Simple but General Solenoid Model

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Motivation

We would like a model that is sufficiently accurate without having to use a field data file. For linear optics, this would achieve two worthwhile goals: the lens characterization is efficient, requiring only 3 parameters rather than a large file of field data, and the computation is efficient.

Note added Aug. 2021: One of the 3 parameters is not actually free at all, if the number of turns is known. This note has been modified to reflect this fact.

Degrees of Freedom

The first order transfer matrix of a symmetric ($B(s) = B(-s)$) solenoid depends upon only 3 parameters. These can be thought of as the rotation angle, the effective strength, and the effective length.

It is well-known that the rotation angle is

$$\theta = \frac{\int B(s)ds}{2B\rho} = \frac{\mu_0 NI}{2B\rho} \tag{1}$$

where $B(s)$ is the on-axis field, $B\rho$ is the particle momentum per charge. The RHS arises from Ampere's law: $\mu_0 = 4\pi \times 10^{-7}$ Tesla-metres per Ampere, the permeability of free space (N is the number of turns in the coil, I is the current).

Applying a rotation of $-\theta$ to the matrix completely decouples it resulting in two identical 2×2 matrices for x and y. These matrices have only 4 elements, but symplecticity means the determinant is 1, and symmetry means the diagonal elements are equal, leaving only two degrees of freedom. These can be identified as a length and a strength, just as in the quadrupole case.

The matrix can thus be written:

$$\begin{pmatrix} \cos KL & \sin KL/K \\ -K\sin KL & \cos KL \end{pmatrix} \qquad (2)$$

It remains to find a method to distill the known function $B(s)$ down to the two parameters K and L.

Tophat Case

By "tophat" is meant that the solenoid field $B(s) = B_0$ is a constant within a length L and zero outside. This is the solenoid that appears in such codes as `TRANSPORT`. It can be shown that in this case

$$KL = \theta = \frac{B_0 L}{2B\rho} \tag{3}$$

Thus this model only has only two parameters (B_0 and L) that determine both the focusing and the rotation. and so cannot describe the general case.

Thin Lens Case

This can be a sufficiently good approximation if $KL \ll 1$. Then

$$\frac{1}{f} = K^2 L \tag{4}$$

so roughly speaking, it also applies if $f \gg L$. Here again, there are only two parameters, f and θ. Insufficient in the general case where it may happen that $f \sim L$.

Elsewhere, I show that the equation of motion for r is

$$r'' + \frac{B^2}{(2B\rho)^2} r = 0 \tag{5}$$

and thus when the thin lens approximation is applicable, we find

$$\frac{1}{f} = \frac{\int B^2 ds}{(2B\rho)^2} \tag{6}$$

General Case

The equation of motion is a Hill's equation, and there is no closed-form solution for arbitrary $B(s)$ except (see Magnus in Wikipedia) as an infinite series of nested commutators.

But it is a simple matter to solve by Runge-Kutta, as is done in TRANSOPTR (first order) and COSY-∞ (any order). Let us say that through this technique, the matrix element R_{21} is known for the given $B(s)$.

$$R_{21} = -K \sin KL \tag{7}$$

It does not matter that the integration range is arbitrary, since R_{21} is insensitive to drifts either before or after the solenoid. It is only necessary that the RK integration start and end where $B \ll B_0$.

Capitalizing on the known result that in the limit $KL \to 0$, $R_{21} = -\frac{\int B^2 ds}{(2B\rho)^2}$, we find:

$$\frac{\sin KL}{KL} = \frac{-R_{21}(2B\rho)^2}{\int B^2 ds} \tag{8}$$

Equations 7 and 8 constitute two equations in the two unknowns K and L.

From K, we deduce B_0 and model the solenoid exactly as if it is a tophat solenoid with length L and field B_0. This causes the wrong rotation angle $\theta_{\rm TH} = \frac{B_0 L}{2B\rho}$. But we know the correct one from Ampere's law, so we simply follow the tophat solenoid with a rotation by angle $\theta - \theta_{\rm TH}$.

Parameterizing

A general symmetric solenoid, we need only B_0, L, and $\int B ds$ or equivalently, NI. But perhaps more intuitively, we can express the integral as a "rotational effective length" $L_{\rm rot} = \int B ds / B_0 = \mu_0 NI/B_0$, which is independent of excitation. Thus we have:

1. a scale parameter that converts current to effective field B_0,
2. L the effective length for focusing,
3. $L_{\rm rot}$ the effective length for rotation.

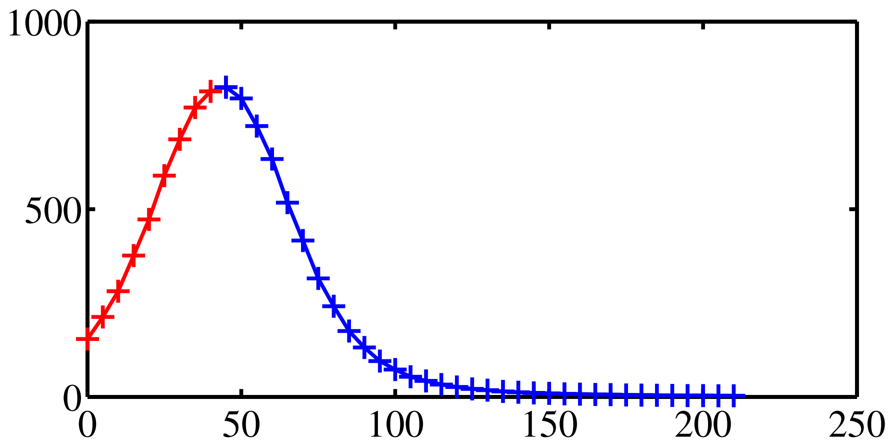
This is a handy scheme, as the first is on order of the maximum of $B(s)$, and the two effective lengths are on order of the solenoid's physical length.

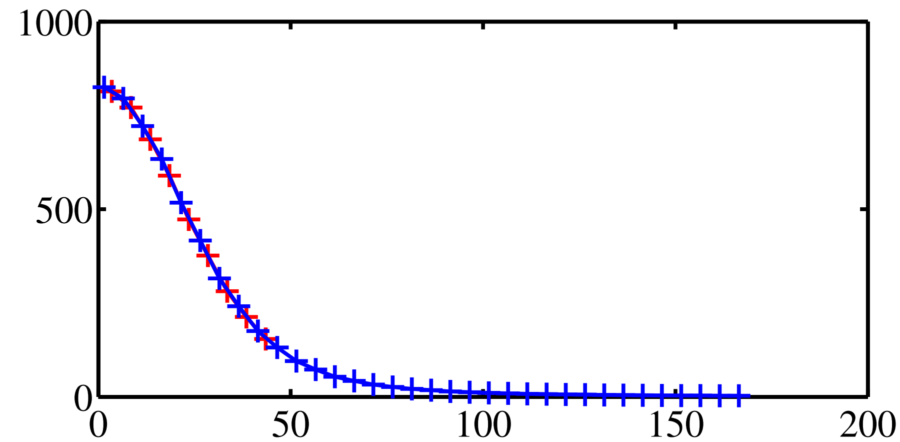
In a tophat code, solenoid has length L, field B_0 and a subsequent applied rotation of $\Delta\theta = \frac{B_0(L_{\rm rot}-L)}{2B\rho} = \frac{\mu_0 NI - B_0 L}{2B\rho}$.

Example: Egun Solenoid at 5 Amps, with New Clamps

Note: In December 2014, the EGUN solenoid was modified, with new field clamps fixed to the solenoid body. This had the effect of increasing the focal power while leaving the rotation practically unchanged.

The measured data are 43 values of B_z at equally spaced z, every 5mm. The solenoid centre is found by taking the leftside data (red, below) and reflecting about a point and adjusting this point until these data coincide as closely as possible with the rightside data (blue).





This results in data from -43.5 mm to $+166.5$ mm. The right side data from $+43.5$ mm to $+166.5$ mm are then used to augment the negative side, resulting in data from -166.5 mm to $+166.5$ mm.

Then the 9 rightmost points are used to fit to $B_z = c_1(c_2^2 + z^2)^{-3/2}$, the field from an isolated current loop, to extend the tails to $|z| = 400$ mm.

Numerical (Simpson's rule) integration then gives $\int B^2 ds = 2.763 \times 10^6$ G^2-cm. For $\int B ds$ we find 5104 G-cm, implying $N = 812.33$ turns. This should be an integer, but is uncertain by more than 0.1% so not clear wheter to round up or down. Regrettably, the manufacturer of the coil did not provide a precise value for N (the actual number on the drawing is 722, but later was claimed to be 10% higher than this). This is a missed opportunity. Hopefully, we can disassemble the solenoid at some point and simply count the turns.

Integrating equations of motion using TRANSOPTR, which uses cubic spline interpolation of the field data, over this total $\Delta z = 800$ mm, we find for 300 keV electrons, $R_{21} = -K \sin KL = -0.13449$/cm. The thin lens value for R_{21} is $-K^2 L = -\frac{\int B^2 ds}{(2B\rho)^2} = -0.15653$/cm, so we have $\frac{\sin KL}{KL} = 0.8592$ which resolves to $KL = 0.9396$.

Finally, effective length for focusing:

$$L = 0.9396^2/0.15653\,\text{cm} = 5.640\,\text{cm}. \tag{9}$$

The effective tophat field is

$$B_0 = 2B\rho K = 699.9\,\text{G} \tag{10}$$

The effective length for rotation is

$$L_{\mathrm{rot}} = \frac{5104\,\text{G-cm}}{B_0} = 7.292\,\text{cm}. \tag{11}$$

Matrices Comparison

These 3 parameters were used in the TRANSOPTR calculation to compare with the direct equation of motion integration. In each case, a 40 cm negative drift was attached before and after, to make this a zero-insertion-length matrix. Results below; These matrices are for (x, x', y, y'), in metres and radians.

Integration through field:

0.3371	0.0017	0.9067	0.0045
−4.6874	0.3369	−12.6061	0.9061
−0.9067	−0.0045	0.3371	0.0017
12.6061	−0.9061	−4.6874	0.3369

3-parameter ('Tophat') model:

0.3379	0.0016	0.9086	0.0042
−4.6874	0.3379	−12.6060	0.9086
−0.9086	−0.0042	0.3379	0.0016
12.6060	−0.9086	−4.6874	0.3379

And here is the same calculation for the field scaled by $1/5$, corresponding to the solenoid at 1 Amp excitation:

Integration through field:

0.9706	0.0002	0.2406	0.0000
−0.6041	0.9706	−0.1497	0.2406
−0.2406	−0.0000	0.9706	0.0002
0.1497	−0.2406	−0.6041	0.9706

3-parameter (‘Tophat’) model:

0.9706	0.0002	0.2406	0.0000
−0.6042	0.9706	−0.1497	0.2406
−0.2406	−0.0000	0.9706	0.0002
0.1497	−0.2406	−0.6042	0.9706

Current Loop Model

Axial field from a current loop:

$$B(s) = \frac{B_0}{(1 + (s/a)^2)^{3/2}}$$

can be integrated analytically:

$$\int B ds = 2aB_0, \quad \int B^2 ds = \frac{3\pi}{8} a B_0^2.$$

Using the numerical integrals of the measured $B(s)$ and solving for the two parameters, find:

$a = 27.77$ mm, and $B_0 = 919$ Gauss per 5 Amps.

Surprisingly in spite of having only two parameters rather than 3, this is a very good model in first order. COSY-∞ procedure CMR can directly find the transfer matrix. Here are the matrices at 300 keV for 5 Amps and 1 Amp respectively. Compare with matrices above: at 5 Amps, errors are no larger than 2%.

0.3382	0.0015	0.9100	0.0039
−4.7553	0.3383	−12.7937	0.9100
−0.9100	−0.0039	0.3382	0.0015
12.7937	−0.9100	−4.7553	0.3383

0.9706	0.0001	0.2406	0.0000
−0.6045	0.9706	−0.1499	0.2406
−0.2406	−0.0000	0.9706	0.0001
0.1499	−0.2406	−0.6045	0.9706

The matrix contains precisely correct rotation, so to make it exact would require a slight thin lens correction. As it is, it's a nice model: quite accurate, analytic, and only 2 parameters.

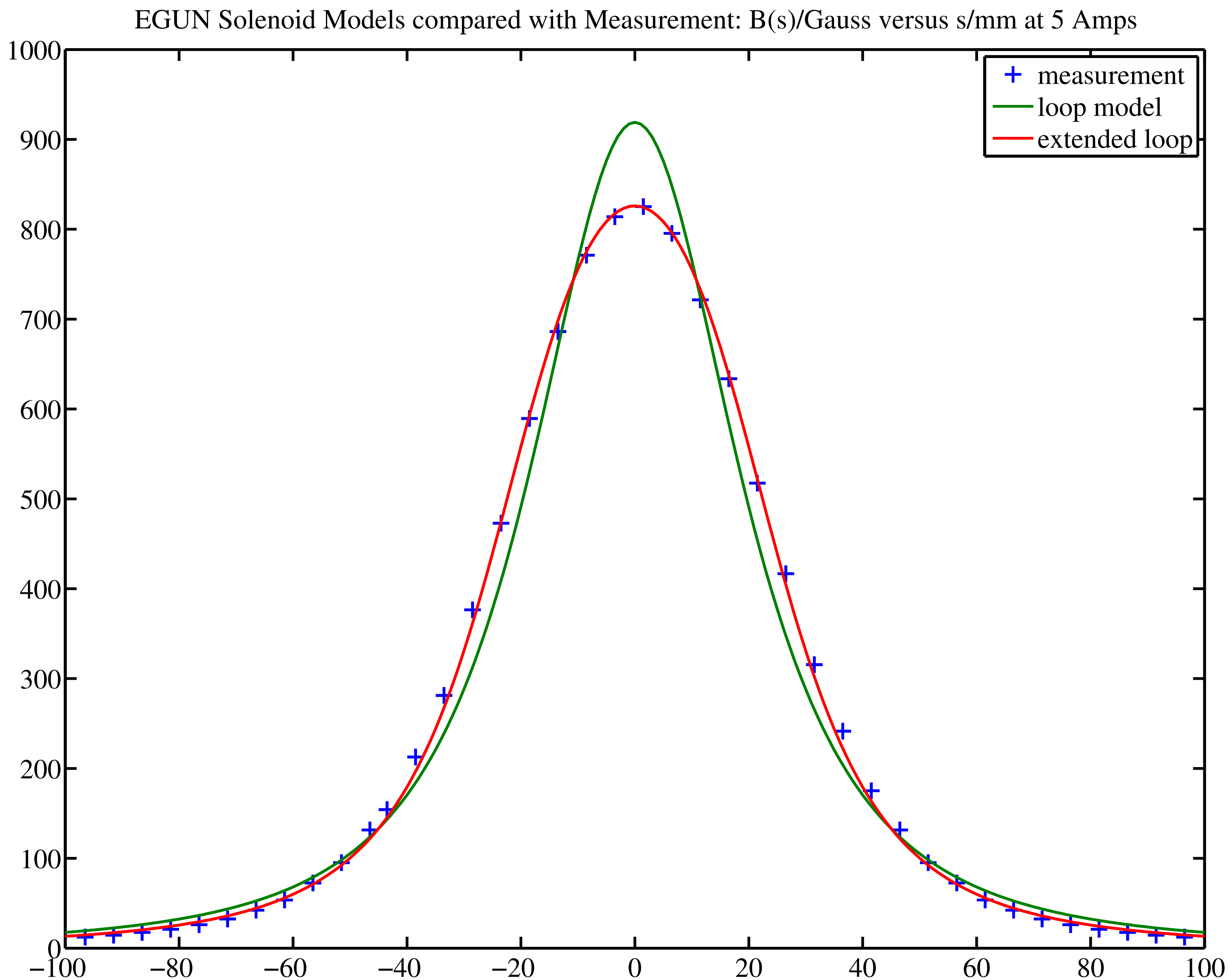
Extended Current Loop Model

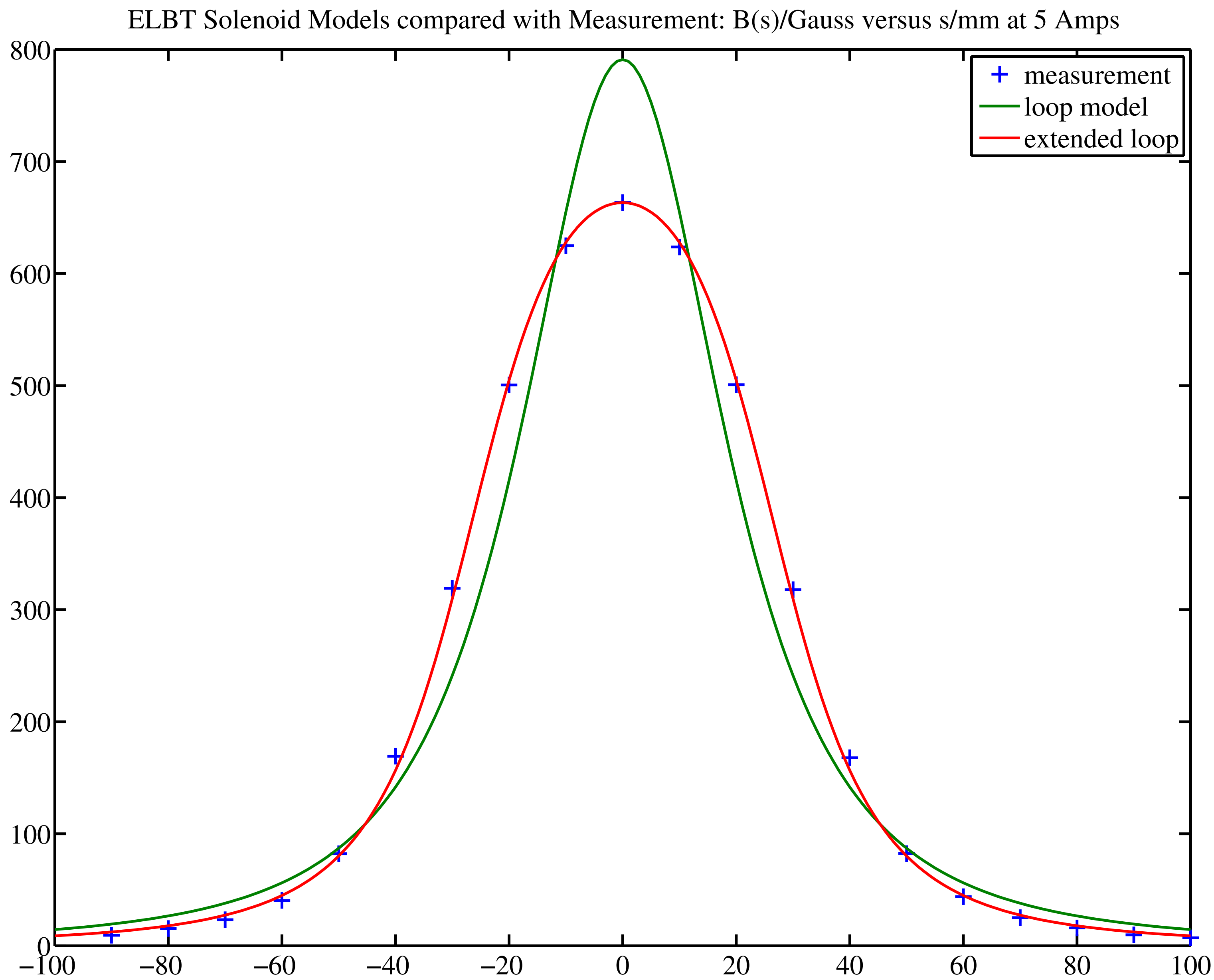
Lastly, here is another exact model. If the loop is extended in the s direction and we leave the extension as a free parameter, it should be clear that we can get an exact model in first order. Here is the on-axis field (a is the radius and b the half-length):

$$B(s) = \frac{B_0\sqrt{a^2+b^2}}{2b}\left(\frac{s+b}{\sqrt{(s+b)^2+a^2}} - \frac{s-b}{\sqrt{(s-b)^2+a^2}}\right)$$

Not surprisingly, if we match the integrals of $B(s)$ and $B^2(s)$, the measurements will agree very closely with the model (see next slide). It's because the actual solenoid is much more like this model than the current loop. The extracted parameters for the EGUN solenoid are

$a = 22.72$ mm, and $b = 20.94$ mm, and $B_0 = 826.$ Gauss


EGUN Solenoid Models compared with Measurement: B(s)/Gauss versus s/mm at 5 Amps
measurement
loop model
extended loop
1000
900
800
700
600
500
400
300
200
100
0
−100
−80
−60
−40
−20
0
20
40
60
80
100



ELBT Solenoid Models compared with Measurement: B(s)/Gauss versus s/mm at 5 Amps
measurement
loop model
extended loop
800
700
600
500
400
300
200
100
0
−100
−80
−60
−40
−20
0
20
40
60
80
100

Here are the EGUN solenoid matrices for 5 A and 1 A.

0.3373	0.0016	0.9073	0.0044
−4.7001	0.3374	−12.6411	0.9073
−0.9073	−0.0044	0.3373	0.0016
12.6411	−0.9073	−4.7001	0.3374

0.9706	0.0002	0.2406	0.0000
−0.6042	0.9706	−0.1498	0.2406
−0.2406	−0.0000	0.9706	0.0002
0.1498	−0.2406	−0.6042	0.9706

This model has no advantage in first order against the tophat model, since the matrices are practically identical.

For Higher Order, use COSY

But for higher order, this model has a decided advantage: it can be expected to give very accurate higher order maps in COSY-∞.

In that code, it is called as follows:

```
a:=0.02272;b:=0.020937;len:=b*2;
dl -b;
cmsi 5. 812.33/len a len ;
dl -b;
```

The arguments of CMSI are current, number of turns per unit coil length, radius a, length $2b$. The current loop model has of course the same number of turns. Here is the code:

```
cmr 5*812.33 0.02777 ;
```

Here for example is the third order map for 5 Amps excitation. The same calculation for the current loop model gives aberrations 30% too large.

```
  0.3372466     -4.700077      0.9073092     -12.64111      0.0000000E+00 100000
  0.1629359E-02 0.3374204      0.4382251E-02 0.9072446      0.0000000E+00 010000
 -0.9073092      12.64111      0.3372466     -4.700077      0.0000000E+00 001000
 -0.4382251E-02-0.9072446      0.1629359E-02 0.3374204      0.0000000E+00 000100
  -159.5256     -223.8481       39.04732     -7122.574      0.0000000E+00 300000
   3.540171     -226.7230     -0.1984800      25.07852      0.0000000E+00 210000
 -0.3713960      8.897569     -0.3504820E-01 -2.297772      0.0000000E+00 120000
  0.1116121E-02-0.3528063      0.8082290E-02 0.7336164E-01  0.0000000E+00 030000
  -39.04732      7122.574      -159.5256     -223.8481      0.0000000E+00 201000
   21.74026     -156.0170       2.511158      281.7267      0.0000000E+00 111000
 -0.2373793      22.36744      -1.073608      3.113791      0.0000000E+00 021000
  -21.54178      130.9385       1.029013     -508.4496      0.0000000E+00 200100
  0.2724211     -20.06967      0.7022112      5.783778      0.0000000E+00 110100
 -0.7910480E-02-0.7336372E-01 0.1127485E-02-0.3528057      0.0000000E+00 020100
  -159.5256     -223.8481       39.04732     -7122.574      0.0000000E+00 102000
   1.029013     -508.4496       21.54178     -130.9385      0.0000000E+00 012000
   2.511158      281.7267      -21.74026      156.0170      0.0000000E+00 101100
  0.7022112      5.783778     -0.2724211      20.06967      0.0000000E+00 011100
  -1.073608      3.113791      0.2373795     -22.36744      0.0000000E+00 100200
  0.1122955E-02-0.3528057      0.7901817E-02 0.7336342E-01  0.0000000E+00 010200
  -39.04732      7122.574      -159.5256     -223.8481      0.0000000E+00 003000
  0.1984800     -25.07852       3.540171     -226.7230      0.0000000E+00 002100
  0.3504888E-01  2.297772     -0.3713962      8.897569      0.0000000E+00 001200
 -0.8114806E-02-0.7336092E-01 0.1125576E-02-0.3528059      0.0000000E+00 000300
```

Parameters for our solenoids

This is for the 3-parameter tophat model.

Solenoid	I	N	B_0/I	L	$L_{\rm rot}$
Egun(old clamps)	5.0 Amp	812.6 turns	121.2 G/A	6.32 cm	8.42 cm
	0.5 Amp	815.6 turns			
Egun(new clamps)	5.0 Amp	812.3 turns	140.0 G/A	5.64 cm	7.29 cm
ELBT	10.0 Amp	689.9 turns	119.4 G/A	5.78 cm	7.26 cm
	5.0 Amp	685.3 turns	116.9 G/A	5.85 cm	7.37 cm
	1.0 Amp	684.3 turns			

$N = \int B ds/(\mu_0 I)$, numerically, if the integral over current is in units of Gauss-cm/A, the number of turns is found by dividing by $0.4\pi = 1.2566$.

Measurements made at low excitation (0.5 and 1 Amp) cannot give reliable values for L because KL is small and $\sin KL/KL$ is very nearly 1. For example, changing a few field values in the measured field by amounts equal to the Hall probe uncertainty could change L by 100%. This just reflects the fact that at the weak end, this parameter is of diminishing importance for modelling the solenoid; it is simply a thin lens with a rotation and the rotation is accurately known from $\int Bds$.

Interestingly, the EGUN solenoid with new field clamps is now much more consistent with the ELBT type. It only has more field per current, and this can be explained by the larger number of turns.

Parameter Summary, all 3 models

Model	Parameter	EGUN	ELBT
Tophat	B_0/I	140.0 G/A	116.9 G/A
	L	5.64 cm	5.85 cm
	L_{rot}	7.29 cm	7.37 cm
Current Loop	B_0/I	183.8 G/A	158.2 G/A
	a	2.78 cm	2.73 cm
Ideal Solenoid	B_0/I	165.2 G/A	132.7 G/A
	a	2.27 cm	1.95 cm
	b	2.09 cm	2.61 cm

Do Tophat Results Depend upon Strength?

Performed the following exercise. Fitted the 3 parameters for the ELBT solenoid at 1 Amp, but scaled the measured field by factor α.

α	B_0/α/G	L/cm	L_{rot}/cm
1.0	121.72	5.206	7.065
2.0	115.23	5.809	7.463
3.0	114.19	5.915	7.531
4.0	113.89	5.946	7.551
5.0	113.82	5.954	7.556
6.0	113.85	5.951	7.554
7.0	113.94	5.941	7.548
8.0	114.08	5.927	7.539
9.0	114.26	5.908	7.526
10.0	114.48	5.885	7.512
11.0	114.75	5.857	7.494
12.0	115.06	5.826	7.474
13.0	115.43	5.789	7.450
14.0	115.85	5.747	7.423
15.0	116.34	5.698	7.392
16.0	116.92	5.642	7.356
17.0	117.59	5.578	7.314
18.0	118.37	5.504	7.265
19.0	119.30	5.419	7.208
20.0	120.41	5.320	7.142
21.0	121.75	5.204	7.064

I did same again for the 5 Amp case:

5α	$B_0/(5\alpha)$/G	L/cm	$L_{\rm rot}$/cm
1.0	128.09	4.870	6.723
2.0	118.92	5.651	7.242
3.0	117.48	5.790	7.331
4.0	117.05	5.833	7.358
5.0	116.91	5.847	7.366
6.0	116.91	5.847	7.366
7.0	116.99	5.839	7.361
8.0	117.12	5.826	7.353
9.0	117.30	5.808	7.342
10.0	117.52	5.786	7.328
11.0	117.79	5.759	7.311
12.0	118.11	5.728	7.291
13.0	118.48	5.692	7.268
14.0	118.92	5.651	7.242
15.0	119.43	5.603	7.211
16.0	120.02	5.547	7.175
17.0	120.72	5.484	7.134
18.0	121.54	5.410	7.086
19.0	122.51	5.324	7.029
20.0	123.68	5.224	6.963
21.0	125.09	5.107	6.884

This extends all the way to the point that $KL = \pi$, far beyond what anyone would do with such a quad. In fact, our working range will be zero to about 4 Amps. In this range, it is a good approximation to use just the value at $\alpha = 4$, or more directly the value for the field survey done at $I = 5$ Amps. The fact that lower excitations give somewhat shorter effective lengths is not a concern since at this limit, the solenoid is more nearly a thin lens. But it is very important that the product of B_0 and L_{rot} be correct; it does not depend upon excitation.

Thus for either solenoid, it is a sufficiently good model to use the parameters fitted for the 5 Amp survey.