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Feasibility study for the cylindrically symmetric magnetic inflector

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Abstract: This note studied the magnetic inflector with the cylindrically symmetric structure for axial injection into the cyclotron. A mirror like field is designed in the injection hole to steer and focus the beam.

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Contents

1 Introduction

The spiral inflector steers the beam from the bore in the main magnet into the median plane to achieve the axial injection with an external ion source. In a conventional electrostatic inflector, the injection beam energy is limited by the breakdown voltage on the electrodes. While the injection intensity is also limited by the small aperture in the electrostatic inflector. Magnetic inflector is promising to overcome these disadvantages.

Recently, There are two types of magnetic inflector. One is the passive type which uses the iron in the injection hole to produce the required magnetic field.[1] The other is the active one which uses a permanent magnet array.[3] The passive type is more robust because there is no concern about the magnet degaussing under the high beam loss in the injection hole. But it is only a concept, that has no existing design. To demonstrate the technology, we studied the inflection conditions and focal property of the passive magnetic inflector with a cylindrically symmetric structure.

2 Reference orbit

2.1 Motion equations

In a cylindrically symmetric system. The magnetic vector potential A only consists of the azimuthal component A_θ. Thus, the hamiltonian using cylindrical coordinates is written as

$$H = \sqrt{P_r^2 c^2 + P_z^2 c^2 + c^4 m_0^2 + \frac{c^2 \left(P_\theta - qr\,\mathrm{A}_\theta\left(r,z\right)\right)^2}{r^2}} \tag{1}$$

Where the canonical momenta are

$$\begin{aligned} P_r &= p_r \\ P_\theta &= \gamma m_0 \theta' r^2 + qrA_\theta \\ P_z &= p_z \end{aligned} \tag{2}$$

Since the hamiltonian is independent from theta because of the cylindrical symmetric. We can easily find that the canonical momentum in the azimuthal direction is a constant. That results the azimuthal momentum p_θ a function of r-z coordinate. Define a potential function as

$$U = \frac{\left(P_\theta - qr\,\mathrm{A}_\theta\left(r,z\right)\right)^2}{2\gamma m_0 r^2} \tag{3}$$

Using the defined potential function U, the motion equation could be written in the following form, essentially as obtained by Glaser [1]

$$
\begin{aligned}
P_r' &= \frac{\partial U}{\partial r} \\
P_\theta' &= 0 \\
P_z' &= \frac{\partial U}{\partial z}
\end{aligned} \tag{4}
$$

Substitled eqt. (2) into eqt.(4). The motion equation is written as

$$
\begin{aligned}
&\theta' = \theta_0' r_0^2/r^2 + \frac{q}{\gamma m_0 r^2}(r_0 \,\mathrm{A}_{\theta 0} - r\,\mathrm{A}_\theta) \\
&\gamma m_0 r'' - \frac{\partial U}{\partial r} = 0 \\
&\gamma m_0 z'' - \frac{\partial U}{\partial z} = 0
\end{aligned} \tag{5}
$$

2.2 Possible analytical solution for the reference orbit

Define a magnetic vector potential to decouple the motion in the 3 direction

$$
A_\theta = \frac{A_0 r_0}{r} + B_0 \left(z - z_0\right) \tag{6}
$$

The motion equation is written as

$$
\begin{aligned}
P_r' &= \frac{q}{r^3 \gamma m_0} \left(P_\theta - q A_0 r_0 - q B_0 r \left(z - z_0\right)\right) \left(P_\theta - q A_0 r_0\right) \\
P_\theta' &= 0 \\
P_z' &= \frac{q}{r \gamma m_0} B_0 \left(P_\theta - q A_0 r_0 - q r B_0 \left(z - z_0\right)\right)
\end{aligned} \tag{7}
$$

The equation 2 is solved using the initial condition $p_\theta = 0, \theta = 0$

$$
P_\theta = q A_0 r_0 \tag{8}
$$

substitult P_θ into the other 2 equations, the motion equation is totally decoupled, in combination with , the momenta and coordinates are easily solved as

$$
\begin{aligned}
r &= \frac{p_{r0}}{m_0\gamma} t + r_0 \\
p_r &= p_{r0} \\
\theta &= \int_0^t p_\theta dt \\
p_\theta &= -\frac{p_{z0}}{p_{r0}t + m_0\gamma r_0} \sin(\frac{qB_0}{m_0\gamma} t) \\
z &= \frac{p_{z0}}{qB_0} \sin(\frac{qB_0}{m_0\gamma} t) + z_0 \\
p_z &= p_{z0} \cos(\frac{qB_0}{m_0\gamma} t)
\end{aligned}
\tag{9}
$$

The momenta in z direction arrives at 0 at the exit of the magnetic inflector. We replace the $\frac{qB_0}{m_0\gamma}t$ with b, p_{r0}/qB_0 with R_r and p_{z0}/qB_0 with R_z, the motion equation is written as

$$
\begin{aligned}
r &= R_r b + r_0 \\
\theta &= \int_0^b \frac{\sin{(b)}}{\frac{R_r}{R_z} b + \frac{r_0}{R_z}} db \\
z &= R_z sin(b) + z_0
\end{aligned}
\tag{10}
$$

The magnetic field is

$$
\begin{aligned}
B_r &= -\frac{\partial}{\partial z} \mathrm{A}_\theta \left(r, z\right) \\
B_z &= -\frac{\partial}{\partial r} \mathrm{A}_\theta \left(r, z\right) - \frac{\mathrm{A}_\theta \left(r, z\right)}{r}
\end{aligned}
\tag{11}
$$

The magnetic field along the orbit is written as

$$
\begin{aligned}
B_r &= -B_0 \\
B_z &= -B_0 \frac{R_z sin(b)}{R_r b + r_0}
\end{aligned}
\tag{12}
$$

2.3 Numerical solution for a mirror like magnetic vector potential

The magnetic mirror is a component that used to confine the charged particles. The vector potential used to define the axial symmetric magnetic field in a mirror field is given as

$$
A_\theta = \frac{A_1 \beta r}{2} - A_2 I_1(\beta r) \cos \beta z \tag{13}
$$

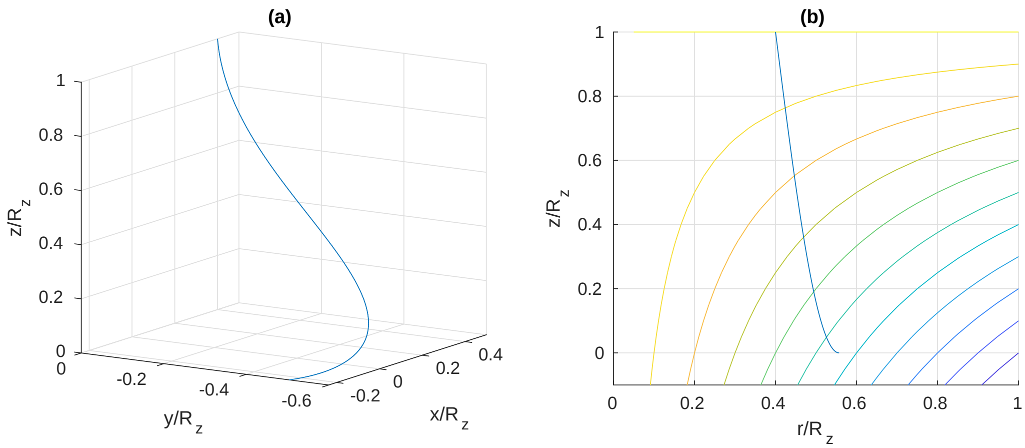

Figure 1: Analytical result of the reference orbit.

Where π/β is the mirror length, $\beta(A_1 + A_2)/(A_1 - A_2)$ is the mirror ratio.

The given magnetic field satisfies the Laplace's equation, which ensures the curl of the magnetic field zero. The linear approximation of the vector potential is given as

$$A_{theta} = \frac{\beta r}{2}(A_1 - A_2 \cos \beta z) \tag{14}$$

With the initial condition

$$\begin{aligned} r(0) &= r_0 \\ z(0) &= 0 \\ \theta(0) &= 0 \\ r'(0) &= r'_0 \\ z'(0) &= z'_0 \\ \theta'(0) &= \theta'_0 \end{aligned} \tag{15}$$

The canonical momenta P_θ is derived as

$$\gamma m_0 r_0^2 \theta' + q r_0^2 \beta/2(A_1 - A2) \tag{16}$$

The motion equation in theta direction could be written as

$$\begin{aligned} \theta' &= \theta'_0 r_0^2/r^2 + \frac{q}{\gamma m_0 r^2}(r_0 \, \mathrm{A}_{\theta 0} - r \, \mathrm{A}_\theta) \\ &= r_0^2/r^2(\theta'_0 + \frac{\beta q}{2\gamma m_0}(A_1 - A_2)) - \frac{\beta q}{2\gamma m_0}(A_1 - A_2 \cos \beta z) \end{aligned} \tag{17}$$

Under the initial condition $r'_0 = 0$, $\theta'_0 = -\frac{\beta q}{\gamma m_0}(A_1 - A_2)$, the motion equation is simplified as

$$
\begin{aligned}
\theta' &= -\frac{\beta q}{2\gamma m_0}\left(A_1 - A_2\cos(\beta z)\right)\left(1 + r_0^2/r^2\right) \\
\gamma m_0 r'' &- (\frac{\beta q}{2\gamma m_0}\left(A_1 - A_2\cos(\beta z)\right))^2 r = 0 \\
\gamma m_0 z'' &- \frac{\beta q}{2\gamma m_0}\left(A_1 - A_2\cos(\beta z)\right)\frac{\beta^2 q}{2\gamma m_0} A_2 \sin(\beta z) r^2 = 0
\end{aligned}
\tag{18}
$$

Where B_r and B_z is the magnetic field in z and r direction respectively

$$
\begin{aligned}
B_r &= -(\beta^2 r A_2/2) sin(\beta z) \\
B_z &= \beta(A_1 - A_2 cos(\beta z))
\end{aligned}
\tag{19}
$$

We use the TR100[2] main magnet model as a testbench to study the injection. Figure 2(a) shows the conceptual model. By tracking the particle reversely from the median plane to the injection point with different Pitch angles, the different reference orbits are shown in figure 2(b). The single B_r bump field near the median plane could reduce the pitch angle by about 20° from the injection point to the median plane.

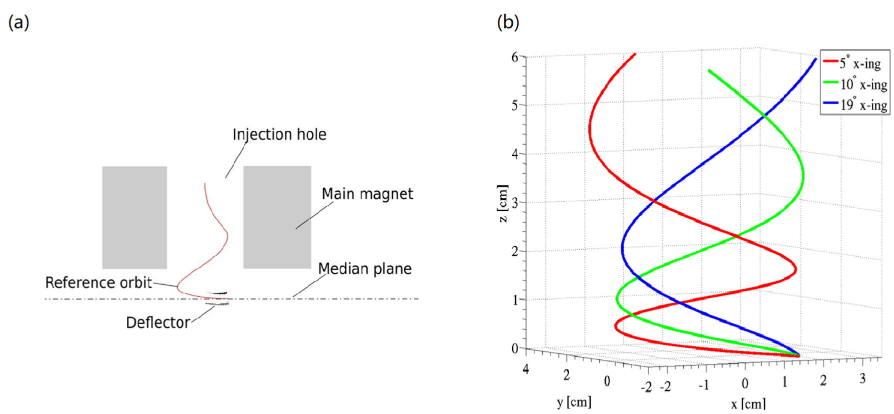

Figure 2: Reference orbit in the injection hole.

3 Beam optics

3.1 Coordinates Transformation

In this report, we use the coordinate (α, β, γ) in the optical coordinate system, which moves along the reference orbit as shown in figure 1 [?]. The γ direction is the same with the velocity of the reference particle. The β direction is perpendicular to the γ direction and parallel to the median plane. At the same time, the cross product of the unit vector of the γ direction and β

direction should have a positive projection on z-axis. The α direction is defined by the cross product of the unit vector of the γ direction and β direction.

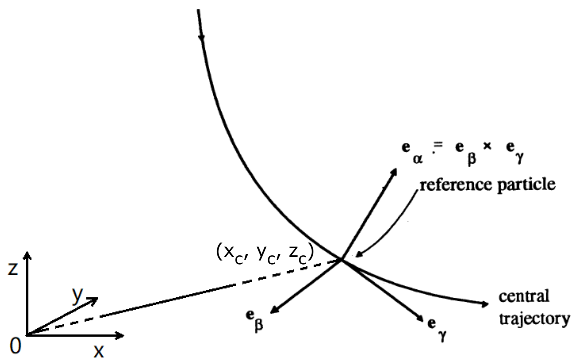

Figure 3: The moving optical coordinate system.

The position vector $\vec{c}$ of the point on the reference orbit under cartesian is written as

$$\vec{c} = (x_c(s), y_c(s), z_c(s)) \tag{20}$$

Where s, the distance along the reference orbit, is the independent variable. The base vector $\vec{e} = (\vec{e_\alpha}, \vec{e_\beta}, \vec{e_\gamma})$ of the moving optical coordinate on the reference orbit is written as

$$\begin{aligned} \vec{e_\alpha} &= (x_\alpha(s), y_\alpha(s), z_\alpha(s)) \\ \vec{e_\beta} &= (x_\beta(s), y_\beta(s), z_\beta(s)) \\ \vec{e_\gamma} &= (x_\gamma(s), y_\gamma(s), z_\gamma(s)) \end{aligned} \tag{21}$$

The transformation from cartesian coordinates (x, y, z) to the moving coordinates (α, β, γ) is written as

$$\begin{aligned} x &= x_c(s) + \alpha x_\alpha(s) + \beta x_\beta(s) + \gamma x_\gamma(s) \\ y &= y_c(s) + \alpha y_\alpha(s) + \beta y_\beta(s) + \gamma y_\gamma(s) \\ z &= z_c(s) + \alpha z_\alpha(s) + \beta z_\beta(s) + \gamma z_\gamma(s) \end{aligned} \tag{22}$$

The transformation matrix from the moving coordinates to the cartesian coordinates is written as

$$
\mathrm{M} = \begin{bmatrix} x_\alpha(s) & x_\beta(s) & x_\gamma(s) \\ y_\alpha(s) & y_\beta(s) & y_\gamma(s) \\ z_\alpha(s) & z_\beta(s) & z_\gamma(s) \end{bmatrix} \tag{23}
$$

The inverse transformation matrix $\mathrm{M}' = \mathrm{M}^T$, as the $(e_\alpha, e_\beta, e_\gamma)$ are orthogonal bases. Choosing a possible generating function that is consistent with Eqs. 3

$$
\begin{aligned}
G = &-P_x[x_c(s) + \alpha x_\alpha(s) + \beta y_\alpha(s) + \gamma z_\alpha(s)] \\
&- P_y[y_c(s) + \alpha x_\beta(s) + \beta y_\beta(s) + \gamma z_\beta(s)] \\
&- P_z[z_c(s) + \alpha x_\gamma(s) + \beta y_\gamma(s) + \gamma z_\gamma(s)]
\end{aligned} \tag{24}
$$

The new canonical momenta is derived from the given generating function

$$
\begin{aligned}
P_\alpha &= -\frac{\partial G}{\partial \alpha} = P_x x_\alpha(s) + P_y x_\beta(s) + P_z x_\gamma(s) \\
P_\beta &= -\frac{\partial G}{\partial \beta} = P_x y_\alpha(s) + P_y y_\beta(s) + P_z y_\gamma(s) \\
P_\gamma &= -\frac{\partial G}{\partial \gamma} = P_x z_\alpha(s) + P_y z_\beta(s) + P_z z_\gamma(s)
\end{aligned} \tag{25}
$$

The canonical momenta under the cartesian coordinate system is given by

$$
\begin{aligned}
P_x &= m_0 v_x + qA_x = m_0 v_0 x' + qA_x \\
P_y &= m_0 v_y + qA_y = m_0 v_0 y' + qA_y \\
P_z &= m_0 v_z + qA_z = m_0 v_0 z' + qA_z
\end{aligned} \tag{26}
$$

where the prime denotes differentiation with respect to s, v_0 is the velocity. Substitute eqt.7 into eqt.6, the new canonical momenta is written as

$$
\begin{bmatrix} P_\alpha \\ P_\beta \\ P_\gamma \end{bmatrix} = m_0 v_0 \mathrm{M}^T \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} + q\mathrm{M}^T \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \tag{27}
$$

Thus, the canonical momenta on the reference trajectory is

$$
\begin{bmatrix} P_{\alpha 0} \\ P_{\beta 0} \\ P_{\gamma 0} \end{bmatrix} = m_0 v_0 \mathrm{M}^T \begin{bmatrix} x_c' \\ y_c' \\ z_c' \end{bmatrix} + q\mathrm{M}^T \begin{bmatrix} A_{x0} \\ A_{y0} \\ A_{z0} \end{bmatrix} \tag{28}
$$

To make the canonical variable small quantities, we subtract eqt.9 from eqt.8. Thus, the generating function becomes

$$
\begin{aligned}
G = &-P_x[x_c(s) + \alpha x_\alpha(s) + \beta y_\alpha(s) + \gamma z_\alpha(s)] \\
&- P_y[y_c(s) + \alpha x_\beta(s) + \beta y_\beta(s) + \gamma z_\beta(s)] \\
&- P_z[z_c(s) + \alpha x_\gamma(s) + \beta y_\gamma(s) + \gamma z_\gamma(s)] \\
&+ \alpha P_{\alpha 0} + \beta P_{\beta 0} + \gamma P_{\gamma 0}
\end{aligned} \tag{29}
$$

The new momenta is given by

$$\begin{bmatrix} P_\alpha \\ P_\beta \\ P_\gamma \end{bmatrix} = m_0 v_0 \mathrm{M}^T \begin{bmatrix} x' - x'_c \\ y' - y'_c \\ z' - z'_c \end{bmatrix} + q\mathrm{M}^T \begin{bmatrix} A_x - A_{x0} \\ A_y - A_{y0} \\ A_z - A_{z0} \end{bmatrix} \tag{30}$$

Using eqt.3 and eqt.11, and expand the transform matrix M into a 6×6 matrix the transformation from (x, P_x, y, P_y, z, P_z) to $(\alpha, P_\alpha, \beta, P_\beta, \gamma, P_\gamma)$ is given by

$$\begin{bmatrix} \alpha \\ P_\alpha \\ \beta \\ P_\beta \\ \gamma \\ P_\gamma \end{bmatrix} = \mathrm{M}^T \begin{bmatrix} x - x_c \\ m_0 v_0 (x' - x'_c) \\ y - y_c \\ m_0 v_0 (y' - y'_c) \\ z - z_c \\ m_0 v_0 (z' - z'_c) \end{bmatrix} + q\mathrm{M}^T \begin{bmatrix} 0 \\ A_x - A_{x0} \\ 0 \\ A_y - A_{y0} \\ 0 \\ A_z - A_{z0} \end{bmatrix} \tag{31}$$

3.2 Transfer Matrix

In order to calculate the transfer matrix, we need to run 6 particles with initial coordinates and momenta given as

$$\begin{aligned} (\alpha, P_\alpha, \beta, P_\beta, P_\gamma) &= (1, 0, 0, 0, 0, 0) \\ (\alpha, P_\alpha, \beta, P_\beta, P_\gamma) &= (0, 1, 0, 0, 0, 0) \\ (\alpha, P_\alpha, \beta, P_\beta, P_\gamma) &= (0, 0, 1, 0, 0, 0) \\ (\alpha, P_\alpha, \beta, P_\beta, P_\gamma) &= (0, 0, 0, 1, 0, 0) \\ (\alpha, P_\alpha, \beta, P_\beta, P_\gamma) &= (0, 0, 0, 0, 1, 0) \\ (\alpha, P_\alpha, \beta, P_\beta, P_\gamma) &= (0, 0, 0, 0, 0, 1) \end{aligned} \tag{32}$$

Substitute eqt.13 into eqt.12, the coordinates and momenta under cartesian system is given as

$$\begin{bmatrix} x \\ P_x \\ y \\ P_y \\ z \\ P_z \end{bmatrix} = \mathrm{M} \begin{bmatrix} \alpha \\ P_\alpha \\ \beta \\ P_\beta \\ \gamma \\ P_\gamma \end{bmatrix} - q \begin{bmatrix} 0 \\ A_x - A_{x0} \\ 0 \\ A_y - A_{y0} \\ 0 \\ A_z - A_{z0} \end{bmatrix} + \begin{bmatrix} x_c \\ m_0 v_0 x'_c \\ y_c \\ m_0 v_0 y'_c \\ z_c \\ m_0 v_0 z'_c \end{bmatrix} \tag{33}$$

Tracking 6 particles with the given initial conditions, the coordinates and momenta at the exit of the inflector is calculated. They are given in cartesian coordinates and could be easily transformed in to the moving coordinates.

Thus, the transfer matrix through the inflector can be easily calculated using the transformed coordinates.

By tracking the 6 particles in Comsol, figure 1 shows the five orbits in cartesian coordinate system.

After transforming the coordinates at the end of the orbit into the moving coordinates with the unit (mm,mrad,mm,mrad,mm,mrad), the transfer matrix is calculated as

$$\mathrm{R} = \begin{bmatrix} 1.9899 & 0.1493 & -1.6822 & -0.0167 & 0.3753 & 0.1340 \\ -5.0231 & 0.1862 & -0.2335 & -0.1780 & 3.8668 & 0.2139 \\ 0.5800 & 0.0223 & 0.8386 & 0.0260 & -0.5524 & -0.0134 \\ -13.7973 & -0.3713 & -8.0835 & 0.3925 & 1.9409 & -0.6353 \\ -0.0311 & 0.0394 & -0.3041 & 0.0195 & 0.6095 & 0.0989 \\ 5.3240 & 0.2201 & -12.4786 & 0.2672 & -5.4282 & 0.8583 \end{bmatrix} \tag{34}$$

Test the symplectic of the transfer matrix,

$$\mathrm{R}^{\mathrm{T}}\mathrm{JR} - \mathrm{J} = \begin{bmatrix} 0 & -0.0043 & -0.0258 & 0.0364 & 0.0082 & -0.0084 \\ 0.0043 & 0 & -0.0151 & 0.0012 & -0.0026 & -0.0001 \\ 0.0258 & 0.0151 & 0 & -0.0024 & 0.0014 & 0.0036 \\ -0.0364 & -0.0012 & 0.0024 & 0 & 0.0007 & -0.0006 \\ -0.0082 & 0.0026 & -0.0014 & -0.0007 & -0.0000 & -0.0008 \\ 0.0084 & 0.0001 & -0.0036 & 0.0006 & 0.0008 & 0 \end{bmatrix} \tag{35}$$

Where J is given as

$$\mathrm{J} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \tag{36}$$

The error is between 10^{-3} and 10^{-2}, which may be resulted from the non-linear of the motion and the noise when tracking the particles numerically.

3.3 Beam envelopes

The beam envelope is studied in the $\alpha - \beta - \gamma$ moving frame. Figure 2 shows the horizontal (β) and vertical (α) envelopes with different magnetic field parameters. A proper beam focusing in both directions could be achieved by adjusting the mirror length and the mirror ratio.

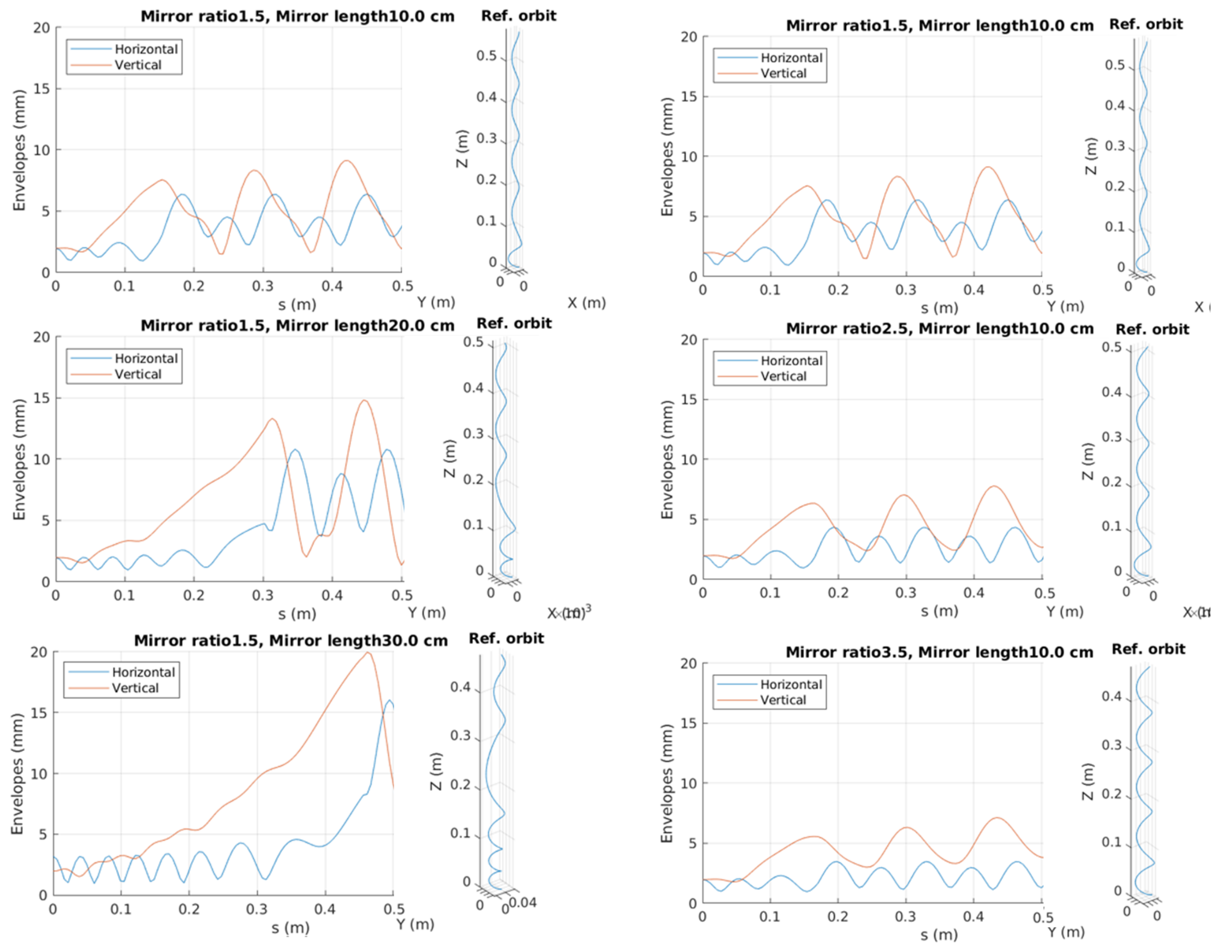

Figure 4: Beam envelopes with different mirror length and mirror ratio. On the right side of each envelopes plot is the reference orbit inside the injection hole.

4 Magnet design

4.1 A 2D main magnet model

I use a 2D model to calculate the magnetic field in the injection hole, but with the pseudo material which use a lumped factor k to calculate the B-H curve, different k means different width ratio of the hill, for a 4 sector magnet with the sector width of 45 degree, k is 0.5. For the yoke, k is 1, thus the B-H curve is that of the real steel. Figure 2 shows the B-H curve with different width ratio k.

Calculate the magnetic field using FEA software, figure 3 shows the magnetization of the iron and figure 4 shows the magnetic field in the injection hole along a vertical line.

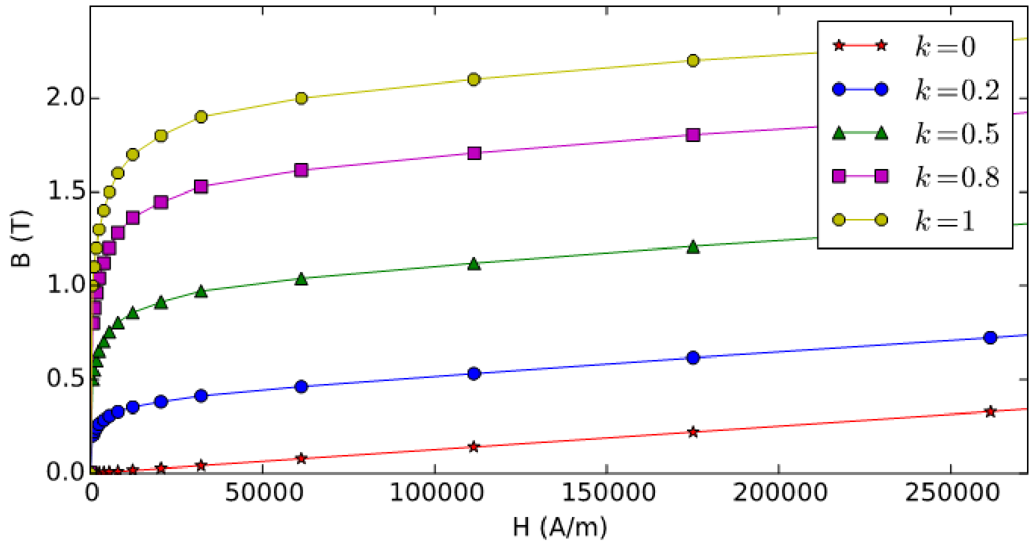

Figure 5: B-H curve of the pseudo material with different sector width ratio.

4.2 Steel plug to adjust the mirror field

Figure 8 shows the structure of the central plug that we used to optimize the mirror field in the injection hole. Figure 9 shows the magnetic field of the optimal magnet model.

5 High intensity simulation

Figure 10 shows the Comsol simulation of the high intensity beam.

6 Conclusion

To maintain the median plane symmetry of the magnet, an electrostatic plate should be placed at the end of the magnetic inflector, which will finally deflect the beam into the median plane with 0 vertical momenta. The envelope study suggests that The beam could be focused both horizontally and vertically by optimizing the mirror ratio and mirror length. A steel plug in the center region is designed to optimize the parameters of the magnetic inflector field.

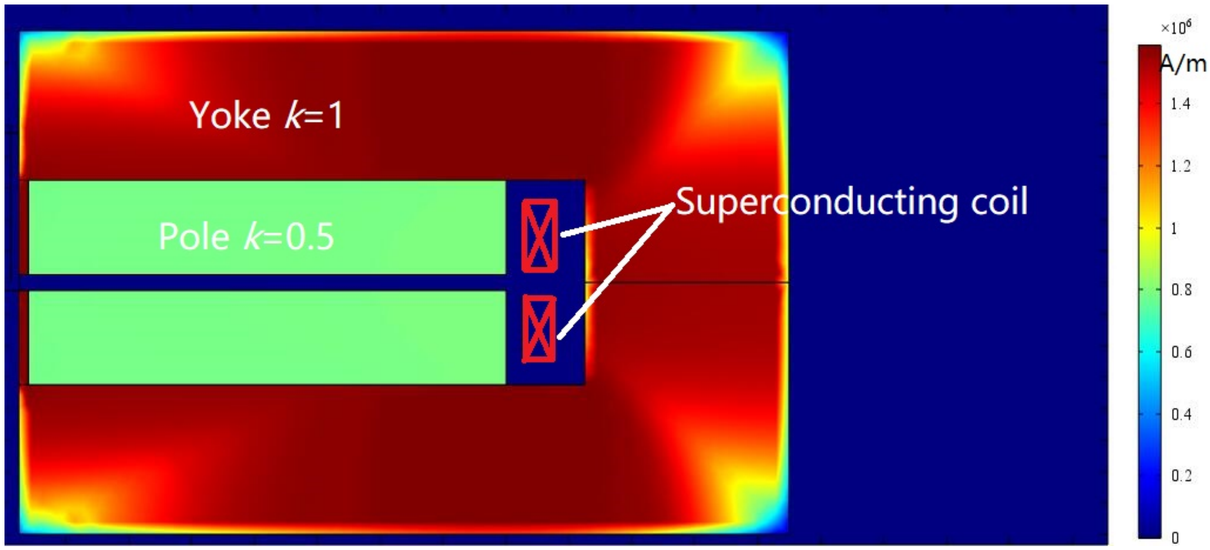

Figure 6: Cross section of the main magnet.The color map shows the magnetization M, unit is A/m.

References

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[2] Y.-N. Rao, R.A. Baartman, Y. Bylinskii, T. Planche, and L.G. Zhang. Conceptual Design of TR100$^+$: An Innovative Superconducting Cyclotron for Commercial Isotopes Production. In *Proc. Cyclotrons'19*, number 22 in International Conference on Cyclotrons and their Applications, pages 298–301. JACoW Publishing, Geneva, Switzerland, jun 2020.

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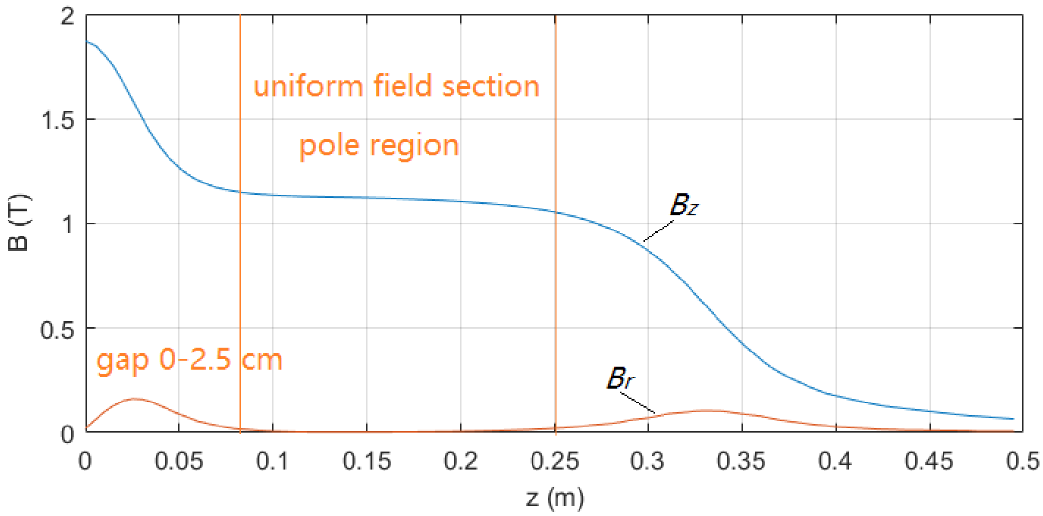

Figure 7: Magnetic field along r=1 cm.The field could be recognized as 3 sections, there is a uniform B_z section, and B_r in this section is 0.

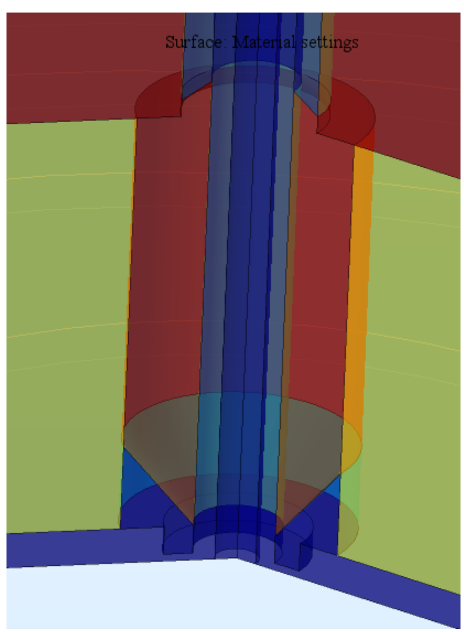

Figure 8: Steel plug in the injection hole.

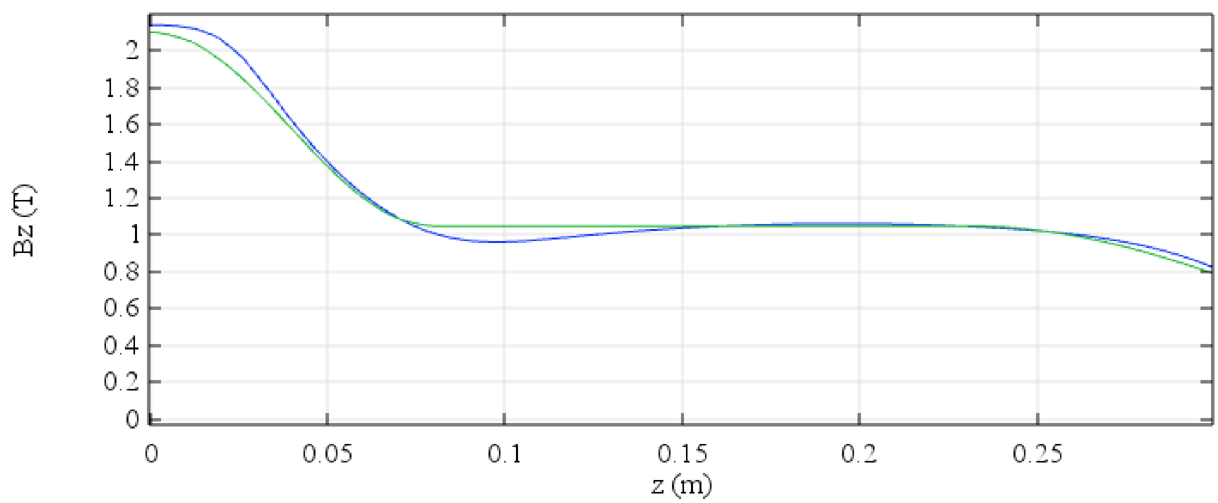

Figure 9: On-axis magnetic field in the injection hole after optimizing the shape of the center plug. Green line is the objective field that could properly focus the beam in the inflector. Blue line is the magnetic field in the FEA model of the designed main magnet with central plug.

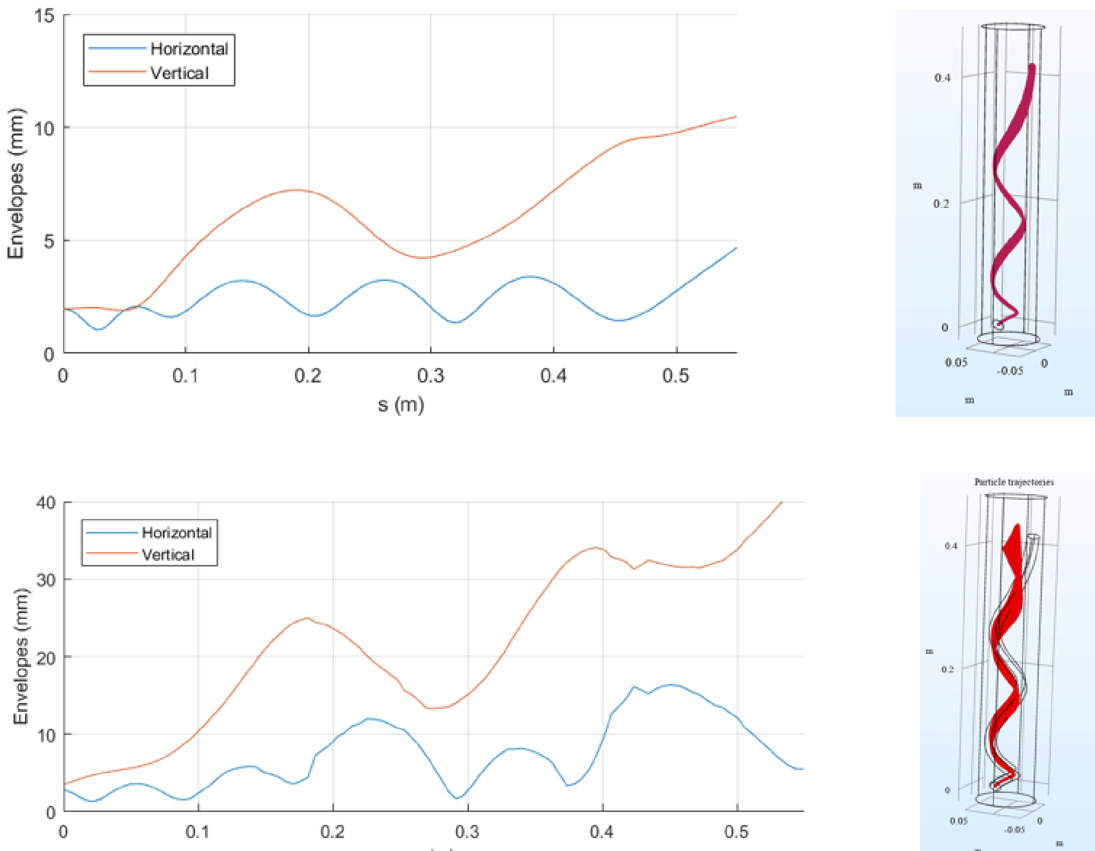

Figure 10: Envelope simulation considering space charge. The upper plots shows the simulation under 1 nA injection beam. The lower one shows that of the 10 mA injection beam. The frame of a spiral pipe in the lower 3D beam plot shows the reference beam path without considering the space charge effect. Obviously, the reference orbit is changed by the space charge. A further design study of a shielding structure is needed to remove the repulsive force from different turns.