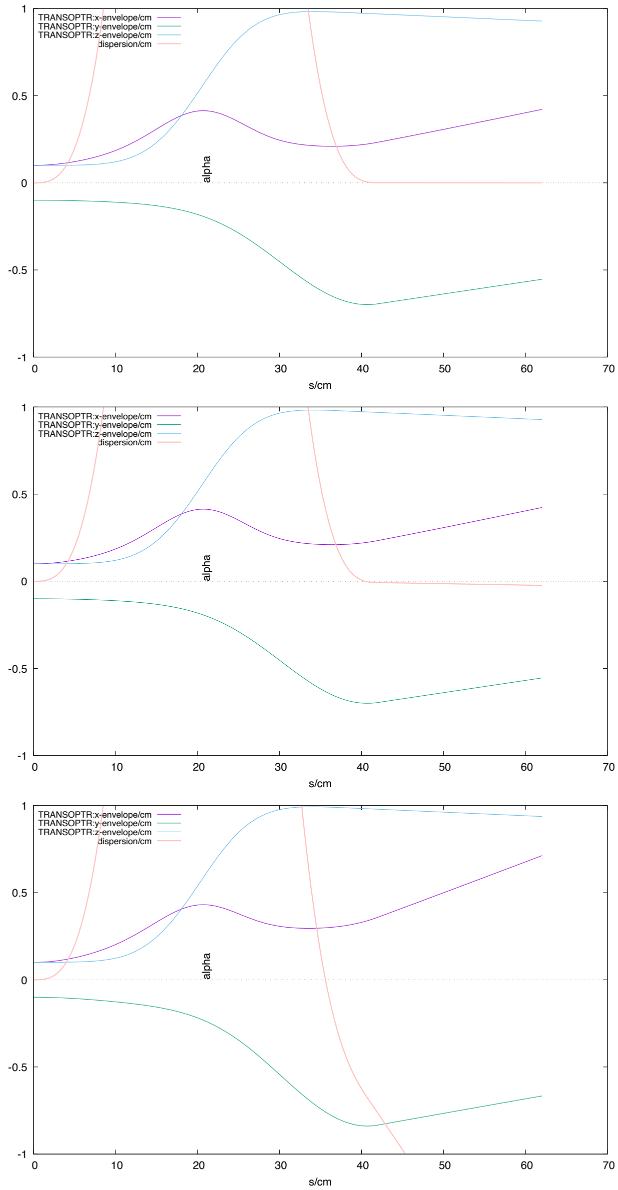

Figure 3: Beam envelope of a 2 MeV *(top)* 0.2, *(middle)* 2, *(bottom)* 200 pC electron beam simulated by TRANSOPTR for $G = -0.01\,\mathrm{T/cm}$.

References

[1] H. A. Enge, Achromatic Magnetic Mirror for Ion Beams, Review of Scientific Instruments 34 (4) (1963) 385–389. arXiv:https://pubs.aip.org/aip/rsi/article-pdf/34/4/385/8348776/385_1_online.pdf, doi:10.1063/1.1718372.
URL https://doi.org/10.1063/1.1718372

[2] J. Lewellen, F. Krawczyk, et al., Simulation of alpha magnet elements in dipole-only tracking codes, Proceeding of FEL2014, Basel, Switzerland, THP023 (2014) 735–738.

[3] H. W. Koay, Alpha magnet, Tech. rep., TRI-BN-23-25, November 29 (2023).

[4] Electron beam dynamics in the 3d magnetic field of alpha magnet at the pbp-cmu electron linac laboratory, Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment 916 (2019) 102–115. doi:https://doi.org/10.1016/j.nima.2018.11.034.
URL https://www.sciencedirect.com/science/article/pii/S0168900218315808

A Changing the independent variable

In any case if ϕ given in [1] is used, the independent variable can be changed from s to ϕ. This causes a change in the F matrix too.

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\phi} = \frac{\mathrm{d}\sigma}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}\phi} = F_\phi\sigma + \sigma F_\phi^T$$
$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}\phi} = F_\phi\mathbf{X}$$

Note that F_ϕ indicates the use of ϕ as its independent variable. The transfer matrix becomes $M = I + F_\phi\,\mathrm{d}\phi$. F_ϕ can be obtained from F_s by applying

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}\phi} = \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x_c}\frac{\mathrm{d}x_c}{\mathrm{d}\phi}$$
$$F_\phi\mathbf{X} = \left(v\frac{\mathrm{d}t}{\mathrm{d}x_c}\frac{\mathrm{d}x_c}{\mathrm{d}\phi}\right)F_s\mathbf{X}$$
$$F_\phi = \left(v\frac{\mathrm{d}t}{\mathrm{d}x_c}\frac{\mathrm{d}x_c}{\mathrm{d}\phi}\right)F_s \tag{8}$$

Substituting the derivative of 1 and $x_c(\phi) = \cos\phi\left[\frac{2\gamma mv(1+a)}{Gq}\right]^{1/2}$. into 8,

$$F_\phi = \left(v\frac{1}{v\sqrt{1-Q(\phi)^2}}(-\sin\phi)\left[\frac{2\gamma mv(1+a)}{Gq}\right]^{1/2}\right)F_s$$
$$F_\phi = \left(\frac{-\sin\phi}{\sqrt{1-Q(\phi)^2}}\left[\frac{2\gamma mv(1+a)}{Gq}\right]^{1/2}\right)F_s \tag{9}$$

The parenthesis of 9 is

$$\frac{\mathrm{d}s}{\mathrm{d}\phi} = \frac{-\sin\phi}{\sqrt{1-Q(\phi)^2}}\left[\frac{2\gamma mv(1+a)}{Gq}\right]^{1/2}$$

where $Q(\phi) = a\sin^2\phi - \cos^2\phi$ and $a = \sin\alpha$. Substituting everything into the F-matrix in [3],

$$F_\phi = \frac{-\sin\phi}{\sqrt{1-Q(\phi)^2}}\left[\frac{2\gamma mv(1+a)}{Gq}\right]^{1/2}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{Gq(Q(\phi)-x_c(\phi)\Omega(\phi))}{\mathrm{P}} & 0 & 0 & 0 & 0 & \Omega(\phi) \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{GqQ(\phi)}{\mathrm{P}} & 0 & 0 & 0 \\ -\Omega(\phi) & 0 & 0 & 0 & 0 & \frac{1}{\gamma^2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{10}$$

where $\Omega(\phi) = \frac{qGx_c(\phi)}{P}$.